**Crux Mathematicorum** is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

https://publications.cms.math.ca/cruxbox/

**Crux Mathematicorum** est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n’est pas une revue scientifique. Soumission en ligne:

https://publications.cms.math.ca/cruxbox/

The Canadian Mathematical Society grants permission to individual readers of this publication to copy articles for their own personal use.

© CANADIAN MATHEMATICAL SOCIETY 2022. ALL RIGHTS RESERVED.
ISSN 1496-4309 (Online)

La Société mathématique du Canada permet aux lecteurs de reproduire des articles de la présente publication à des fins personnelles uniquement.

© SOCIÉTÉ MATHÉMATIQUE DU CANADA 2022. TOUS DROITS RÉSERVÉS.
ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
- University of the Fraser Valley

**Editorial Board**

**Editor-in-Chief**  
Kseniya Garaschuk  
University of the Fraser Valley

**MathemAttic Editors**  
John Grant McLoughlin  
University of New Brunswick
Shawn Godin  
Cairine Wilson Secondary School

**Olympiad Corner Editors**  
Alessandro Ventullo  
University of Milan
Anamaria Savu  
University of Alberta

**Articles Editor**  
Robert Dawson  
Saint Mary’s University

**Associate Editors**  
Edward Barbeau  
University of Toronto
Chris Fisher  
University of Regina
Edward Wang  
Wilfrid Laurier University
Dennis D. A. Epple  
Berlin, Germany
Magdalena Georgescu  
BGU, Be’er Sheva, Israel
Chip Curtis  
Missouri Southern State University
Philip McCartney  
Northern Kentucky University

**Guest Editors**  
Yagub Aliyev  
ADA University, Baku, Azerbaijan
Andrew McEachern  
York University
Vasile Radu  
Birchmount Park Collegiate Institute
Chi Hoi Yip  
University of British Columbia
Didier Pinchon  
Toulouse, France

**Translators**  
Rolland Gaudet  
Université de Saint-Boniface
Frédéric Morneau-Guérin  
Université TÉLUQ

**Editor-at-Large**  
Bill Sands  
University of Calgary
IN THIS ISSUE / DANS CE NUMÉRO

584 MathemAttic: No. 40
584 Problems: MA196–MA200
587 Solutions: MA171–MA175
590 Teaching Problems: No. 19 John McLoughlin
594 Mathemagical Puzzles: No. 2 Tyler Somer
597 From the bookshelf of... John McLoughlin
600 Mathematics from the Web: No. 5
601 Pell’s Equation and Problem Solving Amit Kumar Basistha

611 Olympiad Corner: No. 408
611 Problems: OC606–OC610
613 Solutions: OC581–OC585
623 The number of 2s in prime factorization of superfactorials and Vandermonde determinants Yagub Aliyev
628 Focus On . . . : No. 53 Michel Bataille
633 Problems: 4791–4800
637 Solutions: 4741–4750

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum
with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin
MATHEMATTIC

No. 40

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by February 15, 2023.

MA196. If a cue ball placed at the coordinates (22, 55) was evenly struck so it hit the 1st wall at the point (44, 77), and bounced off with no spin, what are the coordinates when the ball strikes the 6th wall?

MA197. A rectangle has length 4 and width 6. A new shape is formed by taking the set of all points that lie within one unit of a point on the boundary of the rectangle. Compute the area of this new shape.

MA198. Two points P and Q are randomly selected in the interval [0, 2]. What is the probability that P and Q are within a distance of $\frac{1}{3}$ from each other?

MA199. Proposed by Aravind Mahadevan.

If $a$, $b$, and $c$ are the roots of the equation $x^3 + 6x^2 - 52x + 8 = 0$, find the value of

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

In the figure below, $ABCD$ is a square. Arcs $BD$ and $AC$ intersect at $E$. Determine the exact value of $\frac{AE}{EC}$.

![Image of a square with arcs](image)

*Les problèmes dans cette section sont appropriés aux étudiants de l’école secondaire.*

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 février 2023.*

MA196. Une boule de billard située aux coordonnées $(22, 55)$ est frappée de plein fouet de manière uniforme. Elle heurte le 1er mur au point $(44, 77)$ et rebondit sans effectuer un mouvement rotationnel. Quelles sont les coordonnées du point de contact de la boule avec le 6e mur?

![Image of a pool table](image)
MA197. On considère un rectangle de longueur 4 et de largeur 6. Une nouvelle figure est produite en prenant l’ensemble de tous les points qui se trouvent à moins d’une unité d’un point sur la limite du rectangle. Calculez l’aire de cette nouvelle figure.

MA198. On choisit au hasard deux points \( P \) et \( Q \) dans l’intervalle \([0, 2]\). Quelle est la probabilité que la distance entre \( P \) et \( Q \) soit inférieure à \( \frac{1}{3} \)?

MA199. Soumis par Aravind Mahadevan.
Si \( a, b \) et \( c \) désignent les racines de l’équation \( x^3 + 6x^2 - 52x + 8 = 0 \), déterminez la valeur de
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.
\]

Dans la figure ci-dessous, \( ABCD \) est un carré. Les arcs \( BD \) et \( AC \) se rencontrent en \( E \). Déterminez la valeur exacte de \( \frac{AE}{EC} \).

\[\text{Diagram of a square with arcs}\
\text{Labelled points A, B, C, D, and E.}]

---

*Crux Mathematicorum*, Vol. 48(10), December 2022
MATHEMATTIC SOLUTIONS


MA171. PBCQ is a trapezoid in which PQ : BC = 2 : 3. If the area of triangle ABC is 36, then determine the area of PBCQ.

Let D and E be points on PQ and BC respectively such that DE is perpendicular to PQ (and BC) and such that A lies on DE. Then AD and AE are the altitudes of the triangles APQ and ABC respectively. Since PQ is parallel to BC, the triangles APQ and ABC are similar. Therefore

\[
\frac{AD}{AE} = \frac{PQ}{BC} = \frac{2}{3}.
\]

Thus

\[
[PBCQ] = DE \left( \frac{PQ + BC}{2} \right) = (AD + AE) \left( \frac{PQ + BC}{2} \right) = \left( \frac{5}{3} AE \right) \cdot \frac{1}{2} \left( \frac{5}{3} BC \right) = \frac{25}{9} \left( \frac{AE \cdot BC}{2} \right) = \frac{25}{9} |ABC| = \frac{25}{9} \cdot 36 = 100.
\]

Copyright © Canadian Mathematical Society, 2022
MA172. Moe and Joe start together at point $A$ and walk towards point $B$. Moe walks $x$ times as fast as Joe. Moe reaches $B$, then travels back until he meets Joe. Determine the fraction of the distance $AB$ that Joe has travelled at this point.

*Originally question 28 of the 1973 Junior Mathematics Contest, Sponsored by the University of Waterloo.*

We received 10 correct submissions. We feature a standard approach taken by most solvers.

Evidently, $x > 1$. Assume that the distance between $A$ and $B$ is 1 and that Joe has walked distance $d < 1$. The distance $1 + (1 - d)$ travelled by Moe is $xd$, whence $2 - d = xd$. Thus $d = 2/(1 + x)$ is the fraction of the distance $AB$ traversed by Joe.

MA173. You are given an acute-angled triangle $ABC$ in which $J$ is the centre of the ascribed circle which touches $BC$ (and touches $AB$ and $AC$ produced). Calculate the angle $AJC$ in terms of the angles in the triangle.

*Originally question 1 of the 1973 Ontario Senior Mathematics Problems Competition, Sponsored by the University of Waterloo.*

There were 3 correct solutions. We present that of Arvind Mahadevan.

\[
\angle BCJ = \frac{1}{2}(180^\circ - C) \quad \text{and} \quad \angle ACJ = 90^\circ + \frac{C}{2}.
\]

Therefore,

\[
\angle AJC = 180^\circ - \angle JAC - \angle ACJ = 180^\circ - \frac{A}{2} - 90^\circ - \frac{C}{2} = \frac{180^\circ - (A + C)}{2} = \frac{B}{2}.
\]

MA174. If $a, b, c$ are positive real numbers, find the least value of

\[
\left( \frac{b+c}{a} \right) \left( \frac{c+a}{b} \right) \left( \frac{a+b}{c} \right)
\]

*Originally question 9 of the 1973 Ontario Senior Mathematics Problems Competition, Sponsored by the University of Waterloo.*

We received 13 solutions, of which 9 were correct. We present the solution by Prithwijit De.

Since $a, b, c$ are positive real numbers, $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are well-defined real numbers. Observe that

\[
(\sqrt{a} - \sqrt{b})^2 \geq 0 \iff a + b \geq 2\sqrt{ab},
\]

with equality holding if and only if $a = b$. Similarly $b + c \geq 2\sqrt{bc}$ and $c + a \geq 2\sqrt{ca}$. Multiplying these inequalities together, we obtain

\[
(b + c)(c + a)(a + b) \geq 8abc,
\]

which is equivalent to the desired inequality.

*Crux Mathematicorum, Vol. 48(10), December 2022*
The least value of \(\frac{b+c}{a}\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right)\) is 8, and it is attained if and only if \(a = b = c\).

**MA175.** Given that \(p\) and \(q\) are two consecutive odd primes, show that their sum has three or more prime factors.

*Originally question 10 of the 1973 Ontario Senior Mathematics Problems Competition, Sponsored by the University of Waterloo.*

We received 9 submissions of which 4 were correct and complete. We present the solution by John Liao.

Since both \(p\) and \(q\) are odd integers, then their sum is even, and is therefore a multiple of 2. We let \(p + q = 2 \cdot r\) for some integer \(r\). This means \(r = \frac{p + q}{2}\).

Since \(r\) is the average of \(p\) and \(q\), and \(p < q\), then \(r\) is between \(p\) and \(q\):

\[
p < q \implies p + p < q + p \implies p < \frac{p + q}{2} \implies p < r
\]

and similarly:

\[
p < q \implies p + q < q + q \implies \frac{p + q}{2} < q \implies r < q.
\]

Since \(p\) and \(q\) are consecutive primes, any number between them will not be a prime. We know that \(r\) is between \(p\) and \(q\), so this means that \(r\) is a composite number, meaning that it is divisible by at least two primes or else \(r\) would be a prime itself.

Since \(p + q = 2 \cdot r\), then \(p + q\) would have at least 3 prime divisors: 2 and the prime divisors of \(r\), for which there are at least two of them.
TEACHING PROBLEMS

No. 19

John Grant McLoughlin

Square Dancing

Sixteen dancers each distinctly numbered from 1 through 16 are to be paired so that the sum of the numbers in each pairing is a perfect square. Determine the pairings.

One way to begin this problem would be to list each of the numbers along with the plausible candidates for partners. For example, 1 could be paired with any of 3, 8, or 15. Rather than beginning with such an approach let us make some observations.

- Any number that is half of a perfect square will have less possible partners than it may appear to have at first. For example, 2 cannot be added to another number to get 4, and likewise 8 cannot be added to another number to get 16.

- The largest possible perfect square total is 25 as the upper bound of the total is the sum of the largest two numbers, namely, 15 and 16.

- Hence, the only possible perfect square sums to consider are 4, 9, 16, and 25.

- Since 16 is a perfect square, it must be paired with 9 as only 25 is attainable.

We have 16 and 9 as our first pairing. It makes sense to consider what other numbers may have been possible partners with these numbered individuals. We know that 16 had only one possible partner but 9 could have been paired with 7. Perhaps 7 is limited now by the removal of 9. In fact, now only 2 can be paired with 7. (Check that you understand this logic as 7 + 16 will not reach 25 and 7 + 9 is no longer permissible.)

Proceeding, we may consider a number that could have used 2, as in numbers other than 2 that are two less than perfect squares, namely, 7 and 14. Only 14 remains. We can see that 14 will require 11 as a partner. Proceeding with this idea, you may wish to complete the pairings.

The list below may be helpful with the used numbers struck out, as shown.

\[
\begin{array}{cccccccc}
1 & \cancel{2} & 3 & 4 & 5 & 6 & \cancel{7} & 8 \\
\cancel{9} & 10 & \cancel{11} & 12 & 13 & \cancel{14} & 15 & \cancel{16}
\end{array}
\]
In summary, the pairings thus far are 16 and 9, 2 and 7, along with 14 and 11. You may wish to complete the pairings by considering what number(s) may have needed 11, or alternatively, by making your own observations to complete the process. Do this before proceeding further with the problem.

More dancers

Suppose that another eight dancers were added to the mix with the challenge being to have all dancers numbered 1 through 24 paired in such a way that a perfect square sum is attained for each pairing. Is this possible? What about if another eight people join the dance so that the numbers extend from 1 to 32? And eight more? Play with these larger amounts as an investigation of sorts. What do you find? What observations would be helpful to share?

Suppose the opening challenge was reframed in the following manner.

A total of $8n$ dancers each distinctly numbered from 1 through $8n$ are to be paired so that the sum of the numbers in each pairing is a perfect square. Determine if it is possible to completely pair the $8n$ dancers given this requirement.

Are there values of $n$ for which you could confidently convince another person whether this is plausible (or not) with minimal work? Comment on your observations. Feedback is welcomed.

The Woolly Sheep Problem

Two brothers sold a herd of sheep. For each sheep they received as many dollars as the number of sheep in the original herd. The money was divided as follows: the older brother took 10 dollars; then the younger brother took 10 dollars; next the older brother took another 10 dollars; and the younger brother took another 10 dollars; and so on. At the end of the division, the younger brother, whose turn it was, found that there was less than 10 dollars left for him. He took what remained. In order to even things up, the older brother gave him a promissory note. What was the value of the promissory note?

This problem has appeared in various forms. One version asks for the value of the penknife that is given by the older brother to even things up, as stated in Problems of the Week by Jim Totten in the CMS ATOM series publication. The version above is the one that was introduced to me in 1982 at University of Waterloo in a course with Dean Hoffman. The so-called 100 Problems course referred to the opening handout with 100 problems of which this was one.

Why do I like this problem and why is it here? I’ve included the problem in this feature as it too deals with perfect squares as the number of sheep correspond to the number of dollars paid per sheep, thus, again resulting in a perfect square. The problem is one that has figured into my teaching of many courses over the years, the
most memorable being a course that was central to my doctoral dissertation (Grant McLoughlin, 1997). A problem of its own emerged early along in that course with a wide range of mathematical backgrounds. Some students saw challenges as racing opportunities while others got discouraged. This was the problem that I chose to use that September to stem that tide with a complete class allocated to a single problem. The beauty of this problem is that it is relatively easy to understand. Yet there appears to almost be a lack of information. How could one wrestle with it? Begin by playing and as we learned in class it served well as a collective problem-solving experience.

Returning to the problem, what do we notice? Begin playing with some small numbers and the remainders upon division of the funds. For the sake of it let’s say there were 10 sheep worth 10 dollars. There would be nothing left over. We determine quickly that the number of sheep is not a multiple of 10. How about 11? The resulting 121 dollars would only leave 1 extra dollar for the older brother. This would not satisfy the requirement of having 10 dollars left over for the older brother and some smaller amount for the younger brother.

What do we actually need to find here? What numbers could work? Every 20 dollar amount is divided equally but the left over amount upon division by 20 must exceed 10, in that the older brother gets 10 dollars and the youngest brother some additional dollars less than 10. Here we have the essence in understanding the problem. Now the opportunity for teaching some mathematics may arise as it did in that course. Several students had not been introduced to modular arithmetic and the idea was helpful here in formalizing an approach.

Consider any number of sheep can be represented in the form $10k+r$ where $r$ is the remainder when the number of sheep is divided by 10. If we take $(10k+r)(10k+r)$ representing the amount of money paid for the sheep, then each component of the resulting simplified product will be divisible by 20, aside from the $r^2$ piece.

Now let us look at the possible values of $r$ from 0 through 9 inclusive and the remainders when $r^2$ is divided by 20, namely, $r^2 \pmod{20}$:

<table>
<thead>
<tr>
<th>$r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^2 \pmod{20}$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>5</td>
<td>16</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

The only remainder that can be obtained between 10 and 20 is 16. That is, the amount left over to be divided in the final stage of the money sharing process must have been $16. It follows that the older brother received $10 and the younger brother only $6. The value of the promissory note required to balance the amounts would have been half of the difference, namely, $2.

In conclusion, the value of the problem from a teaching perspective is enhanced by the fact that the problem typically takes time to process. Further, there is a blend of novelty in both the presentation of information and the formal solution that make the problem one that most students will learn from through the experience.
Any feedback on these problems is welcomed via email (johngm@unb.ca). Contributions of your examples of teaching problems can be sent along also for consideration and possible publication here.

Reference
MATHEMAGICAL PUZZLES
No. 2
Tyler Somer
Molten Gold - II

The first article of this series introduced two illusions that give the impression that some figures could be dissected and rearranged such that the new figure has an area which is 1 unit larger. Many presenters will demonstrate one or the other illusion as a form of magic, and will rarely give the viewer the chance to ponder the solution. Without a physical model to handle, or diagram to study, the viewer will likely not investigate the illusion. I hope that many readers used that first article to build their own models of both illusions, and discuss with their own viewers some of the mathematics that form the basis of the illusions.

With magic, the presenter does not want the audience to solve the mystery of the demonstration. With puzzles, by contrast, the designers and builders do indeed want the audience to solve the objects. Even after being solved, some puzzles may still seem “magical” since the solver does not yet understand the principle of the puzzle’s design.

Consider Figure 1: the frame seems to be filled by three pieces. One piece is slender, along the right side of the frame. The other two pieces are roughly triangular, but with an irregularity near the centre of the assembly. In Figure 2, two pieces have been placed in alternative positions, revealing an additional unit of area in the centre.

As discussed in the previous article in this series, the increase in area came about by way of dissecting a particular shape and then rearranging the pieces into a different shape. Here, we have the same shape, yet we also have the paradoxical increase in area. This is astonishing!

The example above is complicated, in that it involves angles of approximately 45 degrees, if the frame is a square. Further, the unit increase here is not quite a square. Figures 3 and 4 correspond to Figures 1 and 2 respectively. With the
additional gridlines, one can see the square frame of the puzzle. The gridlines also illustrate that the diagonal is not quite 45 degrees, and that the unit increase is not quite a square. Note that in order to use pieces with 45-degree angles, and to have a square-unit increase, the frame in this example cannot be a square. The illusion/paradox is most effective when the frame is perceived to be a square. What appears to be a simple puzzle actually disguises the paradox, and so it seems magical.

![Figure 3](image1) ![Figure 4](image2)

Unlike the example above, most area-paradox puzzles have square or rectangular pieces, so the pieces are packed parallel to the sides of the frame. Regardless of the shape of the frame and the shapes of the component pieces, there must be some space allowance – play – for the pieces to shift fractionally in the frame. Most viewers and solvers discount the play as an acceptable flaw in the manufacturing of the puzzle. In truth, the designer has carefully determined the amount of play, by calculating the actual dimensions of the pieces and the frame.

Many such area-paradox puzzles increase by a unit square. The square is sometimes parallel to the sides of the frame, while others have the square at some angle, similar to the example illustrated here. Three puzzles in the *Geometrex*™ series by Rex Games do have a square to insert at some angle. Sadly, this series of puzzles seems to be out of print. An internet search is fruitless at this time.

Returning to our current example: When the puzzle is presented to the solver, the pieces – all but the one enigmatic piece – appear to fill the frame. Many solvers will remove some of these pieces and try to rearrange them: either shifting them (translation) or flipping them (usually by horizontal and/or vertical reflections only). Such a solving strategy leads to failure, as the amount of play in the frame remains essentially unchanged.

The correct method of solution – in most cases – is to rotate the pieces by 90 degrees. With this strategy, the $x$- and $y$-dimensions of the pieces are switched, and the amount of play changes. Some of the pieces may yet need to be flipped over, but the rotation is the critical movement. After some shifting of the pieces, the space which accommodates the unit piece appears. The puzzle is solved.
The sample solution given in Figure 2 – and so Figure 4 as well – is misleading. The more likely solution is given by Figure 5, in which the narrow strip on the right is actually considered part of the frame. The two large near-triangular pieces swap their $x$- and $y$-dimensions by way of a 90-degree rotation, then they are flipped and shifted appropriately to reveal the unit space. Figure 6 is given, to show the result with gridlines.

Note that the illustrations presented in this article are comparatively simple, since this type of puzzle rarely has merely two pieces to rearrange. By contrast, the Geometrex$^\text{TM}$ series, mentioned above, have between six and twelve pieces to rearrange. This is typical of such puzzles. Interested readers are invited to search for The Curry Triangle Puzzle as a starting point for area-paradox puzzles online.

In the next article, we will consider some design calculations for other two-dimensional area paradoxes.

When he was teaching, Tyler often had mechanical puzzles in his classroom. As a freelancer, Tyler has worked with numerous inventors and co-designers to bring dozens of table-top solo-logic puzzle kits to market. He continues to design puzzles, and he spends a good deal of time in his woodshop, building his own and others’ puzzle designs.
From the bookshelf of . . .

John Grant McLoughlin

This new feature of MathemAttic brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to MathemAttic@cms.math.ca. Publishers are also welcome to send along books for possible review.

Mathematical Quickies: 270 Stimulating Problems with Solutions
by Charles W. Trigg
ISBN 0-486-24949-2, 210 + ix pages
Published by Dover Publications, 1985 (2nd ed.)

This book has been on my shelf for years. Curiously it usually sits unopened with its usefulness arising from periodic glances at its material contents. The appeal of the book is exactly that. Quickly opening the book at almost any page offers a problem that one can readily state and play with over time. Instead, it may be that a problem appears curious enough to motivate playing with it as with the selection that arose upon just opening the book now. Problem 132 states:

The nine positive digits can be arranged into 3-by-3 arrays in 9! ways.
Find the sum of the determinants of these arrays.

Curiosity is a critical component of mathematical learning. Frequently a problem selection sparks an interest to either play with the problem or to seek the solution. It seems like a cop out to not solve a problem. However, my experience has been that one of the best ways to learn mathematics while improving problem solving skills comes via thoroughly reading solutions. This is an underutilized method of educating one’s self about mathematics, in my opinion. Insights into the solution process offer value in terms of content as well as giving an idea as to how mathematicians tackle problems. A hallmark of this book is the elegance of solutions, thus, making it a fine reference for learning mathematics through consideration of solutions. Occasionally it is a clever use of a theorem or lesser-known mathematical fact that figures prominently. Heightened awareness of the application of such ideas enhances problem solving prowess.

The context for the problem selection in the book is relevant. Trigg introduced a section of the Problems and Questions feature of Mathematics Magazine entitled Quickies while serving as an editor. Its heading statement appears here, as it sets the tone for the nature of the questions in the book.
From time to time this department will publish problems which may be solved by laborious methods, but which with proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known. (p. vii)

The result is that problems contained in *Mathematical Quickies* offer either elegance or quickness, or both, in their solutions. Hence, it becomes practical to pack 270 different problems along with solutions into just over 200 pages. The conciseness of solutions is magnified when one considers the large amount of space taken up by diagrams and credits in the solutions’ section. Each problem is given a short title with many of these hinting at aspects of the content such as *Pied Product*, *Sine Sum*, *Relatively Prime Integers*, or *Diophantine Duo*.

A final section follows the problems and solutions, as the author honourably credits the contributors of questions and solutions. Veteran readers of *Crux* will surely recognize some of the names from earlier years including the two most prolific contributors of solutions, namely, Leo Moser and Murray Klamkin. In addition, W. J. Blundon and H. S. M. Coxeter appear on the list. Meanwhile the inclusion of Euclid, Pappus, Pythagoras and some others caught me by surprise. The listing is clearly thorough.

In summary, the book is on my shelf because it serves as a quick “go to” for whatever purpose needed – whether that be as a source for an intriguing problem, a quick reference for learning something, or a browse knowing that a blend of the aforementioned purposes will emerge. It is fitting to close with a trio of problems for your consideration, the third of which is the problem that accompanies the cover picture of the book.

17. **Quartic with a Rational Root**

Show that the quartic equation

\[ ()x^4 + ()x^3 + ()x^2 + ()x + () = 0 \]

where the gaps are filled in by any arrangement of the numbers 1, −2, 3, 4, −6 always has a rational root.

138. **Dissection of Triangle into Two Similar Triangles**

Show that any given triangle can be dissected by straight cuts into four pieces which can be arranged to form two triangles similar to the given triangle.

*Crux Mathematicorum*, Vol. 48(10), December 2022
254. Colorful Square Arrays

*In how many distinct patterns can 9 congruent squares – 3 red, 3 white, and 3 blue – be arranged in a square array so that all three colors appear in every column and row?*
MATHEMATICS FROM THE WEB
No. 5

This column features short commentaries or descriptions of mathematical items from the internet that may be of interest to pre-university students and teachers. Your contributions are welcomed and may be sent to mathemattic@cms.math.ca.

Transum – Unfinished Game
https://www.transum.org/Software/MathsMenu/Starter.asp?ID_Starter=46
Transum offers many curricular and recreational mathematics puzzles and problems from the UK. Some of their Advanced Starters are great introductions to more theoretical mathematics concepts like this probability challenge.

A coin is tossed repeatedly. If it comes up heads Pascal gets a point but if it comes up tails Fermat wins a point. The first person to win three points is the winner and receives the prize of £12. Unfortunately the game had to end abruptly after three tosses of the coin. Pascal had two points and Fermat had one point. They decided to share the £12 in a ratio that matched the probability of them winning the game if it had continued. How should they divide the £12?

(Submitted by Carly Ziniuk, teacher, The Bishop Strachan School, Toronto, Ontario.)

Diophantine Equation, problem #66 from the website Project Euler
https://projecteuler.net/problem=66
Readers who enjoyed the article Pell’s Equation and Problem Solving in this issue may enjoy exploring this problem. Starting in 2001, Project Euler currently has more than 800 problems on its website. These problems are typically solved using a combination of mathematical knowledge and computer programming, many of which will be of interest to readers of MathemAttic.

(Submitted by Doddy Kastanya, Oakville, Ontario.)

Performance Assessment Resource Bank - Dog’s Play Area
https://www.performanceassessmentresourcebank.org/bin/performance-tasks
In this larger performance task, you can independently explore the optimal location to tether a dog so that it does not dig in a specific section of the garden with still the largest play area. Originating from Stanford, the problem offers different measurements and constraints, asking you to consider generalizing the problem and then extending it by exploring the relationship between radii and chord, as it affects the distance the dog can travel.

(Submitted by Carly Ziniuk, teacher, The Bishop Strachan School, Toronto, Ontario.)
Pell’s Equation and Problem Solving

Amit Kumar Basistha

In this article I intend to introduce readers to the theory of Pell’s equation and its application in solving a variety of number theoretic problems. Here, I do not intend to prove the majority of the theorems and results and will focus more on problem solving. Interested readers may refer to the references in the bibliography for a more detailed analysis of the theory.

1 The equation

A lot of problems involving natural numbers can be tackled by reducing them to equations of the form $x^2 - dy^2 = 1$. Such an equation where $x$, $y$, and $d$ are natural numbers and $d$ is not a perfect square is called Pell’s equation. Notice that $x^2 - dy^2 = (x + \sqrt{d}y)(x - \sqrt{d}y)$. Also if $m$ is a natural number then we can write $(x_1 + \sqrt{d}y_1)^m = x_m + \sqrt{d}y_m$ and $(x_1 - \sqrt{d}y_1)^m = x_m - \sqrt{d}y_m$. So we have

$$x_m^2 - dy_m^2 = (x_m - \sqrt{d}y_m)(x_m + \sqrt{d}y_m) = (x_1 + \sqrt{d}y_1)^m(x_1 - \sqrt{d}y_1)^m = (x_1^2 - dy_1^2)^m$$

Thus given an ordered pair that is a solution, we can generate infinitely many solutions. But finding the smallest solution can take a lot of computational effort. To have a feel of that the reader may try to find the smallest solution for $d \leq 20$. In fact, for $d = 991$ the smallest possible solution is

$$x = 379 516 400 906 811 930 638 014 896 080,$$

$$y = 12 055 735 790 331 359 447 442 538 767.$$

So the most natural question to ask is does such a solution always exist, to which the answer is yes. I shall only briefly touch that topic but let me get over some formalities first. The least positive solution, $(x_1, y_1)$, of the equation is called the fundamental solution. In fact it can be proved using elementary methods that (you should try proving it) $(x_m, y_m)$ is a solution if and only if

$$x_m + \sqrt{d}y_m = (x_1 + \sqrt{d}y_1)^m$$

or equivalently,

$$x_m = \frac{1}{2} \left( (x_1 + \sqrt{d}y_1)^m + (x_1 - \sqrt{d}y_1)^m \right)$$

$$y_m = \frac{1}{2\sqrt{d}} \left( (x_1 + \sqrt{d}y_1)^m - (x_1 - \sqrt{d}y_1)^m \right)$$
Evidently the $x_i$’s and the $y_i$’s are strictly increasing sequences of positive integers.

So the hurdle is to find the fundamental solution $(x_1, y_1)$. It appears that the fundamental solution is related to the simple continued fraction expansion of $\sqrt{d}$. We have

$$\sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

where $a_0 = [\sqrt{d}]$ and it can be proved that $a_1, a_2, a_3, \ldots$ is a periodic sequence of positive integers. In compact form it is represented as $[a_0; a_1, a_2, a_3, \ldots]$. There is an algorithm to find the continued fraction representation of any irrational number. Since it is of no significant use to us here, I resist my temptation to provide it. One can find the algorithm in [2].

The number

$$C_k = \frac{p_k}{q_k} = [a_0; a_1, a_2, \ldots, a_k]$$

is called the $k$-th *convergent* of the continued fraction. If $m$ is the length of the period of $\sqrt{d} = [a_0; a_1, a_2, a_3, \ldots]$, then the fundamental solution is given by

$$(x_1, y_1) = \begin{cases} (p_{m-1}, q_{m-1}), & \text{if } m \text{ is even} \\ (p_{2m-1}, q_{2m-1}), & \text{if } m \text{ is odd} \end{cases}$$

Now, with this theoretical background, let us get our hands dirty and try solving some problems.

**Problem 1**: Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k.$$

Show that $2x - 1$, $2x$, and $2x + 1$ are the sides of a triangle whose area and inradius are also integers.

**Solution**: Evidently the numbers $2x - 1$, $2x$, and $2x + 1$ satisfy the triangular inequality as $x \geq 1$. Notice that

$$x = \frac{1}{2} \left( (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \right).$$

Let

$$y = \frac{1}{2\sqrt{3}} \left( (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right).$$

Clearly $y$ is also an integer. Notice that $x$ and $y$ satisfy $x^2 - 3y^2 = 1$. The triangle with sides $2x - 1$, $2x$, and $2x + 1$ has semiperimeter $s = 3x$ and area

$$\Delta = \sqrt{3x(x - 1)x(x + 1)}$$

$$= x\sqrt{3(x^2 - 1)}$$

$$= x\sqrt{9y^2} = 3xy$$

*Crux Mathematicorum*, Vol. 48(10), December 2022
Therefore, the inradius is \( r = \frac{A}{s} = \frac{3\sqrt{3}}{4x} = y \), an integer.

**Note:** In fact all the triangles with consecutive side lengths and integer area have side lengths \( 2x - 1, 2x, 2x + 1 \) where \( x = \frac{1}{2} \left( (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \right) \). Hint: Try to derive the equation \( u^2 - 3v^2 = 1 \) from the figure below.

![Triangle Diagram](image)

**Problem 2:** Show that there are infinitely many pairs \((a, b)\) of relatively prime integers such that both the quadratics \( x^2 + ax + b = 0 \) and \( x^2 + 2ax + b = 0 \) have integer roots.

**Solution:** Consider the equation \( x^2 + 4ux + (4u^2 - 1) = 0 \). For any natural number \( u \), \( 4u \) and \( 4u^2 - 1 \) are co-prime and the equation has integer roots \( -2u + 1 \) and \( -2u - 1 \). Let \( a = 2u - 1, b = 4u^2 - 1 \), so \( x^2 + 2ax + b = x^2 + 8ux + (4u^2 - 1) \) has discriminant \( 4(12u^2 + 1) \). For integer roots we must have \( v \) such that \( v^2 - 12u^2 = 1 \). There exist infinitely many solutions to this equation. Therefore there are infinitely many such pairs.

**Problem 3:** If \( m, n, \) and \( p \) are positive integers such that

\[
m + n + p - 2\sqrt{mnp} = 1
\]

then at least one of them is a perfect square.

**Solution:** Write the relation as

\[
(m + n + p - 1)^2 = 4mnp.
\]

Substituting \( a = 2m - 1, b = 2n - 1, \) and \( c = 2p - 1 \) and simplifying we get

\[
a^2 + b^2 + c^2 - 2abc = 1
\]

\[
\Rightarrow a^2 + b^2 - a^2b^2 - 1 = -(c^2 - 2abc + a^2b^2)
\]

\[
\Rightarrow (a^2 - 1)(b^2 - 1) = (c - ab)^2.
\]

Similarly, \( (b^2 - 1)(c^2 - 1) = (a - bc)^2 \), so we must have \( a^2 - 1 = du^2, b^2 - 1 = dv^2, c^2 - 1 = dw^2, |ab - c| = duvw, \) and \( |bc - a| = duvw \).

Let \((x_1, y_1)\) be the fundamental solution to the Pell equation \( x^2 - dy^2 = 1 \) and let \( s = x_1 + y_1\sqrt{d} \). So all the solutions of this equation are of the form

\[
x_k = \frac{1}{2} \left( s^k + \frac{1}{s^k} \right), \quad y_k = \frac{1}{2\sqrt{d}} \left( s^k - \frac{1}{s^k} \right).
\]

Hence, \( a = \frac{1}{2} \left( s^{k_1} + \frac{1}{s^{k_1}} \right), \) \( b = \frac{1}{2} \left( s^{k_2} + \frac{1}{s^{k_2}} \right), \) and \( c = \frac{1}{2} \left( s^{k_3} + \frac{1}{s^{k_3}} \right) \) for some \( k_1, k_2, k_3 \in \mathbb{N}_0 \).
Suppose \( m \geq n \geq p \). So \( k_1 \geq k_2 \geq k_3 \) and \( ab - c = duv \). This implies
\[
c = ab - duv = \frac{1}{4} \left( \left( s^{k_1} + \frac{1}{s^{k_1}} \right) \left( s^{k_2} + \frac{1}{s^{k_2}} \right) - \left( s^{k_1} - \frac{1}{s^{k_1}} \right) \left( s^{k_2} - \frac{1}{s^{k_2}} \right) \right)
\]
\[
\Rightarrow c = s^{k_1-k_2} + \frac{1}{s^{k_1-k_2}}.
\]
Hence \( k_3 = k_1 - k_2 \). So one of the \( k_i \) is even. Suppose \( k_1 \) is even. So
\[
m = \frac{a + 1}{2} = \left( \frac{1}{2} \left( \frac{k_1}{s^{k_1}} + \frac{1}{s^{k_1}} \right) \right)^2.
\]
The other cases can be handled similarly.

**Problem 4:** Prove that there are infinitely many quadruples \((x, y, z, t)\) of positive integers with no common divisor such that \(x^3 + y^3 + z^2 = t^4\).

**Solution:** Notice that
\[
1^3 + 2^3 + 3^3 + \cdots + (n-2)^3 + (n-1)^3 + n^3 = \left( \frac{n(n+1)}{2} \right)^2
\]
and hence
\[
\left( \frac{(n-2)(n-1)}{2} \right)^2 + (n-1)^3 + n^3 = \left( \frac{n(n+1)}{2} \right)^2.
\]
So it suffices to prove that for infinitely many \( n \)
\[
\frac{n(n+1)}{2} = m^2.
\]
Rearranging this equation yields
\[
n^2 + n - 2m^2 = 0
\]
\[
4n^2 + 4n + 1 - 8m^2 = 1
\]
\[
(2n+1)^2 - 8m^2 = 1
\]
So \((2n+1, m)\) are the roots of \(x^2 - 8y^2 = 1\). Also since \( x \) is odd, for each root of \(x^2 - 8y^2 = 1\) we can find a corresponding root of \( n \). So there are infinitely many such quadruples.

**Problem 5:** Prove that the only solution of \(5^a - 3^b = 2\) is \( a = b = 1 \).

**Solution:** Clearly if one of \( a \) or \( b \) is 1, then the other is also 1. So let \( a, b > 1 \).
Consider \( x = 3^b + 1 \) and \( y = 3^{b-1} \times 5^{\frac{a-1}{2}} \), then
\[
x^2 - 15y^2 = (3^b + 1)^2 - 15 \times 3^{b-1} \times 5^{a-1}
\]
\[
= (3^b + 1)^2 - 3^b \times 5^a
\]
\[
= (3^b + 1)^2 - 3^b(3^b + 2) = 1.
\]
So $x$ and $y$ are the roots of the Pell equation $x^2 - 15y^2 = 1$. The roots $(x_n, y_n)$ of this equation satisfy the recurrence

$$y_{n+2} = 8y_{n+1} - y_n; \text{ with } y_1 = 1, y_2 = 8.$$ 

Taking $y_n \pmod{3}$ we get the sequence $\{1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots\}$. Also, taking $y_n \pmod{7}$ we get the sequence $\{1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots\}$. Since $3 \mid y$ so $y = y_{3k}$ for some $k \in \mathbb{N}$. But then $7 \mid y$ which is not possible. So $a, b > 1$ is not possible.

### 2 The general and negative Pell equation

The equation $x^2 - dy^2 = c$ is called the general Pell equation. Note that if $x^2 - dy^2 = c$ is solvable then it has infinitely many solutions. [Hint: Let $u^2 - dv^2 = c$ and $r^2 - ds^2 = 1$, then $(ur \pm dv) = d(us \pm vr)^2 = (u^2 - dv^2)(r^2 - ds^2) = c$.]

A more particular case of this equation $x^2 - dy^2 = -1$, called the negative Pell equation, is of particular importance. It is known that it is solvable if and only if the period of $\sqrt{d}$, when expressed as a continued fraction, is of odd length. The following are some important results on this equation:

- $x^2 - dy^2 = -1$ is not solvable if some prime $p \equiv 3 \pmod{4}$ divides $d$.
- $x^2 - py^2 = -1$ where $p$ is a prime is solvable if and only if $p \equiv 1 \pmod{4}$ or $p = 2$.
- If $(x_1, y_1)$ is the fundamental solution of the equation $x^2 - dy^2 = -1$ and $(u_n, v_n)$ are the solutions of $x^2 - dy^2 = 1$ then the solutions $(x_n, y_n)$ of $x^2 - dy^2 = -1$ are given by
  $$x_n = x_1u_n + dy_1v_n, \quad y_n = y_1u_n + x_1v_n.$$ 
- If $\sqrt{d}$ has an odd period $m$ then the fundamental solution of $x^2 - dy^2 = -1$ is given by $(p_{m-1}, q_{m-1})$, where $\frac{p_{m-1}}{q_{m-1}}$ is the $m - 1^{\text{st}}$ convergent of the continued fraction expansion of $\sqrt{d}$.

**Problem 6:** For the sequence $a_n = \left[\sqrt{n^2 + (n + 1)^2}\right]$, prove that there are infinitely many $n$’s such that $a_n - a_{n-1} > 1$ and $a_{n+1} - a_n = 1$.

**Solution:** Let us consider the case $n^2 + (n + 1)^2 = y^2$, then

$$2n^2 + 2n + 1 - y^2 = 0$$
$$4n^2 + 4n + 1 - 2y^2 = -1$$
$$(2n + 1)^2 - 2y^2 = -1$$

The negative Pell equation $u^2 - 2v^2 = -1$ has infinitely many solutions, and in each case $u$ is odd. So we can let $u_k = 2n + 1$ and $v_k = y$ as the solutions of the
equation. Also we have $a_n = y$. Now $a_n < \sqrt{2(n+1)}$ and $a_{n-1} < \sqrt{2n}$. This implies
\[
\begin{align*}
  a_n - a_{n-1} &= \frac{4n}{a_{n+1} + a_n} \\
  a_n - a_{n-1} &> \frac{4n}{2\sqrt{2n} + \sqrt{2}}.
\end{align*}
\]
For $n > 1$ this expression is bigger than 1, so $a_n - a_{n-1} > 1$. Also
\[
a_{n+1} = \left\lfloor \sqrt{(n+1)^2 + (n+2)^2} \right\rfloor = \left\lfloor \sqrt{y^2 + 4n + 4} \right\rfloor
\]
and $n < y < 2n + 1$. Hence
\[
y + 1 < \sqrt{y^2 + 4n + 4} < y + 2.
\]
Therefore
\[
a_{n+1} - a_n = y + 1 - y = 1.
\]
Since there are infinitely many solutions of the equation $x^2 - 2y^2 = -1$ there are infinitely many such $n$.

**Problem 7:** Let $t$ be a fixed natural number. Prove that there are infinitely many $n \in \mathbb{N}$ for which $(n^2 + 1)^t \mid n!$.

**Solution:** Let $p$ be any prime such that $p \equiv 1 \pmod{4}$ and $\sqrt{p - 1} > 2t$. Consider the equation $x^2 - py^2 = -1$, which has infinitely many solutions. Consider the solutions $(x_k, y_k)$ such that $y_k > p$. Now
\[
x_k^2 = py_k^2 - 1 > (p - 1)y_k^2
\]
\[
\Rightarrow x_k > 2ty_k
\]
and $pt < y_kt < 2ty_k < x_k$.

If we can show that $(py_k^2)^t \mid x_k!$ then $(x_k^2 + 1)^t = (py_k^2)^t \mid x_k!$ and putting $n = x_k$ we are done. So to prove this consider the following cases:

**Case 1:** $p \mid y_k$

Let $y_k = p^a \times m$, with gcd$(m, p) = 1$. We want to prove
\[
(p \times p^{2a} \times m^2)^t = p^{(2a+1)t}m^{2t} \mid x_k!
\]
We have $2ty_k < x_k \Rightarrow 2tp^am < x_k$, so $p, 2p, 3p, \ldots, (2p^{a-1}m)p$ are all the factors of $x_k!$. Therefore, $p^{2tp^{a-1}m} \mid x_k!$. Now we just have to show $2p^{a-1}m > 2a + 1$.

If $a = 1$ then, since $y_k > p$, we have $m > 1$ and then $2p^{a-1}m \geq 4 > 3 = 2a + 1$.

If $a > 1$ then we have $p^{a-1} > 2a + 1$. This is because $\sqrt{p - 1} > 2t \geq 2 \Rightarrow p > 5$.

Hence in both the cases $p^{2tp^{a-1}m} > p^{(2a+1)}$, so $p^{(2a+1)} \mid x_k!$.

Now we have $m, 2m, \ldots, p^a \times m$ are factors of $m$ less than $x_k$. So $m p^a \mid x_k!$. As $2t < \sqrt{p - 1}$, we have $p^t \geq p > 2t$. So $m^{2t} \mid x_k!$ and hence $p^{(2a+1)m^{2t}} \mid x_k! \Rightarrow (py_k^2)^t \mid x_k!$.

_Crux Mathematicorum_, Vol. 48(10), December 2022
Case 2: \( p \nmid y_k \)

Here \( pt < x_k \) and \( 2ty_k < x_k \). So we have \( p, 2p, \ldots, tp \) are factors of \( p \) less than \( x_k \) giving \( p^t \mid x_k! \). Similarly \( y_k^{2t} \mid x_k \). As \( p \) and \( y_k \) are co-prime, we have \( (py_k^t)^t \mid x_k! \).

Hence in both cases \( (py_k^t)^t \mid x_k! \) and we are done.

Although some of the results mentioned before can be used to show that a particular negative Pell equation is not solvable in integers, sometimes some other techniques can be helpful too. It is illustrated in the following problem where I use infinite descent to do the same.

**Problem 8:** Show that the equation \( x^2 - k(k+4)y^2 = -1 \) is not solvable in integers for \( k \geq 2 \).

**Solution:** Let us assume that there are solutions that satisfy this equation. Make the substitutions

\[
x = u - (k+2)v, \quad y = \frac{v}{2}.
\]

As \( x \) and \( y \) are integers clearly \( u \) and \( v \) are integers too. This gives

\[
u^2 + \frac{(k+2)^2v^2}{4} - (k+2)uv - \frac{k(k+4)v^2}{4} + 1 = 0
\]

\[
u^2 - (k+2)uv + v^2 + 1 = 0
\]

(1)

is solvable in integers. Note that if \( u = v \) then \( ku^2 = 1 \) which is not possible for \( k \geq 2 \), so \( u \neq v \). Let \((u_1, v_1)\) be a solution to the equation \((1)\). Without loss of generality we may assume \( u_1, v_1 > 0 \) (\( uv \) must be positive otherwise the \( LHS > 0 \)). Also as the equation is symmetric in \( u \) and \( v \) it is enough to consider the case \( u_1 > v_1 \).

Write \( u_2 = (k+2)v_1 - u_1 \) and \( v_2 = v_1 \). One can easily check that \((u_2, v_2)\) satisfies \((1)\). Also \( u_1 \) and \( u_2 \) are the roots of

\[
u^2 - (k+2)v_1u + v_1^2 + 1 = 0,
\]

so

\[u_1u_2 = v_1^2 + 1\]

\[u_2 = \frac{v_1^2 + 1}{u_1} < \frac{v_1^2 + 1}{v_1} \leq v_1 + 1\]

\[u_2 \leq v_1\]

\[u_2 + v_2 < u_1 + v_1\]

Also \( u_1 + u_2 > 0, u_1u_2 > 0 \). Hence since \( u_1 > 0 \) we have \( u_2 > 0 \).

So for every pair of positive solution of \((1)\) we get another pair of positive solutions, the sum of which is less than the previous pair. Hence by infinite descent \((1)\) can have no solution. Hence we are done.
3 The equation \( au^2 - bv^2 = 1 \)

Now let us consider the more general equation \( au^2 - bv^2 = 1 \) where \( a, b \in \mathbb{N}, \gcd(a, b) = 1, \) and \( ab \) is not a square. We can solve this equation by relating it with the Pell equation \( x^2 - aby^2 = 1 \). To do this let us assume there is a solution \((u_1, v_1)\) to \( au^2 - bv^2 = 1 \) and relate the pair \((u_1, v_1)\) and \((x_1, y_1)\) through

\[
x_1 + y_1 \sqrt{ab} = (\sqrt{a}u_1 + \sqrt{b}v_1)^2.
\]

Which requires

\[
x = au_1^2 + bv_1^2 \quad \text{and} \quad y = 2u_1v_1.
\]

Now let \( au_1^2 - bv_1^2 = 1 \). Make the substitution

\[
u = u_1x + bv_1y \quad \text{and} \quad v = v_1x + au_1y.
\]

So

\[
au^2 - by^2 = a(u_1x + bv_1y)^2 - b(v_1x + au_1y)^2
\]

\[
1 = au_1^2x^2 - bv_1^2y^2 + ab^2v_1^2y^2 - a^2bu_1^2y^2
\]

\[
x^2(au_1^2 - bv_1^2) - aby^2(au_1^2 - bv_1^2) = 1
\]

\[
x^2 - aby^2 = 1
\]

So the equation gets converted into a Pell equation.

**Problem 9:** Show that if \( 3n + 1 \) and \( 4n + 1 \) are perfect squares then \( 56 | n \).

**Solution:** Let \( 3n + 1 = u^2 \) and \( 4n + 1 = v^2 \), so \( 4u^2 - 3v^2 = 1 \). Let \( \{(u_m, v_m)\} \) be the set of solutions of this equation. This equation has fundamental solution \((1, 1)\). Hence make the substitution \( u = x + 3y \) and \( v = x + 4y \) to get the equation \( x^2 - 12y^2 = 1 \) with solutions \((x_m, y_m)\) given by

\[
x_m + \sqrt{12}y_m = (7 + 2\sqrt{12})^m
\]

Now \( n = v_m^2 - u_m^2 = y_m(2x_m + 7y_m) \).

If \( m \) is even then \((2) \Rightarrow 7 | y_m \). If \( m \) is odd then \((2) \Rightarrow 7 | x_m \). So in any case \( 7 | n \).

Also \( 4n + 1 = v^2 \), so \( v \) is odd. Let \( v = 2k + 1 \) implying \( 4n + 1 = 4k^2 + 4k + 1 \Rightarrow n = k(k + 1) \). So \( n \) is even.

So \( u^2 = 3n + 1 \) is odd implying \( u^2 \equiv 1 \mod{8} \Rightarrow 3n \equiv 0 \mod{8} \Rightarrow 8 | n \). Thus \( 7 \times 8 = 56 | n \).

**Problem 10:** If \( x \) and \( y \) are positive integers such that \( x(y + 1) \) and \( y(x + 1) \) are perfect squares then prove that either \( x \) or \( y \) is a square.

**Solution:** Since \( x(y + 1) \) is a perfect square, then \( x = au_1^2 \) and \( y + 1 = au_2^2 \).

*Crux Mathematicorum*, Vol. 48(10), December 2022
Similarly $x + 1 = bv_1^2$ and $y = bv_2^2$, where either $a$ and $b$ are 1 or square-free integers. Also,

$$au_1^2 - bv_1^2 = -1 \quad \text{and} \quad au_2^2 - bv_2^2 = 1.$$  

Clearly $\gcd(a, b) = 1$. Let $z = u_2v_1 - v_2u_1$, so

$$(v_2u_1)^2 = (u_2v_1 - z)^2$$  
$$(bv_2^2)(au_1^2) = ab(u_2v_1 - z)^2$$  
$$(au_2^2 - 1)(bv_1^2 - 1) = ab(u_2v_1 - z)^2$$  
$$au_2^2 + bv_2^2 + abz^2 - 1 = 2\sqrt{(au_2^2)(bv_2^2)(abz^2)}$$

Problem 3 tells us that at least one of $au_2^2$, $bv_1^2$, $abz^2$ is a square. Since $\gcd(a, b) = 1$ and, for $a, b > 1$, $a$ and $b$ are square-free, at least one of $a$ and $b$ is 1.

If $a = 1$ then $x = u_1^2$ is a perfect square.

If $b = 1$ then $y = v_2^2$ is a perfect square.

### 4 Problems for practice

The key to master any topic is to practice problems. Here I provide ten problems for the readers to ponder over. I hope you enjoy them.

1. Prove that there are infinitely many triples $(a, b, c)$ of positive integers such that $a^4 + b^3 = c^2$.

2. Prove that if $m$ is an integer where $m = 2 + 2\sqrt{28n^2} + 1$ for positive integer $n$, then $m$ is a perfect square.

3. Find the smallest integer $n > 1$ for which

$$\sqrt{1^2 + 2^2 + 3^2 + \cdots + n^2}$$

is a natural number.

4. Let $a_0 = 0$, $a_1 = 4$, and $a_{n+1} = 18a_n - a_{n-1}$ for $n \geq 1$. Prove that $5a_n^2 + 1$ is a perfect square for all $n$.

5. Let $m$, $n$, $p$ be positive integers such that $m + n + p - 1 = 2\sqrt{mpn}$. Prove that at least one of the following must be true

$$m \mid (n + p - 1)^2, \quad n \mid (p + m - 1)^2, \quad p \mid (m + n - 1)^2.$$  

6. Prove that there exists two strictly increasing sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n(a_n + 1)$ divides $b_n^2 + 1$, for all $n \in \mathbb{N}$.

7. Find all positive integers $n$ such that both $2n + 1$ and $3n + 1$ are perfect squares. Also show that for all such $n$, $40 \mid n$.  

Copyright © Canadian Mathematical Society, 2022
8. Find all pairs \((k, m)\) of positive integers such that \(k < m\) and
\[
1 + 2 + 3 + 4 + \cdots + k = (k + 1) + (k + 2) + \cdots + m
\]

9. Prove that if \(\frac{a^2 + 1}{b^2} + 4\) is a perfect square, then the square must be 9.

10. Find all pairs \((m, n)\) of positive integers such that both \(mn + n\) and \(mn + m\) are squares.

References


---

Amit Kumar Basistha is currently a first year undergraduate student doing a Bachelors in Mathematics from Indian Statistical Institute, Bangalore. He is mostly interested in pure mathematics and is an avid number theory fan. His first exposure to number theory was in an Indian National Mathematics Olympiad Training Camp held by Assam Academy of Mathematics. Apart from mathematics Amit is a passionate sports fan following cricket and football. He likes listening to music and reading novels – mainly classics.
OLYMPIAD CORNER
No. 408

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by February 15, 2023.

**OC606.** Determine the number of triples of positive integers \((a, b, c)\) such that

\[ a + ab + abc + ac + c = 2017. \]

**OC607.** Find the largest possible number of integers that can be selected from the set \(\{1, 2, 3, \ldots, 100\}\) so that there are no two of them that differ by 2 or by 5.

**OC608.** Prove that \(2^{-x} + 2^{-1/x} \leq 1\) for all real numbers \(x > 0\).

**OC609.** Let \(n\) be a positive integer, \(n \equiv 4 \pmod{8}\). The numbers

\[ 1 = k_1 < k_2 < \ldots < k_m = n \]

are all positive divisors of \(n\). Prove that if the number \(i \in \{1, 2, \ldots, m-1\}\) is not divisible by 3, then \(k_{i+1} \leq 2k_i\).

**OC610.** The perpendicular bisector of side \(BC\) intersects the circumcircle of triangle \(ABC\) at points \(P\) and \(Q\), with points \(A\) and \(P\) on the same part of side \(BC\). Point \(R\) is the orthogonal projection of point \(P\) on the straight line \(AC\). Point \(S\) is the midpoint of the segment \(AQ\). Prove that points \(A, B, R\) and \(S\) lie on a circle.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’une olympiade mathématique régionale ou nationale.
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 février 2023.

OC606. Déterminer le nombre de triplets d’entiers positifs \((a, b, c)\) tels que
\[a + ab + abc + ac + c = 2017.\]

OC607. Déterminer le plus grand nombre possible d’entiers appartenant à l’ensemble \(\{1, 2, 3, \ldots, 100\}\), tels qu’aucuns deux d’entre eux diffèrent par 2 ou 5.

OC608. Démontrer que \(2^{-x} + 2^{-1/x} \leq 1\) pour tout nombre réel \(x > 0\).

OC609. Soit \(n\) un entier positif tel que \(n \equiv 4 \pmod{8}\) et supposons que
\[1 = k_1 < k_2 < \ldots < k_m = n\]
sont les diviseurs positifs de \(n\). Démontrer que si le nombre \(i\) appartenant à \(\{1, 2, \ldots, m-1\}\) n’est pas divisible par 3, alors \(k_{i+1} \leq 2k_i\).

OC610. La bissectrice orthogonale du côté \(BC\) rencontre le cercle circonscrit du triangle \(ABC\) en \(P\) et \(Q\), où les points \(A\) et \(P\) se trouvent du même côté de la ligne \(BC\). Aussi, \(R\) est la projection orthogonale du point \(P\) vers la ligne \(AC\); enfin, \(S\) est le point milieu du segment \(AQ\). Démontrer que les points \(A, B, R\) et \(S\) se trouvent sur un même cercle.
OC581. Find the greatest positive integer \( n \) such that \( n + 3 \) divides \( 1^3 + 2^3 + \cdots + n^3 \).

*Originally Problem 10 from 2021 Indonesia International Mathematics Competition Keystage 3 Individual.*

We received 16 submissions, of which 12 were correct and complete. We present 3 solutions.

**Solution 1, by Roy Barbara.**

The answer is \( n = 15 \).

Since \( 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4} \), the given condition is equivalent to

\[
4(n + 3) \mid n^2(n + 1)^2. \tag{1}
\]

If \( p \) is a prime divisor of \( n + 3 \), then, \( p \mid n \) or \( p \mid (n + 1) \), so that \( p \mid (n + 3) - n = 3 \) or \( p \mid (n + 3) - (n + 1) = 2 \). Hence, \( p \in \{2, 3\} \), which implies \( n + 3 = 2^r \cdot 3^s \), where \( r \) and \( s \) are non-negative integers. Condition (1) reduces to

\[
2^{r+2} \cdot 3^s \mid (2^r \cdot 3^s - 3)^2(2^r \cdot 3^s - 2)^2.
\]

Let \( A = (2^r \cdot 3^s - 3)^2(2^r \cdot 3^s - 2)^2 \). We claim that \( r \leq 1 \) and \( s \leq 2 \).

(i) If \( r \geq 2 \), then \( 2^r \cdot 3^s - 3 \equiv 1 \pmod{2} \) and \( (2^r \cdot 3^s - 2)^2 = 2^2(2^{r-1} \cdot 3^s - 1)^2 \), where \( 2^{r-1} \cdot 3^s - 1 \equiv 1 \pmod{2} \). Hence, the largest power of 2 that divides \( A \) is \( 2^2 \), while \( 2^4 \mid 2^{r+2} \cdot 3^s \), contradiction.

(ii) If \( s \geq 3 \), then \( (2^r \cdot 3^s - 3)^2 = 3^2(2^r \cdot 3^{s-1} - 1)^2 \), where \( 2^r \cdot 3^{s-1} - 1 \equiv 2 \pmod{3} \), and \( 2^r \cdot 3^s - 2 \equiv 1 \pmod{3} \). Hence, the largest power of 3 that divides \( A \) is \( 3^2 \), while \( 3^3 \mid 2^{r+2} \cdot 3^s \), contradiction.

We conclude that the largest possible value for \( n + 3 \) is \( 2^1 \cdot 3^2 = 18 \), corresponding to \( n = 15 \). Finally, it is easy to see that \( n = 15 \) is indeed a solution of (1).

**Solution 2, by Oliver Geupel.**

In view of the identity

\[
1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2,
\]

Copyright © Canadian Mathematical Society, 2022
we have to find the greatest \( n \) with the property that \( n + 3 \) divides \( \frac{1}{4} n^2(n + 1)^2 \). By inspection \( n = 15 \) has the required property, and we are going to prove that it is the greatest such number. For the sake of obtaining a contradiction, suppose that some \( n > 15 \) has the desired property. We consider the cases of even and odd numbers \( n \) in succession.

First suppose that \( n \) is even, that is \( n = 2m \) where \( m \geq 8 \). Then \( 2m + 3 \) divides \( m^2(2m + 1)^2 \). Observing that the numbers \( 2m + 1 \) and \( 2m + 3 \) are coprime, we obtain \( 2m + 3 \mid m^2 \), but

\[
(2m + 3)(m - 2) - 2m^2 = -m - 6 < 0 < m - 3 = (2m + 3)(m - 1) - 2m^2
\]

implies that

\[
\frac{m - 2}{2} < \frac{m^2}{2m + 3} < \frac{m - 1}{2},
\]
a contradiction.

Next suppose that \( n \) is odd, \( n = 2m - 1 \) where \( m \geq 9 \). Then \( 2(m + 1) \mid (2m - 1)^2 m^2 \). Hence \( m \) is even, \( m = 2q \) where \( q \geq 5 \). Then \( 2(2q + 1) \mid (4q - 1)^2 \cdot 4q^2 \). Since \( 2q + 1 \) is coprime to \( 4 \) and to \( q \), we obtain \( 2q + 1 \mid (4q - 1)^2 \). But

\[
(2q + 1)(8q - 8) - (4q - 1)^2 = -9 < 0 < 2q - 8 = (2q + 1)(8q - 7) - (4q - 1)^2
\]
gives

\[
8q - 8 < \frac{(4q - 1)^2}{2q + 1} < 8q - 7,
\]
a contradiction. The proof is complete.

Solution 3, by Theo Koupelis.

We have

\[
1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4} = k \cdot (n + 3),
\]

where \( k \) is a positive integer, and thus

\[
4k = \frac{n^2(n + 1)^2}{n + 3} = n^3 - n^2 + 4n - 12 + \frac{36}{n + 3}.
\]

Therefore, \( n + 3 \mid 36 \) and thus \( n \in \{1, 3, 6, 9, 15, 33\} \). But only \( n \in \{3, 6, 15\} \) lead to an integer value for \( k \), and thus \( n_{\text{max}} = 15 \).

**OC582.** Two plane mirrors \( OP \) and \( OQ \) are inclined at an acute angle (diagram is not to scale). A ray of light \( XY \) parallel to \( QO \) strikes mirror \( OP \) at \( Y \). The ray is reflected and hits mirror \( OQ \). Then it is reflected again and hits mirror \( OP \). Finally, it is reflected for a third time and strikes mirror \( OQ \) at right angles at \( R \), as shown. The distance \( OR \) is 5 cm. The ray \( XY \) is \( d \) cm from the mirror \( OQ \). What is the value of \( d \)?

*Crux Mathematicorum*, Vol. 48(10), December 2022
Originally Question 29 from the 2022 Online Practice Test for Grade 11, Math Kangaroo Contest.

We received 15 submissions, of which 13 were correct and complete. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

Let $S,T$ and $Z$ be as shown in the figure below.

Since $\angle RSO = \angle TSY$, then the right-angles triangles $OSR$ and $YST$ are similar. Therefore

$$\frac{RS}{RO} = \frac{ST}{TY}.$$ 

The triangles $RST$ and $ZYT$ are also similar giving

$$\frac{RS}{ST} = \frac{ZY}{YT}.$$ 

Therefore

$$\frac{ZY}{YT} = \frac{RS}{ST} = \frac{RO}{TY}$$

giving $d = ZY = RO = 5$. 
Let the reflected ray of \( XY \) strike \( OQ \) at \( V \) and let its reflected ray strike \( OP \) at \( U \). Let the line \( XY \) intersect the line \( RU \) at \( T \) (see figure above). Let \( \theta = \angle POQ \).

We determine \( \angle VYU \) in two ways. First, we have \( \angle VYU = \angle UYT \) (since lines \( VY \) and \( YT \) are symmetrical about \( OP \)) and \( \angle UYT = \theta \) (since \( YT \parallel OQ \)), hence \( \angle VYU = \theta \). On the other hand, the internal bisector of \( \angle RUV \) is perpendicular to \( OU \) and \( UR \perp OQ \), hence \( \frac{1}{2} \angle RUV = \theta \). It follows that \( \angle RVU = 90^\circ - 2\theta \), hence \( \angle QVY = 90^\circ - 2\theta \) and \( \angle UVY = 4\theta \). Since \( \angle YUV = \angle OUR = 90^\circ - \theta \), we deduce that \( \angle UYV = 90^\circ - 3\theta \).

Therefore, we must have \( 90^\circ - 3\theta = \theta \) and therefore \( \theta = 22.5^\circ \).

Now, \( RU = OR \cdot \tan(22.5^\circ) = 5 \cdot \tan(22.5^\circ) \) and \( \Delta URV \) is right-angled and isosceles, hence \( UV = 5\sqrt{2} \cdot \tan(22.5^\circ) \). Also \( UT = UV \) since the triangles \( TYU \) and \( VYU \) are congruent. Finally,

\[
d = RT = RU + UT = 5 \tan(22.5^\circ) + 5\sqrt{2} \cdot \tan(22.5^\circ) \\
= 5(1 + \sqrt{2}) \tan(22.5^\circ) \\
= 5(\sqrt{2} + 1)(\sqrt{2} - 1) \\
= 5.
\]

**OC583.** Reduce the following expression to a simplified rational

\[
\cos \frac{\pi}{9} + \cos \frac{5\pi}{9} + \cos \frac{7\pi}{9}.
\]

*Originally Problem 32 from the 2021 Stanford Math Tournament.*

*We received 19 submissions, of which 18 were correct and complete. We present 3 solutions.*
Solution 1, by Theo Koupelis.

Let \( z = e^{i\theta} \), where \( \theta = \frac{\pi}{9} \). Then \( z^9 + 1 = 0 = (z + 1) \cdot \sum_{k=0}^{8}(-1)^k z^k \). Clearly \( z \neq -1 \) and thus \( \sum_{k=0}^{8}(-1)^k z^k = 0 \). We have

\[
\cos \theta = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z},
\]

and similarly, using the expression \( z^9 = -1 \), we get

\[
\cos(5\theta) = \frac{z^{10} + 1}{2z^5} = \frac{1 - z}{2z^5}, \quad \text{and} \quad \cos(7\theta) = \frac{z^{14} + 1}{2z^7} = -\frac{z^4 + 1}{2z^2}.
\]

Also,

\[
\cos(3\theta) = \cos \frac{\pi}{3} = \frac{1}{2} = \frac{z^6 + 1}{2z^3},
\]

and thus \( z^6 + 1 = z^3 \). Therefore, using repeatedly the expression \( z^9 = -1 \), we get

\[
I = \cos^7 \theta + \cos^7 (5\theta) + \cos^7 (7\theta)
\]

\[
= \frac{1}{128} \left[ \frac{(z^2 + 1)^7}{z^7} + \frac{(1 - z)^7}{z^{35}} - \frac{(z^4 + 1)^7}{z^{14}} \right]
\]

\[
= -\frac{1}{128} \cdot \left[ z^2(z^2 + 1)^7 - z(1 - z)^7 + z^4(z^4 + 1)^7 \right]
\]

\[
= -\frac{1}{128} \cdot \left[ 43(z^8 - z^7 + z^6 - z^5 + z^4 - z^3 + z^2 - z) + 20(z^6 - z^3) \right]
\]

\[
= -\frac{1}{128} \cdot (-43 - 20) = \frac{63}{128}.
\]

Solution 2, by the Missouri State University Problem Solving Group.

Let \( \alpha = \cos \frac{\pi}{9}, \beta = \cos \frac{5\pi}{9}, \) and \( \gamma = \cos \frac{7\pi}{9} \).

Since

\[
\cos 3\theta = 4\cos^3 \theta - 3\cos \theta
\]

and \( \cos(\pi/3) = \cos(5\pi/3) = \cos(7\pi/3) = 1/2, \alpha, \beta, \) and \( \gamma \) are the roots of

\[
x^3 - \frac{3}{4} x - \frac{1}{8} = 0.
\]

By Vieta’s formulas, we have

\[
\alpha + \beta + \gamma = 0
\]

\[
\alpha \beta + \beta \gamma + \gamma \alpha = -\frac{3}{4}
\]

\[
\alpha \beta \gamma = -\frac{1}{8}.
\]
Let
\[ a_n = \alpha^n + \beta^n + \gamma^n. \]

It is well known (and easy to verify) that the \( a_n \) satisfy a recurrence relation that mirrors the polynomial of which \( \alpha, \beta, \) and \( \gamma \) are the roots, namely
\[ a_n = \frac{3}{4}a_{n-2} + \frac{1}{8}a_{n-3}. \]

The initial values of this recurrence are
\[ a_0 = 3, \quad a_1 = 0, \quad a_2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{3}{2}. \]

Using our recurrence, we find that
\[ a_3 = \frac{3}{4}, \quad a_4 = \frac{9}{8}, \quad a_5 = \frac{15}{32}, \quad a_6 = \frac{57}{64}, \quad a_7 = \frac{63}{128}, \quad a_8 = \frac{93}{128}, \quad a_9 = \frac{123}{256}, \ldots \]

The answer to the question asked is therefore \( a_7 = \frac{63}{128} \).

**Solution 3, by Michel Bataille.**

We show that \( \cos^7 \frac{\pi}{9} + \cos^7 \frac{5\pi}{9} + \cos^7 \frac{7\pi}{9} = \frac{63}{128} \).

Since \( \cos \frac{3\pi}{9} = \frac{1}{2} \) and \( \cos \frac{6\pi}{128} = \frac{1}{2} - \frac{1}{2^7} \), all amounts to proving that \( S = \frac{1}{2} \) where
\[ S = \cos^7 \frac{\pi}{9} + \cos^7 \frac{3\pi}{9} + \cos^7 \frac{5\pi}{9} + \cos^7 \frac{7\pi}{9} = \cos^7 \frac{\pi}{9} - \cos^7 \frac{2\pi}{9} + \cos^7 \frac{3\pi}{9} - \cos^7 \frac{4\pi}{9}. \]

[Note that \( \cos \frac{5\pi}{9} = \cos (\pi - \frac{4\pi}{9}) = -\cos \frac{4\pi}{9} \) and similarly \( \cos \frac{7\pi}{9} = -\cos \frac{2\pi}{9} \).]

The result \( S = \frac{1}{2} \) is the particular case \( n = 4 \) of the following general formula that holds for any positive integer \( n \):
\[ \sum_{k=1}^{n} (-1)^{k+1} \left( \cos \frac{k\pi}{2n+1} \right)^{2n-1} = \frac{1}{2}. \quad (1) \]

**Proof of (1).** From the binomial theorem, we deduce that
\[ (1 + x)^{2n+1} - (1 - x)^{2n+1} = 2xP(x^2), \]

where
\[ P(x) = \sum_{k=0}^{n} \left( \begin{array}{c} 2n+1 \\ 2k+1 \end{array} \right) x^k. \]

*Crux Mathematicorum*, Vol. 48(10), December 2022
Since \((1 + x)^{2n+1} = (1 - x)^{2n+1}\) if and only if
\[
\frac{1 + x}{1 - x} = \exp\left(\frac{2k\pi i}{2n + 1}\right)
\]
for some \(k \in \{0, 1, \ldots, 2n\}\), the complex roots of \((1 + x)^{2n+1} - (1 - x)^{2n+1}\) are 0 and the numbers \(i \tan x_k, -i \tan x_k\) \((k = 1, 2, \ldots, n)\) where \(x_k = \frac{k\pi}{2n+1}\). It readily follows that
\[
P(x) = \prod_{k=1}^{n} (x + \tan^2 x_k).
\]
The roots \(w_k = -\tan^2 x_k\) of \(P(x)\) being distinct, we have the following decomposition into partial fractions:
\[
\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{1}{P'(w_k)} \cdot \frac{1}{x - w_k}. \tag{2}
\]
Now, by differentiation, we have
\[
(2n + 1)((1 + x)^{2n} + (1 - x)^{2n}) = 2P(x^2) + 4x^2 P'(x^2),
\]
which, with \(x = i \tan x_k\), provides
\[
(2n + 1) \cdot \frac{2(-1)^k \cos x_k}{(\cos x_k)^{2n}} = -4 \tan^2 x_k P'(w_k)
\]
(since \(1 + i \tan x_k = \frac{x_k}{\cos x_k}\) and \(\cos(2nx_k) = (-1)^k \cos x_k\)).

Since \(P(0) = 2n + 1\), taking \(x = 0\) in (2) yields
\[
\frac{1}{2n + 1} = \sum_{k=1}^{n} (-1)^{k+1} \sin^2 x_k \cos^{2n-3} x_k \cdot \frac{1}{\tan^2 x_k}
= \sum_{k=1}^{n} (-1)^{k+1} (\cos x_k)^{2n-1}
\]
and (1) follows.

**OC584.** We say that two sequences \(x, y : \mathbb{N} \to \mathbb{N}\) are completely different if \(x(n) \neq y(n)\) holds for all \(n \in \mathbb{N}\). Let \(F\) be a function assigning a natural number to every sequence of natural numbers such that \(F(x) \neq F(y)\) for any pair of completely different sequences \(x, y\), and for constant sequences we have \(F((k, k, \ldots)) = k\). Prove that there exists \(n \in \mathbb{N}\) such that \(F(x) = x(n)\) for all sequences \(x\).

*Originally Problem 1 from the 2020 Miklós Schweitzer Memorial Competition in Mathematics.*

*We received 2 correct submissions. We present the solution by UCLan Cyprus Problem Solving Group.*
Let $a$ be the sequence $(1, 2, 3, \ldots)$ defined by $a(n) = n$ for each $n \in \mathbb{N}$ and let $c_i$ be the constant $i$ sequence $(i, i, i, \ldots)$ defined by $c_i(n) = i$ for each $n \in \mathbb{N}$.

Assume that $F(a) = k$. We will show that $F(x) = x(k)$ for all sequences $x$.

Pick any $\ell \neq k$ and let $b_\ell$ be the sequence $(k, \ldots, k, \ell, k, \ldots)$ defined by $b_\ell(n) = k$ for $n \neq k$ and $b_\ell(k) = \ell$. Since $b_\ell$ is completely different from $a$, then $F(b_\ell) \neq k$. Since for each $m \neq k, \ell$ the sequence $b_\ell$ is completely different from $c_m$, then $F(b_\ell) \neq m$. Therefore we must have $F(b_\ell) = \ell$.

For $m \in \mathbb{N}$, let $S_m$ be the set of all sequences $z$ for which $z(k) = m$ and $z(n) \neq k$ for each $n \neq k$. Note that any $z \in S_m$ is completely different from every $b_\ell$ with $\ell \neq m$. Thus $F(z) \neq \ell$ for every $\ell \neq m$. It follows that $F(z) = m$ for each $z \in S_m$.

Now for any sequence $x$ and every $m \neq x(k)$ we can find a $z \in S_m$ which is completely different from $x$. Indeed we have $z(k) = m \neq x(k)$ and we can define the sequence arbitrarily at the other indices as long as $z(n) \neq k, x(n)$ for each $n \neq k$. Thus $F(x) \neq F(z) = m$ for each $m \neq x(k)$. It follows that $F(x) = x(k)$ as required.

**OC585.** Let $n \geq 3$ be a fixed integer. The number 1 is written $n$ times on a blackboard. Below the blackboard, there are two buckets that are initially empty. A move consists of erasing two of the numbers $a$ and $b$, replacing them with the numbers 1 and $a+b$, then adding one stone to the first bucket and $\gcd(a, b)$ stones to the second bucket. After some finite number of moves, there are $s$ stones in the first bucket and $t$ stones in the second bucket, where $s$ and $t$ are positive integers. Find all possible values of the ratio $t/s$.

*Originally Problem 5 from the XXXII Asian Pacific Mathematics Olympiad, March 2020.*

*We received only 1 submission.*

*We present the solution by UCLan Cyprus Problem Solving Group.*

Suppose that after a positive number of steps we have $1 = a_1 \leq a_2 \leq \cdots \leq a_n$ stones in the buckets. We claim that

$$t < (a_2 - 1) + 2(a_3 - 1) + \cdots + (n - 1)(a_n - 1).$$

We proceed by induction on the number of steps. The result is immediate if we perform one step as then $t = 1, a_1 = \cdots = a_{n-1} = 1, a_n = 2$.

Suppose that in the next step we replace $a_i, a_j$ with $i < j$ by $1, a_i + a_j$. Let $a_1', \ldots, a_n'$ be the new sequence. So these are $1, a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_i + a_j'$.
{

\[ a_j, \ldots, a_n \text{ in some order. By the rearrangement inequality we have} \]

\[ (a_2 - 1) + 2(a_3 - 1) + \cdots + (n - 1)(a_n - 1) \]

\[ \geq (a_1 - 1) + 2(a_2 - 1) + \cdots + (i - 1)(a_{i-1} - 1) + i(a_{i+1} - 1) + \cdots \]

\[ + (j - 2)(a_{j-1} - 1) + (j - 1)(a_j - 1) + j(a_{j+1} - 1) + \cdots + (n - 1)(a_{n-1}) \]

\[ > t + (a_1 - 1) + (a_2 - 1) + \cdots + (a_{i-1} - 1) + (j - 1)a_i \]

\[ \geq t + 0 + \cdots + 0 + a_i = t + a_i \geq t + \gcd(a_i, a_j) = t'. \]

This proves our claim. From this it follows that

\[ t < (n - 1)(a_1 + a_2 + a_3 + \cdots + a_{n-1} - n) = (n - 1)s. \]

In particular we always have \( t/s < n - 1 \). It is also immediate that \( t/s \geq 1 \).

We claim that every rational number \( p/q \in [1, n-1) \) can occur. Obviously \( p/q = 1 \) can occur so assume that \( 1 < \frac{p}{q} < n - 1 \).

**Claim.** Given \( a_1, \ldots, a_{n-1} \in \mathbb{N} \), we can arrange so that after a few steps the buckets have \( 1, 1, \ldots, 1, a_1 \cdots a_{n-1} \) stones and also that

\[ t = \left( n - 1 - \frac{1}{a_1} - \cdots - \frac{1}{a_{n-1}} \right) a_1 \cdots a_{n-1}. \]

**Proof of Claim.** We proceed by induction on \( n \). For \( n = 2 \) the claim is that \( t = a_1 - 1 \) which is easy to see as all moves are forced. Assume now that it is true for \( n = k \) and we want to prove it for \( n = k + 1 \). We proceed by induction on \( a_k \).

If \( a_k = 1 \), we need to prove that we can arrange for

\[ t = \left( k - 1 - \frac{1}{a_1} - \cdots - \frac{1}{a_{k-1}} - 1 \right) a_1 \cdots a_{k-1} \]

which follows from the case \( n = k \). Assume now that it is true when \( a_k = r \) and we want to prove it for \( a_k = r+1 \). After some steps we can have \( 1, 1, \ldots, 1, a_1 \cdots a_{k-1}r \) stones and

\[ t = \left( k - 1 - \frac{1}{a_1} - \cdots - \frac{1}{a_{k-1}} - \frac{1}{r} \right) a_1 \cdots a_{k-1}r \]

\[ = \left( k - 1 - \frac{1}{a_1} - \cdots - \frac{1}{a_{k-1}} \right) a_1 \cdots a_{k-1}r - a_1 \cdots a_{k-1} \]

After some more steps we can have \( 1, 1, \ldots, 1, a_1 \cdots a_{k-1}, a_{k-1} \cdots a_{k-1}r \) with \( t \) being increased by

\[ \left( k - 1 - \frac{1}{a_1} - \cdots - \frac{1}{a_{k-1}} \right) a_1 \cdots a_{k-1}. \]

In the last step we merge the last two buckets to get \( 1, 1, \ldots, 1, a_1 \cdots a_{k-1}(r + 1) \) coins with \( t \) further increased by \( a_1 \cdots a_{k-1} \). The total increase in the value of \( t \) is

\[ \left( k - 1 - \frac{1}{a_1} - \cdots - \frac{1}{a_{k-1}} \right) a_1 \cdots a_{k-1} \]

Copyright © Canadian Mathematical Society, 2022
and therefore the final value of $t$ is

$$
\left( k - \frac{1}{a_1} - \cdots - \frac{1}{a_k} \right) a_1 \cdots a_{k-1}(r+1) - a_1 \cdots a_{k-1} \\
= \left( k - \frac{1}{a_1} - \cdots - \frac{1}{a_{k-1}} - \frac{1}{r+1} \right) a_1 \cdots a_{k-1}(r+1).
$$

This completes the proof of the claim. □

In particular, if we take $a_1 = \cdots = a_{n-1} = a$ in the claim, we end up with $1, 1, \ldots, 1, a^{n-1}$ stones in the buckets and

$$
t = \left( n - 1 - \frac{n-1}{a} \right) a^{n-1} = (n-1)(a-1)a^{n-2}.
$$

In this case we have $s = a^{n-1} - 1$. After $\ell + 1$ steps it is easy to get $s = a^{n-1} + \ell$ and $t = (n-1)(a-1)a^{n-2} + \ell + 1$. We want to choose $a, \ell$ such that

$$
p = \frac{(n-1)(a-1)a^{n-2} + \ell + 1}{a^{n+1} + \ell} \iff p(a^{n-1} + \ell) = q((n-1)(a-1)a^{n-2} + \ell + 1)
$$

$$
\iff \ell = \frac{q((n-1)(a-1)a^{n-2} + 1) - pa^{n-1}}{p - q}
$$

$$
\iff \ell = \frac{(q(n-1) - p)a^{n-1} + q - q(n-1)a^{n-2}}{p - q}.
$$

Note that $p > q$. Now picking $a \equiv 1 \mod (p - q)$ we have that $\ell$ is an integer. Furthermore, since $q(n-1) - p \geq 1$, picking $a > q(n-1)$ we have that $\ell > 0$. In particular, for such choices of $a$ and $\ell$ the ration $p/q$ is achieved.
The number of 2s in prime factorization of superfactorials and Vandermonde determinants

Yagub Aliyev

The following question was extensively discussed on problem-solving websites [7,8]:

**Problem 1.** Prove that \( \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i} \) is an integer for arbitrary integers \( x_1, x_2, \ldots, x_n \).

This also appeared as a problem in The American Mathematical Monthly at least twice: Problem E2637 and Problem E2949. We will not give a detailed solution of this interesting problem because one can find in the mentioned references several solutions based on different ideas. Instead, a sketch of the proof will be given here.

**Solution of Problem 1.** We first note that \( V_n = \prod_{1 \leq i < j \leq n} x_j - x_i \) is equal to Vandermonde determinant of \( x_1, x_2, \ldots, x_n \):

\[
V_n = V(x_1, x_2, \ldots, x_n) = \begin{vmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\
1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1}
\end{vmatrix}
\]

In particular, \( P_n = \prod_{1 \leq i < j \leq n} (j - i) = (n - 1)!(n - 2)! \ldots 2!1! \) is Vandermonde determinant of 1, 2, 3, \ldots, \( n \):

\[
P_n = V(1, 2, 3, \ldots, n) = \begin{vmatrix}
1 & 1 & 1^2 & \cdots & 1^{n-1} \\
1 & 2 & 2^2 & \cdots & 2^{n-1} \\
1 & 3 & 3^2 & \cdots & 3^{n-1} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & n & n^2 & \cdots & n^{n-1}
\end{vmatrix}
\]

We need to prove that

\[
\frac{V_n}{P_n} = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)}{\prod_{1 \leq i < j \leq n} (j - i)} = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i},
\]

is an integer, or which is the same, to prove that \( P_n | V_n \). For this, it is sufficient to show that \( V_n/P_n \) is equal to a determinant with integer entries. Indeed,

\[
\prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{j - i} = \begin{vmatrix}
1 & (x_1^1) & (x_1^2) & \cdots & (x_1^{n-1}) \\
1 & (x_2^1) & (x_2^2) & \cdots & (x_2^{n-1}) \\
1 & (x_3^1) & (x_3^2) & \cdots & (x_3^{n-1}) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & (x_n^1) & (x_n^2) & \cdots & (x_n^{n-1})
\end{vmatrix},
\]
every entry of which is a binomial coefficient \( \binom{x}{k} = \frac{x(x-1)(x-2)\ldots(x-k+1)}{k!} \), for \( x \geq k \), and \( \binom{x}{k} = 0 \) for \( x < k \), and therefore an integer. One can find this equality in linear algebra problem books and problem-solving websites. See for example Problem 269 in [6], or [7], [8]. The proof is complete.

There are also combinatorial interpretations and generalizations [3-5] of the last determinant. Since \( P_n[V_n] \), and \( P_n = V_n \), when, for example, \( x_i = i \) for all \( i = 1, 2, \ldots, n \), we can ask and easily answer the following interesting question about the minimum possible value of \( V_n \).

**Problem 2.** For what choice of distinct integers \( x_1, x_2, \ldots, x_n \) is the number of 2s in the prime factorization of \( V_n = \prod_{1 \leq i < j \leq n} x_i x_j \) minimal?

**Answer.** The minimum occurs when the numbers \( x_i \) are 1, 2, \ldots, \( n \) or any other arithmetic sequence with difference 1.

The same can be said about the number of 3s, 5s or any other prime \( p \). The number of \( ps \) in the prime factorization of Vandermonde determinant \( V(x_1, x_2, \ldots, x_n) \) is minimal when integers \( x_1, x_2, \ldots, x_n \) are, for example, 1, 2, 3, \ldots, \( n \). This can be expressed as

\[
\min_{x_1, x_2, \ldots, x_n \in \mathbb{Z}} \text{val}_p(V(x_1, x_2, \ldots, x_n)) = \text{val}_p(V(1, 2, 3, \ldots, n)),
\]

where \( \text{val}_p(x) \) is the \( p \)-adic valuation function, counting the number of \( ps \) in \( x \) (see [10], p. 3 where it is denoted as \( \text{ord}_p(x) \)). In the remaining part of the paper, we will explore some properties of the sequence \( \text{val}_2(V(1, 2, 3, \ldots, n)) \).

The sequence \( P_{n+1} = n!(n-1)\ldots2! \) (also called superfactorials and denoted as \( sf(n) \)) is the product of first \( n \) factorials and appears as A000178 in the Online Encyclopedia of Integer Sequences (OEIS) [1] (See also [2]). We denote the number of 2s in the prime factorization of \( P_{n+1} \) by \( d_n = d(n) \). The sequence \( d_n \) appears in OEIS as A174605. Let us also denote by \( t_n = t(n) \) and \( s_n = s(n) \) the number of 2s in the prime factorizations of \( n! \) and \( n \), respectively. The sequences \( t_n \) and \( s_n \) are also in OEIS (A011371 and A007814, respectively).

\[
\begin{array}{cccccccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\
 s_n & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 4 & 0 & 1 & 0 \\
 t_n & 0 & 1 & 1 & 3 & 3 & 4 & 4 & 7 & 7 & 8 & 8 & 10 & 10 & 11 & 11 & 15 & 15 & 16 & 16 \\
 d_n & 0 & 1 & 2 & 5 & 8 & 12 & 16 & 23 & 30 & 38 & 46 & 56 & 66 & 77 & 88 & 103 & 118 & 134 & 150 \\
\end{array}
\]

**Table 1**

We will start with some basic identities involving these functions.

**Exercise 1.** Prove that \( d_n \) is the partial sum of \( t_n \), which in turn is the partial sum of \( s_n \):

\[
d_n = t_n + d_{n-1}, t_n = s_n + t_{n-1},
\]

\[
d_n = t_1 + t_2 + \ldots + t_n, t_n = s_1 + s_2 + \ldots + s_n.
\]

**Exercise 2.** Prove that \( t_n \) can be expressed as a sum, using the greatest integer function:

\[
t_n = \lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + \ldots + \lfloor n/2^k \rfloor + \ldots
\]

_Crux Mathematicorum_, Vol. 48(10), December 2022
This formula also gives the number of trailing zeros of $n!$ written 2-base number system. It is a special case of Legendre’s formula [9, 11] for the number of $p$s in the prime factorization of $n!$, for the prime $p$,

$$\text{val}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots + \left\lfloor \frac{n}{p^k} \right\rfloor + \ldots$$

The sum always terminates, as the $k$th term is zero for $p^k > n$. In particular, $\text{val}_5(n!)$ counts the number of trailing zeros of $n!$, written in the decimal number system. Since $\text{val}_2(n!) \geq \text{val}_5(n!)$, the maximal power of $10 = 2 \cdot 5$, which divides $n!$, is determined only by $\text{val}_5(n!)$. Proof of the last inequality is left as an exercise.

**Exercise 3.** Prove the following formula for $d(n)$:

$$d(n) = s(P_{n+1}) = s(n!(n-1)! \ldots 2!1!)= s(n!) + s((n-1)!)+ \ldots + s(2!) + s(1!),$$

$$d_n = \sum_{i=1}^{n} i \cdot s_{n+1-i} = 1 \cdot s_n + 2 \cdot s_{n-1} + \ldots + n \cdot s_1.$$  

**Exercise 4.** Express $d_n$ as a sum using greatest integer function.

**Exercise 5.** Prove that $d_n \leq \binom{n}{2}$.

**Solution.** Since $t(n) \leq n-1$ (see [1], A011371), we can write

$$d_n = t(1) + t(2) + t(3) + \ldots + t(n) \leq 0 + 1 + 2 + \ldots + (n - 1) = n(n - 1)/2.$$

**Remark.** According to [1] (A174605),

$$d(n) = \frac{1}{2} n(n + 1) - \text{(Total number of 1’s in binary expansions of 0, \ldots, n),}$$

$$d(n) = \frac{1}{2} \left( (n + 1)^2 - \sum_{i=1}^{k} (e_i + 2i - 1)2^{e_i} \right),$$

where $n + 1 = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_k}$, $e_1 > e_2 > \ldots > e_k$, $e_i$ are nonnegative integers.

It seems that there is no simple explicit formula for $d(n)$ itself. But it is possible to find a simple formula for $d(2^n)$, $d(2^n \pm 1)$, $d(2^n \pm 2)$ and similar sequences.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$d(2^n - 2)$</th>
<th>$d(2^n - 1)$</th>
<th>$d(2^n)$</th>
<th>$d(2^n + 1)$</th>
<th>$d(2^n + 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>12</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>77</td>
<td>16</td>
<td>23</td>
<td>38</td>
<td>98</td>
</tr>
<tr>
<td>4</td>
<td>390</td>
<td>88</td>
<td>103</td>
<td>134</td>
<td>510</td>
</tr>
<tr>
<td>5</td>
<td>1767</td>
<td>416</td>
<td>447</td>
<td>7560</td>
<td>2014</td>
</tr>
<tr>
<td>6</td>
<td>7560</td>
<td>1824</td>
<td>1887</td>
<td>7680</td>
<td>8062</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2
Problem 3. Find $d(2^n - 1)$.

Solution. First, note that $s(2n) = s(n) + 1, s(2n + 1) = 0, s(2^n) = n$. So, $t(2^{n+1}) = 2t(2^n) + 1$. Therefore, $t(2^n) = 2^n - 1$. Consequently, $s((2^n)!) = 2^n - 1$ and

$$d(2^n - 1) = 2d(2^{n-1} - 1) + 2^{n-1}s((2^{n-1})!) = 2d(2^{n-1} - 1) + 2^{n-1}(2^{n-1} - 1).$$

In particular, $d(2^2 - 1) = 2d(2 - 1) + 2s((2^1)!) = 0 + 2 = 2$. Using this recursive formula again and again we obtain

$$d(2^n - 1) = 2(2d(2^{n-2} - 1) + 2^{n-2}(2^{n-2} - 1)) + 2^{n-1}(2^{n-1} - 1)$$
$$= 2^2d(2^{n-2} - 1) + 2^{n-1}(2^{n-2} - 1) + 2^{n-1}(2^{n-1} - 1)$$
$$= 2^3d(2^{n-3} - 1) + 2^{n-1}((2^{n-3} - 1) + (2^{n-2} - 1) + (2^{n-1} - 1))$$
$$= \ldots$$
$$= 2^{n-2}d(2^2 - 1) + 2^{n-1}((2^2 - 1) + (2^3 - 1) + \ldots + (2^{n-1} - 1))$$
$$= 2^{n-1}((2^0 - 1) + (2^1 - 1) + \ldots + (2^{n-1} - 1)) = 2^{n-1}(2^n - 1 - n).$$

Problem 4. Find $d(2^n)$ and $d(2^n - 2)$.

Solution. Obviously, $d(2^n) = d(2^n - 1) + s((2^n)!)$. Since

$$d(2^n) = 2^{n-1}(2^n - 1 - n) + s(2^n) + s(2^n - 1) + s(2^n - 2) + \ldots + s(1)$$
$$= 2^{n-1}(2^n - 1 - n) + t(2^n)$$
$$= 2^{n-1}(2^n - 1 - n) + 2^n - 1,$$
$$= 2^{n-1}(2^n + 1 - n) - 1.$$

Similarly, since $t(2^n - 1) = t(2^n) - s(2^n) = 2^n - 1 - n$,

$$d(2^n - 2) = d(2^n - 1) - t(2^n - 1) =$$
$$= 2^{n-1}(2^n - 1 - n) - (2^n - 1 - n) = (2^{n-1} - 1)(2^n - 1 - n).$$

Exercise 6. Find $d(2^n + 1)$ and $d(2^n + 2)$ and compare the results with Table 2.

Acknowledgment

Many thanks to the anonymous reviewer who suggested many corrections and improvements for this paper.

References


*Crux Mathematicorum*, Vol. 48(10), December 2022


[7] AoPS Online, High School Olympiads

[8] Showing $\prod_{i<j} \frac{x_j - x_i}{j-i}$ is an integer, StackExchange post


From Focus On... No. 49

1. For $\alpha, \beta > 0$, prove that

$$\int_0^1 (1 - x^\alpha)^{\frac{1}{\beta}} \, dx = \int_0^1 (1 - x^\beta)^{\frac{1}{\alpha}} \, dx.$$ 

The change of variables $x = t^{\frac{1}{\alpha}}$ yields

$$\int_0^1 (1 - x^\alpha)^{\frac{1}{\beta}} \, dx = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} (1-t)^{1+\frac{1}{\beta}-1} \, dt = \frac{1}{\alpha} B \left( \frac{1}{\alpha}, 1 + \frac{1}{\beta} \right) = \frac{1}{\alpha} \cdot \frac{\Gamma \left( \frac{1}{\alpha} \right) \Gamma \left( 1 + \frac{1}{\beta} \right)}{\Gamma \left( \frac{1}{\alpha} + \frac{1}{\beta} + 1 \right)}.$$ 

Recalling that $\Gamma(x+1) = x\Gamma(x)$ for positive $x$, we finally obtain

$$\int_0^1 (1 - x^\alpha)^{\frac{1}{\beta}} \, dx = \frac{1}{\alpha\beta} \cdot \frac{\Gamma \left( \frac{1}{\alpha} \right) \Gamma \left( \frac{1}{\beta} \right)}{\Gamma \left( \frac{1}{\alpha} + \frac{1}{\beta} + 1 \right)}.$$ 

The desired result follows from the symmetry in $\alpha, \beta$ in the right-hand side.

2. Let $a \in (0,1)$. For $n \in \mathbb{N}$, let $I_n(a) = \int_0^\infty \frac{dt}{(1 + t^{1/a})^n}$. Find $\lim_{n \to \infty} n^a I_n(a)$. 

[Hint: use substitution $t = (1 - u)^a u^{-a}$.]

The change of variables $t = (1 - u)^a \cdot u^{-a}$ readily leads to

$$I_n(a) = aB(a,n-a) = a\Gamma(a) \cdot \frac{\Gamma(n-a)}{\Gamma(n)}.$$ 

Since

$$(x+1)(x+2) \cdots (x+n) \sim \frac{n^x n!}{\Gamma(x)}$$ 

as $n \to \infty$ 

and

$$\Gamma(n-a) = \Gamma(1-a) \prod_{j=1}^{n-1} (j-a)$$

we obtain

$$\frac{\Gamma(n-a)}{\Gamma(n)} \sim n^{-a}$$ 

as $n \to \infty$ 

and therefore

$$\lim_{n \to \infty} n^a I_n(a) = a\Gamma(a).$$
3. Let $a, b, c, d, k$ be positive real numbers such that $k = a + b = c + d$. Prove that

$$\prod_{n=0}^{\infty} \frac{(kn + a)(kn + b)}{(kn + c)(kn + d)} = \frac{\sin(\pi a/k)}{\sin(\pi c/k)}.$$ 

For $N \to \infty$ we have

$$\prod_{n=0}^{N} (n + (a/k)) \sim \frac{N^{a/k} N!}{\Gamma(a/k)},$$

hence

$$\prod_{n=0}^{\infty} \frac{(kn + a)(kn + b)}{(kn + c)(kn + d)} = \lim_{N \to \infty} \frac{N^{a/k} N!}{\Gamma(a/k)} \cdot \frac{N^{b/k} N!}{\Gamma(b/k)} \cdot \frac{\Gamma(c/k)}{\Gamma(c/k)} \cdot \frac{\Gamma(d/k)}{\Gamma(d/k)} = \frac{\Gamma(c/k) \Gamma(d/k)}{\Gamma(a/k) \Gamma(b/k)}.$$ 

From $\frac{d}{k} = 1 - \frac{c}{k}$, we deduce that $\Gamma(c/k) \Gamma(d/k) = \frac{\pi}{\sin(\pi c/k)}$; similarly, we can see that $\Gamma(a/k) \Gamma(b/k) = \frac{\pi}{\sin(\pi a/k)}$ and we finally obtain

$$\prod_{n=0}^{\infty} \frac{(kn + a)(kn + b)}{(kn + c)(kn + d)} = \frac{\sin(\pi a/k)}{\sin(\pi c/k)}.$$

4. Prove that

$$\int_{0}^{\infty} \frac{e^{-x}(1 - e^{-2x})(1 - e^{-4x})(1 - e^{-6x})}{x(1 - e^{-14x})} \, dx = \ln 2$$

We use that for real numbers $a, b, c$ satisfying $a, a + b, a + c, a + b + c > 0$, we have

$$I(a, b, c) := \int_{0}^{1} \frac{t^{a-1}(1 - t^{b})(1 - t^{c})}{(1 - t)(- \ln t)} \, dt = \ln \left( \frac{\Gamma(a + b + c) \Gamma(a)}{\Gamma(a + b) \Gamma(a + c)} \right)$$

(proven in Focus On No 49 p. 39-40).

The substitution $e^{-14x} = t$ readily shows that the given integral is equal to

$$I \left( \frac{1}{14}, \frac{1}{7}, \frac{2}{7} \right) - I \left( \frac{1}{2}, \frac{1}{7}, \frac{2}{7} \right),$$

that is, to

$$\ln \left( \frac{\Gamma(1/2) \Gamma(1/14)}{\Gamma(3/14) \Gamma(5/14)} \right) - \ln \left( \frac{\Gamma(13/14) \Gamma(1/2)}{\Gamma(9/14) \Gamma(11/14)} \right) = \ln \left( \frac{\Gamma(1/14) \Gamma(9/14) \Gamma(11/14)}{\Gamma(3/14) \Gamma(5/14) \Gamma(13/14)} \right).$$

Now, using the formula $\sqrt{\pi} \Gamma(x) = 2^{x-1} \Gamma \left( \frac{x}{2} \right) \Gamma \left( \frac{x+1}{2} \right)$ with $x = \frac{1}{7}$, we obtain

$$\Gamma(1/14) \Gamma(4/7) = 2^{6/7} \sqrt{\pi} \Gamma(1/7).$$
Applying the formula again for \( x = \frac{9}{7}, \frac{11}{7}, \frac{3}{7}, \frac{3}{7}, \frac{13}{7} \), we easily get

\[
\frac{\Gamma(1/4)\Gamma(9/4)\Gamma(11/4)}{\Gamma(3/4)\Gamma(5/4)\Gamma(13/4)} = \frac{\Gamma(1/7)\Gamma(11/7)\Gamma(6/7)\Gamma(10/7)}{\Gamma(4/7)\Gamma(8/7)\Gamma(3/7)\Gamma(13/7)}
\]

and, using

\[
\Gamma(11/7) = \frac{4}{7}\Gamma(4/7), \quad \Gamma(10/7) = \frac{3}{7}\Gamma(3/7), \quad \Gamma(8/7) = \frac{1}{7}\Gamma(1/7), \quad \Gamma(13/7) = \frac{6}{7}\Gamma(6/7),
\]

we see that the previous ratio simplifies to 2. The result follows.

From Focus On... No. 50

1. Let \( A = (a_{i,j}) \in M_4(\mathbb{F}) \) be such that \( a_{i,i} = a^2, \ a_{i,5-i} = b^2, \ a_{i,j} = ab \) for \( j \neq i, 5-i \ (i,j = 1,2,3,4) \). Calculate \( \det(A) \).

First adding rows 2,3,4 to the first one, we obtain

\[
\det(A) = (a+b)^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ ab & a^2 & b^2 & ab \\ ab & b^2 & a^2 & ab \\ b^2 & ab & ab & a^2 \end{vmatrix} = (a+b)^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ ab & a(a-b) & b(b-a) & 0 \\ ab & b(b-a) & a(a-b) & 0 \\ b^2 & b(a-b) & b(a-b) & a^2 - b^2 \end{vmatrix}
\]

so that

\[
\det(A) = (a+b)^2(a^2 - b^2) \begin{vmatrix} 1 & 0 & 0 \\ ab & a(a-b) & b(b-a) \\ ab & b(b-a) & a(a-b) \end{vmatrix}
\]

and therefore

\[
\det(A) = (a+b)^2(a^2 - b^2)[a^2(a-b)^2 - b^2(b-a)^2] = (a^2 - b^2)^4.
\]

2. Let \( A \in M_n(\mathbb{F}) \) satisfy \( A^2 = kA \ (k \in \mathbb{F}) \) and let \( r = r_k(A) \). Prove that \( \det(I_n + A) = (1 + k)^r \).

We may suppose that \( r \geq 1 \) and we first consider the case \( k = 0 \), i.e. \( A^2 = O_n \).

Using a basis \( (U_1, \ldots, U_r, U_{r+1}, \ldots, U_n) \) where \( (U_1, \ldots, U_r) \) is a basis of \( \text{im}(A) \), we obtain that \( I_n + A \) is similar to a triangular matrix whose diagonal entries are all equal to 1 (note that \( AU_i = 0 \) for \( i = 1, \ldots, r \) and that \( AU_i \) is linear combination of \( U_1, \ldots, U_r \) if \( i = r+1, \ldots, n \)). It follows that \( \det(I_n + A) = 1 = (1 + k)^r \) in that case.

From now on, we assume that \( k \neq 0 \). If \( X \in \ker A \cap \text{im}(A) \), then \( AX = 0 \) and \( X = Y \) for some column vector so that \( kX = (kA)Y = A^2Y = AX = 0 \) and \( X = 0 \). This proves that \( \ker A \cap \text{im}(A) = \{0\} \). In addition, we have \( n = \dim(\ker A) + \dim(\text{im}(A)) \), hence \( F^n = \ker A \oplus \text{im}(A) \).

Now, we consider a basis \( (U_1, \ldots, U_r, U_{r+1}, \ldots, U_n) \) where \( (U_1, \ldots, U_r) \) is a basis of \( \text{im}(A) \) and \( (U_{r+1}, \ldots, U_n) \) a basis of \( \ker A \). Observing that \( AU_i = kU_i \) if \( i =
3. Let $A \in \mathcal{M}_n(\mathbb{R})$ satisfy $AA^T = A^2$. Prove that $A^T = A$.

Let $B = A - A^T$. The desired equality $B = O_n$ will follow from $\text{tr}(B^TB) = 0$ that we now show.


But from $AA^T = A^2$, we deduce that $(A^T)^2 = AA^T$ whence

$\text{tr}(B^TB) = \text{tr}(A^TA - (A^T)^2) = \text{tr}(A^TA) - \text{tr}((A^T)^2) = \text{tr}(A^TA) - \text{tr}(AA^T) = 0$

From Focus On... No. 51

1. Let $A \in \mathcal{M}_n(\mathbb{F})$ have $n$ distinct eigenvalues in $\mathbb{F}$ and let $M \in \mathcal{M}_n(\mathbb{F})$ be such that $MA = AM$. Prove that $M = p(A)$ for some polynomial $p(x) \in \mathbb{F}[x]$.

Let $C(A)$ be the set of all matrices $M \in \mathcal{M}_n(\mathbb{F})$ such that $AM = MA$. It is easily seen that $C(A)$ is a subspace of $\mathcal{M}_n(\mathbb{F})$.

From the hypothesis, $A$ is diagonalizable, hence $A = PDP^{-1}$ for some invertible matrix $P$ and $D$ is a diagonal matrix. The equality $AM = MA$ writes as $(PDP^{-1})M = M(PDP^{-1})$, hence $DM' = M'D$ where $M' = P^{-1}MP$. Since the diagonal entries of the diagonal matrix $D$ are distinct (being the eigenvalues of $A$), the matrix $M'$ must be diagonal: $M' = \sum_{i=1}^{n} a_i E_i$ where $a_1, \ldots, a_n$ are the diagonal entries of $M'$ and $E_i$ is the diagonal matrix whose diagonal entries are 0 except the $i$th entry which is 1. Thus, $M = \sum_{i=1}^{n} a_i (PE_i P^{-1})$; this shows that the matrices $PE_i P^{-1}$, $i = 1, \ldots, n$ span the subspace $C(A)$. Since they are also independent (readily proved), we have $\dim(C(A)) = n$.

Now, the $n$ matrices $I_n, A, A^2, \ldots, A^{n-1}$ are obviously all in $C(A)$ and they are independent: indeed, if $\sum_{i=0}^{n-1} \alpha_i A^i = O_n$, then

$$\sum_{i=0}^{n-1} \alpha_i P^{-1} A^i P = \sum_{i=0}^{n-1} \alpha_i D^i = O_n$$

that is, $p(\lambda_1) = \cdots = p(\lambda_n) = 0$ where $p(x) = \sum_{i=0}^{n-1} \alpha_i x^i$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. Since these eigenvalues are distinct, $p$ must be the zero polynomial and therefore $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. It follows that the $n$ matrices $I_n, A, A^2, \ldots, A^{n-1}$ span $C(A)$ and therefore every element of this subspace is a polynomial in $A$. 

Copyright © Canadian Mathematical Society, 2022
2. Let $A \in \mathcal{M}_3(\mathbb{R})$ satisfying $A \neq O_3$ and $A^3 = -A$. Find the minimum polynomial of $A$.

Since $A^3 + A = O_3$, the minimum polynomial $\mu(x)$ of $A$ divides $x^3 + x = x(x^2 + 1)$. The polynomials $x$ and $x^2 + 1$ being irreducible in $\mathbb{R}[x]$, we have either $\mu(x) = x$ or $\mu(x) = x^2 + 1$ or $\mu(x) = x(x^2 + 1)$. Since $A \neq O_3$, we have $\mu(x) \neq x$. The assumption $\mu(x) = x^2 + 1$ implies $A^2 = -I_3$, hence $(\det(A))^2 = \det(-I_3) = -1$, a contradiction since $\det(A)$ is a real number. We conclude that $\mu(x) = x^3 + x$.

3. Let $m, n, p$ be positive integers with $p \geq 2$ and let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^m = I_n$ and $BA = AB^p$. Show that $B$ is diagonalizable.

The matrix $A$ is invertible (with $A^{-1} = A^{m-1}$), hence we have $B = AB^p A^{-1}$. It follows that $B^p = A(B^p)^p A^{-1}$ and $B = A^2 B^p A^{-2}$.

Assuming that $B = A^k B^{p^k} A^{-k}$, we obtain $B^p = A^kB^{p^k+1} A^{-k}$ and deduce that

$$B = A(A^kB^{p^k+1} A^{-k})A^{-1} = A^{k+1} B^{p^k+1} A^{-(k+1)}.$$ 

We have proved by induction that $B = A^k B^{p^k} A^{-k}$ for all positive integers $k$. In particular, $B = A^m B^{p^m} A^{-m} = B^{p^m}$, showing that the non-zero polynomial $P(x) = x^{p^m} - x = x(x^{p^m-1} - 1)$ satisfies $P(B) = O_n$ (note that $p^{m} > 1$ since $p \geq 2$). The complex roots of $P(x)$ are 0 and the $(p^m - 1)$th roots of unity, all simple roots. The roots of the minimum polynomial of $B$, which divides $P(x)$, are also simple roots. Thus, $B$ is diagonalizable.
PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2023**.

4791. **Proposed by George Stoica.**
Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{C}^n$. Prove that if $u$ maximizes $\prod_{i=1}^n |v_i \cdot u|$ over all unit vectors in $\mathbb{C}^n$, then for all $i$, $|v_i \cdot u| \geq \frac{1}{\sqrt{n}}$.

4792. **Proposed by George Apostolopoulos.** (Correction).
The interior bisectors of angles $B$ and $C$ of a triangle $ABC$ with incenter $I$ meet $AC$ at $D$ and $AB$ and $E$, respectively. Suppose that $\text{Area}(BIC) = \text{Area}(AEID)$. Prove that $\angle A \leq 60^\circ$.

4793. **Proposed by Corneliu Manescu-Avram.**
Let $a$ be an even positive integer and let $p$ be an odd prime number such that $\gcd(a^2 - 1, p) = 1$. Prove that $a^{n-1} - 1$ is divisible by $n$, where $n = \frac{a^{2p} - 1}{a^2 - 1}$.

4794. **Proposed by Abhishek Jha.**
Let $P(x) = x^2 + bx + 1$, where $b$ is a non-negative integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \geq 0$. Find all polynomials $P(x)$ such that there exists a positive integer $n > 1$ which divides $x_n$.

4795. **Proposed by Mihaela Berindeanu.**
Let $(x_n)$ and $(y_n)$ be two sequences of natural numbers. If $(x_n)$ is defined by the recurrence relation $x_1 = \frac{1}{9}$ and $x_{n+1} = 9x_n^3$, $\forall n \geq 1$ and $y_n = 9x_n^2 + 3x_n + 1$, calculate:

$$\lim_{n \to \infty} y_1 \cdot y_2 \cdots y_n$$

4796. **Proposed by Vasile Cirtoaje and Leonard Giugiuc.**
Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1^2}{(1-a_1)^2} + \frac{a_2^2}{(1-a_2)^2} + \cdots + \frac{a_n^2}{(1-a_n)^2} \geq \left( \frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n} - \frac{\sqrt{n}}{\sqrt{n+1}} \right)^2.$$
4797. Proposed by Goran Conar.
Let $x, y, z > 0$ be real numbers. Prove the following inequality:
\[ \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq x + y + z. \]

A point $L$ is randomly selected inside circle $\omega$, 6 points $A_1, A_2, A_3, B_1, B_2, B_3$ ($As$ and $Bs$ are in clockwise order) lies on $\omega$ such that $\angle LA_1A_2 = \angle LA_2A_3 = \angle LB_1B_2 = \angle LB_2B_3$ Prove that $A_1B_3, A_2B_2, A_3B_1$ are concurrent or parallel.

4799. Proposed by Ovidiu Furdui and Alina Sîntămărian.
Calculate
\[ \lim_{n \to \infty} \int_1^2 \sqrt[n]{|x^n|} \, dx, \]
where $\lfloor x \rfloor$ denotes the floor of $x \in \mathbb{R}$.

4800. Proposed by Michel Bataille.
Two circles $\Gamma_1, \Gamma_2$, with centres $O_1, O_2$ and distinct radii, intersect in $A_1$ and $A_2$. Let $C_1$ on $\Gamma_1$ and $C_2$ on $\Gamma_2$ be such that the line $C_1C_2$ is parallel to $O_1O_2$ with $A_1, A_2, O_1, O_2$ not on $C_1C_2$. Let lines $C_1O_1$ et $C_2O_2$ intersect at $M$ and lines $MA_1, MA_2$ intersect $C_1C_2$ in $B_1, B_2$, respectively. Prove that the circumcircles of $\triangle MB_1B_2$ et $\triangle MC_1C_2$ are tangent.

Crux Mathematicorum, Vol. 48(10), December 2022
Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 février 2023.

4791. Soumis par George Stoica.
Soient $v_1, \ldots, v_n$ des vecteurs unitaires de $\mathbb{C}^n$. Montrez que si $u$ maximise $\prod_{i=1}^{n} |v_i \cdot u|$ sur tous les vecteurs unitaires de $\mathbb{C}^n$, alors pour tout $i$ on a $|v_i \cdot u| \geq \frac{1}{\sqrt{n}}$.

4792. Soumis par George Apostolopoulos. (Correction.)
Les bissectrices intérieures des angles $B$ et $C$ d’un triangle $ABC$ dont le centre du cercle inscrit est $I$ rencontrent $AC$ et $AB$ respectivement en $D$ et $E$. Supposons que $\text{Aire}(BIC) = \text{Aire}(AEID)$. Montrez que $\angle A \leq 60^\circ$.

4793. Soumis par Corneliu Manescu-Avram.
Soit $a$ un entier positif pair et de plus soit $p$ un nombre premier impair vérifiant $\text{PGCD}(a^2 - 1, p) = 1$. Montrez que $a^{n-1} - 1$ est divisible par $n$, où $n = \frac{a^{2p} - 1}{a^2 - 1}$.

4794. Soumis par Abhishek Jha.
Soit $P(x) = x^2 + bx + 1$, où $b$ est un entier non négatif. Posons $x_0 = 0$ et $x_{i+1} = P(x_i)$ pour tout entier $i \geq 0$. Trouvez tous les polynômes $P(x)$ pour lesquels il existe un entier positif $n > 1$ divisant $x_n$.

4795. Soumis par Mihaela Berindeanu.
Soient $(x_n)$ et $(y_n)$ deux suites de nombres naturels. Si la suite $(x_n)$ est définie par la relation de récurrence $x_1 = \frac{1}{9}$ et $x_{n+1} = 9x_n^3$, $\forall n \geq 1$ et si $y_n = 9x_n^2 + 3x_n + 1$, calculez:
$$\lim_{n \to \infty} y_1 \cdot y_2 \cdots y_n$$

Soient $a_1, a_2, \ldots, a_n$ des nombres réels positifs vérifiant $a_1 + a_2 + \cdots + a_n = 1$. Montrez que
$$\frac{a_1^2}{(1-a_1)^2} + \frac{a_2^2}{(1-a_2)^2} + \cdots + \frac{a_n^2}{(1-a_n)^2} \geq \left( \frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \cdots + \frac{a_n}{1-a_n} - \frac{\sqrt{n}}{\sqrt{n+1}} \right)^2.$$
4797. \textit{Soumis par Goran Conar.}

Soient \(x, y, z > 0\) des nombres réels. Démontrons l'inégalité suivante :
\[
\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq x + y + z.
\]

4798. \textit{Soumis par Jason Fang.}

Un point \(L\) est choisi au hasard à l'intérieur d'un cercle \(\omega\). Six points \(A_1, A_2, A_3, B_1, B_2\) et \(B_3\) (les \(A_i\) et les \(B_j\) sont ordonnés dans le sens horaire) appartiennent à \(\omega\) et vérifient \(\angle LA_1A_2 = \angle LA_2A_3 = \angle LB_1B_2 = \angle LB_2B_3\) Montrez que les droites \(A_1B_3, A_2B_2\) et \(A_3B_1\) sont concourantes ou parallèles.

\[
\[
\text{Diagramme:}

4799. \textit{Soumis par Ovidiu Furdui et Alina Sîntămărian.}

Calculez
\[
\lim_{n \to \infty} \int_1^2 \sqrt[n]{x^n} \, dx,
\]
on où \([x]\) désigne la partie entière de \(x \in \mathbb{R}\).

4800. \textit{Soumis par Michel Bataille.}

Considérons des cercles \(\Gamma_1\) et \(\Gamma_2\) de rayons distincts et ayant respectivement pour centres les points \(O_1\) et \(O_2\). Ces deux cercles se rencontrent en \(A_1\) et \(A_2\). Soient \(C_1\) sur \(\Gamma_1\) et \(C_2\) sur \(\Gamma_2\) des points tels que la droite \(C_1C_2\) est parallèle à \(O_1O_2\) avec \(A_1, A_2, O_1\) et \(O_2\) n’appartenant pas à \(C_1C_2\). Soit \(M\) le point d’intersection des droites \(C_1O_1\) et \(C_2O_2\) et soient \(B_1\) et \(B_2\) les points d’intersection respectifs des droites \(MA_1\) et \(MA_2\) avec \(C_1C_2\). Montrez que les cercles circonscrits à \(\Delta MB_1B_2\) et \(\Delta MC_1C_2\) sont tangents.

\textit{Crux Mathematicorum, Vol. 48(10), December 2022}
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4741. Proposed by Alexander Bloom.

A large pond has 100 lily pads lying in a straight line, numbered consecutively 1 to 100, with a frog making jumps between adjacent lily pads. At any given lily pad \(x\), \(1 \leq x \leq 99\), the probability of the frog moving forwards to the next lily pad is \(\frac{1}{x}\) and the probability of retreating to the previous lily pad is \(\frac{x-1}{x}\). For some \(y\), if the frog starts on lily pad \(y\), then the probability that the frog reaches lily pad 100 without ever touching lily pad \(y-1\) can be written as \(\frac{9!}{10 \cdot 98!}\). Find the value of \(y\).

We received 7 solutions. However, the problem statement is erroneous, so only 5 of the received solutions are correct. Our apologies for the omission. We present here the solution by the UCLan Cyprus Problem Solving Group that proposes an alternative statement of the problem.

We first modify the problem as we believe it was intended to be and then find the probabilities for the original formulation.

Let \(p_n\) be the probability that starting from the lily pad \(n\) we reach lily pad 100 such that in our first step we move forwards and we never move two consecutive steps backwards.

We have \(p_{99} = \frac{1}{99}\) and

\[
p_n = \frac{p_{n+1}}{n} \left( 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right) = \frac{(n+1)p_{n+1}}{n^2}
\]

for \(n = 1, 2, \ldots, 98\). This is because from lily pad \(n\) we have to move to lily pad \(n+1\). Then go backwards and forwards \(k\) times for some \(k\), and then move from lily pad \(n\) to lily pad 100 such that in our first step we move forwards and we never move two consecutive steps backwards. For a fixed \(k\) this probability is

\[
\frac{1}{n} \left( \frac{n}{n+1} \cdot \frac{1}{n} \right)^k p_n.
\]

By (reverse) induction it is easy to get that

\[
p_n = \frac{(n-1)!}{n \cdot 98!}
\]

So we have \(p_{10} = \frac{9!}{10 \cdot 98!}\).
We now solve the original formulation. We let $q_n$ be the probability that starting from lily pad $n$ we reach lily pad 100 without ever stepping on lily pad $n - 1$. We have $q_{99} = \frac{1}{99}$ and

$$q_n = \frac{q_{n+1}}{n} + \frac{(1-q_{n+1})q_n}{n}$$

for each $n = 1, 2, \ldots, 98$. Here, the first part of the formula is equal to the probability that we move first to lily pad $n + 1$ and then never even returning to lily pad $n$ again, while the second part of the formula denotes the probability that we move to lily pad $n + 1$, at some point of the process we move back to lily pad $n$, but we never touch lily pad $n - 1$. This gives

$$q_n = \frac{q_{n+1}}{n-1+q_{n+1}}$$

and so

$$\frac{1}{q_n} = 1 + \frac{n-1}{q_{n+1}}$$

Inductively it is now easy to show that

$$\frac{1}{q_2} = 1 + 1 + 2 + 6 + \cdots + (n-3)! + \frac{(n-2)!}{q_n}$$

for each $n = 3, 4, \ldots, 99$. Thus

$$\frac{(n-2)!}{q_n} = \frac{(n-2)! + (n-1)! + \cdots + 96! + 97!}{q_{99}}$$

$$= (n-2)! + (n-1)! + \cdots + 96! + 99 \cdot 97!$$

Thus

$$q_n = \frac{(n-2)!}{(n-2)! + (n-1)! + \cdots + 97! + 98!}$$

for each $n = 2, 3, \ldots, 99$.

It is not difficult to check that $q_2 < q_3 < \cdots < q_9 < \frac{q_9}{10 \cdot 98!} < q_{10} < \cdots < q_{99}$.

4742. Proposed by Michel Bataille.

Let $m, n$ be non-negative integers. Prove that

$$\sum_{k=0}^{n} \binom{2n+2m}{k+2m} \binom{2n-k}{k} 4^{-k} = \binom{4n+4m}{2n} 4^{-n}.$$ 

We received 7 solutions, all of which were correct. We present the solution by Ivan Hadinata.

We approach the problem by a finite generating function. In other words, we will compare the corresponding coefficients of two expansions of the same polynomial.
In the expansion of $$(1 + x)^{4m+4n}$$, the coefficient of $$x^{2n}$$ is $$\binom{4m+4n}{2n}$$. 
Another way of writing the polynomial is 

$$(1 + x)^{4m+4n} = (1 + x^2 + 2x)^2m+2n = \sum_{a+b+c=2m+2n} \frac{(2m+2n)!}{a!b!c!} \cdot 1^a(x^2)^b(2x^c) = \sum_{a+b+c=2m+2n} \frac{(2m+2n)!}{a!b!c!} \cdot 2^c x^{2b+c}. $$

The coefficient of $$x^{2n}$$ in this expansion is 

$$\sum_{2b+c=2n} \frac{(2m+2n)!}{(2m+2n - b - c)!b!c!} \cdot 2^c = \sum_{b=0}^{n} \frac{(2m+2n)!}{(2m+b)!(2n-2b)!} \cdot 2^{2n-2b} \cdot 2^c = \sum_{k=0}^{n} \frac{(2m+2n)!}{(2m+k)!k!(2n-2k)!} \cdot 4^{n-k} = \sum_{k=0}^{n} \binom{2n+2m}{k+2m} \binom{2n-k}{k} \cdot 4^{n-k}.$$ 

Thus, we find that 

$$\sum_{k=0}^{n} \binom{2n+2m}{k+2m} \binom{2n-k}{k} \cdot 4^{n-k} = \binom{4m+4n}{2n},$$

which is equivalent to the stated proposition.

4743. Proposed by Cristian Chiser.

Let $$ABC$$ be an acute triangle. Suppose that $$D, E, F$$ are points on sides $$BC, CA$$ and $$AB$$, respectively, such that $$FD$$ is perpendicular to $$BC$$, $$DE$$ is perpendicular to $$CA$$, and $$EF$$ is perpendicular to $$AB$$. Let $$a, b, c$$ be the side lengths of the triangle $$ABC$$ and let $$R_{DEF}$$ be the circumradius of triangle $$DEF$$. Show that

$$R_{DEF} = \frac{a+b+c}{9}$$

if and only if triangle $$ABC$$ is equilateral.

We received 13 solutions, all correct. The following solution by Theo Koupelis is one of the shortest proofs.

We have $$\angle FDE = \angle FDC - \angle CDE = 90^\circ - (90^\circ - \angle C) = \angle C$$, and similarly $$\angle DEF = \angle A$$, and $$\angle EFD = \angle B$$. Thus, $$\triangle ABC$$ and $$\triangle DEF$$ are similar and

$$\frac{FE}{c} = \frac{R_{DEF}}{R_{ABC}}.$$
But
\[
a = BD + DC = \frac{FD}{\tan B} + \frac{DE}{\sin C},
\]
\[
b = CE + EA = \frac{DE}{\tan C} + \frac{FE}{\sin A},
\]
\[
c = AF + FB = \frac{FE}{\tan A} + \frac{FD}{\sin B}.
\]

Solving the above system in terms of the sides of triangle \(DEF\) we get
\[
FE \cdot (1 + \cos A \cos B \cos C) = \sin A \cdot (b + c \cdot \cos B \cos C - a \cdot \cos C),
\]
with similar expressions for \(DF\) and \(DE\). Substituting the well-known expressions for \(\cos A, \cos B, \cos C\) in terms of \(a, b, c\), we get
\[
1 + \cos A \cos B \cos C = \frac{(a^2 + b^2 + c^2) \cdot 16E^2}{8a^2b^2c^2},
\]
and
\[
b + c \cdot \cos B \cos C - a \cdot \cos C = \frac{4E^2}{a^2b},
\]
where \(E\) is the area of \(\triangle ABC\). Therefore, using \(a = 2R_{ABC} \cdot \sin A\), we get
\[
FE = \frac{abc^2}{R_{ABC} \cdot (a^2 + b^2 + c^2)},
\]
and thus
\[
R_{DEF} = \frac{abc}{a^2 + b^2 + c^2} \leq \frac{a + b + c}{9},
\]
because from AM-GM we get \((a^2 + b^2 + c^2)(a + b + c) \geq 9abc\), with equality when \(a = b = c\). Therefore, if \(R_{DEF} = \frac{a+b+c}{9}\) then \(a = b = c\) and \(\triangle ABC\) is equilateral; on the other hand, if the triangle is equilateral, then \(R_{DEF} = \frac{a^3}{3a^2} = \frac{a}{3} = \frac{a+b+c}{9}\).
4744. Proposed by Olimjon Jalilov.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that the equation

$$\tan^2 x \cdot f''(x) + (\tan^3 x + 4 \tan x)f'(x) + 2f(x) = 0, \quad x \neq \pm \frac{\pi}{2}$$

has at least one root in the open interval $(-\pi, \pi)$.

We received 6 correct solutions, indicating that the problem admits the trivial solution $x = 0$. Two answers even show that for certain choice of the function $f(x)$ ($f(x) = x$ for example), $x = 0$ is the only solution.

4745. Proposed by Marius Stănean.

Let $a, b, c$ be nonnegative real numbers such that $ab + bc + ca = 4$. Prove that

$$\left( a^2 + b^2 + c^2 + 1 \right) \left( \frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \right) \geq \frac{45}{8}.$$ 

When does equality hold?

14 solutions were submitted from 13 individuals; 6 of them were correct, 4 did not identify all the conditions for equality, and the remainder were incorrect. We present 2 solutions.

Solution 1, by Mohamed Amine Ben Ajiba.

Let $p = a + b + c, q = ab + bc + ca = 4, r = abc, a^2 + b^2 + c^2 = p^2 - 2q = p^2 - 8$ and $a^2b^2 + b^2c^2 + c^2a^2 = q^2 - 2pr = 16 - 2pr$.

Since $(a + b + c)^2 \geq 3(ab + bc + ca)$, then $p^2 \geq 3q = 12$.

By the Schur Inequality

$$0 \leq a(a - b)(a - c) + b(b - a)(c - a) + c(c - a)(c - b)$$

$$= (a + b + c)^3 - 4(a + b + c)(ab + bc + ca) + 9abc$$

$$= p^3 - 4pq + 9r = p^3 - 16p + 9r.$$ 

The desired inequality is successively equivalent to

$$\left( a^2 + b^2 + c^2 \right) \left( \frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \right) \geq \frac{45(a^2 + b^2 + c^2)}{8(a^2 + b^2 + c^2 + 1)}$$

$$\frac{c^2}{a^2 + b^2} + \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + 3 \geq \frac{45(a^2 + b^2 + c^2)}{8(a^2 + b^2 + c^2 + 1)}$$

$$\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{21(a^2 + b^2 + c^2) - 24}{8(a^2 + b^2 + c^2 + 1)}.$$ 

Copyright © Canadian Mathematical Society, 2022
Applying the Cauchy-Schwarz inequality to the vectors
\[ (a(b^2 + c^2)^{-1/2}, b(c^2 + a^2)^{-1/2}, c(a^2 + b^2)^{-1/2}) \]
and
\[ (a(b^2 + c^2)^{1/2}, b(c^2 + a^2)^{1/2}, c(a^2 + b^2)^{1/2}) , \]
we note that
\[ \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \geq \frac{(a^2 + b^2 + c^2)^2}{2(a^2 b^2 + b^2 c^2 + c^2 a^2)} , \]
so that it suffices to prove that
\[ \frac{(a^2 + b^2 + c^2)^2}{a^2 b^2 + b^2 c^2 + c^2 a^2} \geq \frac{21(a^2 + b^2 + c^2) - 24}{4(a^2 + b^2 + c^2 + 1)} , \]
which is equivalent to
\[ 2(p^2 - 8)(p^2 - 7) - (21p^2 - 192)(8 - pr) \geq 0 , \]
or
\[ 2p^6 - 46p^4 + 184p^2 + 640 + 3pr(7p^2 - 64) \geq 0 . \]
Observe that, since \( p^2 \geq 12 \), \( 7p^2 - 64 > 0 \).

When \( p^2 \geq 16 \), the left side of the foregoing equation exceeds
\[ 2p^6 - 46p^4 + 184p^2 + 640 \geq 2(p^2 - 16)(p^2 + 9) + 124 \geq 0 . \]

It remains to deal with the case that \( 12 \leq p^2 \leq 16 \). In this case
\[
2p^6 - 46p^4 + 184p^2 + 640 + 3pr(7p^2 - 64) \\
\geq 2p^6 - 46p^4 + 184p^2 + 640 + \frac{p^2(16 - p^2)(7p^2 - 64)}{3} \\
= (16 - p^2)(p^2 - 12)(p^2 - 10) \geq 0 .
\]

Equality occurs if and only if \( a = b = c = \frac{2}{3} \sqrt{3} \) or \( \{a, b, c\} = \{2, 2, 0\} \).

Solution 2, by Theo Koupelis.

Let \( s = a^2 + b^2 + c^2 = p^2 - 8 \). The second factor of the left side is equal to
\[
\frac{1}{s - c^2} + \frac{1}{s - b^2} + \frac{1}{s - a^2} = \frac{3s^2 - 2s^2 + t}{s^3 - s^3 + st - t^2} \\
= \frac{(p^2 - 8)^2 + (16 - 2pr)}{(p^2 - 8)(16 - 2pr) - t^2}.
\]

*Crux Mathematicorum*, Vol. 48(10), December 2022
The proposed inequality is equivalent to $F \geq 0$, where
\[
F = 8(p^2 - 7)[(p^2 - 8)^2 + (16 - 2pr)] - 45[(p^2 - 8)(16 - 2pr) - r^2] \\
= 8p^6 - 184p^2 + 816p^2 + 1280 + 74p^3r - 608pr + 45r^2 \\
= 8(p^2 - 16)(p^4 - 7p^2 - 10) + pr(74p^2 - 608) + 45r^2.
\]
Since $p^2 \geq 12$, then $p^4 - 7p^2 - 10 = p^2(p^2 - 12) + 5(p^2 - 2)$ and $74p^2 - 608$ are both positive.

$F \geq 0$ is clearly true when $p^2 \geq 16$. Since $p^2 \geq 12$, we need to deal only with the situation that $12 \leq p^2 \leq 16$.

The numbers $a$, $b$, $c$ are roots of the cubic polynomial $f(x) = x^3 - px^2 + 4x - r$. The roots of the derivative $f'(x) = 3x^2 - 2px + 4$ are given by $\frac{1}{3}(p \pm u)$, where $u = \sqrt{p^2 - 12}$. Since $f(\frac{1}{3}(p + u)) \leq 0$, we find that
\[
r \geq \frac{1}{27}(p^3 + 3p^2u + 3pu + u^3) - \frac{p}{9}(p^2 + 2pu + u^2) + \frac{4}{3}(p + u) \\
= \frac{1}{27}[-2p^3 - 3p^2u + u^3 + 36p + 36u] = \frac{1}{27}[-2p^3 - 3(p^2 - 12)u + 36p + u^3] \\
= \frac{2}{27}[-p^3 + 18p - (p^2 - 12)\sqrt{p^2 - 12}] \geq 0.
\]
(The quantity $p(18 - p^2) - (p^2 - 12)\sqrt{p^2 - 12}$ decreases from a maximum of $8\sqrt{3}/9$ to a minimum of 0 as $p^2$ increases from 12 to 16.)

It suffices to verify that $F \geq 0$ holds when $r = (2/27)[-p^3 + 18p - u^3]$. The inequality is equivalent to
\[
(p^2 - 12)(61p^4 - 444p^2 - 1440) \geq (p^2 - 12)p(101p^2 - 732)\sqrt{p^2 - 12}.
\]
Since both sides are nonnegative, we can square to get the equivalent inequality
\[
(p - 12)^2(16 - p^2)(4p^2 + 1)(5p^4 - 88p^2 + 400) \geq 0,
\]
which is true. If equality occurs we must that either $p^2 = 12$ or $p^2 = 16$. In the former case, $f'(x) = (\sqrt{3}x - 2)^2$ and $a = b = c = 2/\sqrt{3}$ and in the latter case $r = 0$ and exactly one of $a$, $b$, $c$ vanishes and the remaining two equal 2.

Comments from the editor. All the solvers required quite a bit of labour to solve this problem; two resorted to software. Several obtained the polynomial $F$ as a quadratic in $r$ to be minimized, and two evoked the pqr-lemma (see The pqr-method: part I, by Steven Chow, Howard Halim, Victor Rong Crux Math. 43:5 (May, 2017), 210-214) which allowed one to assume that two of the variables were equal. Walther Janous generalized the situation to show that under the condition $ab + bc + ca = t^2$ with $t \geq 2$, the right side of the inequality can be replaced by $9(t^2 + 1)/2t^2$. Equality occurs when $a = b = c = t/\sqrt{3}$. However, the other possible case of equality when $abc = 0$ is realized only when $t = 2$. Ivan
Hadinata and Vivek Mehra expressed \( ab, bc \) and \( ca \) as weighted parts of 4 with, for example, \( ab = 4z(x + y + z)^{-1} \) and \( ab = 2xy(xy + yz + zx)^{-1} \) respectively. The left side can be expressed in terms of \( x, y, z \) and the inequality established with the help of the Muirhead inequalities. Marie-Nicole Gras noted that \( a, b, c \) are the roots of the cubic polynomial \( x^3 - px^2 + 4x - r \), whose discriminant \( \Delta(r) = -27r^2 - (4p^3 - 72p)r + 16p^2 - 256 \) must be nonnegative. Using the fact that \( 9r \geq 16p - p^3 \), she noted that \( r \geq A/B \). Since \( r \) must be at least as great as the smaller root \( v \) of the quadratic \( \Delta(r) \), she achieved the result by showing that \( v \geq A/B \).

**4746. Proposed by George Stoica.**

Let \( A_n \) be the \( n \times n \) matrix with elements \( a_{ij} = \binom{n+1}{2i-j} \), \( i, j = 1, \ldots, n \). Prove that \( \det A_n = 2^{n(n+1)/2} \).

We received 4 solutions for this problem, all correct. We present solution by UCLan Cyprus Problem Solving Group.

We proceed by induction on \( n \), the case \( n = 1 \) being trivial.

Let \( B_{n+1} \) be the \( (n+1) \times (n+1) \) matrix with elements \( b_{ij} = \binom{n+1}{2i-j} \) for \( i, j = 1, 2, \ldots, n+1 \). Note that the last row of \( B_{n+1} \) has entries \((0, 0, \ldots, 0, 1)\). Deleting the last row and column of \( B_{n+1} \) we get \( A_n \) and therefore \( \det(B_{n+1}) = \det(A_n) \).

Now let \( C_{n+1} \) be the \( (n+1) \times (n+1) \) matrix with elements

\[
c_{ij} = \begin{cases} 
1 & j = i \in \{1, 2, \ldots, n\} \\
1 & j = i-1 \in \{1, 2, \ldots, n\} \\
(-1)^{n+1-i} \binom{n+1}{i-1} & i \in \{1, 2, \ldots, n\}, j = n+1, \\
n+2 & i = j = n+1 \\
0 & \text{otherwise}
\end{cases}
\]

For example

\[
C_3 = \begin{pmatrix} 
1 & 0 & 1 \\
1 & 1 & -3 \\
0 & 1 & 4 
\end{pmatrix}.
\]

We claim that \( A_{n+1} = B_{n+1}C_{n+1} \). Indeed let us compute that \((i, j)\)-entry of \( B_{n+1}C_{n+1} \).

- For \( j \neq n+1 \), since the only non-zero entries of the \( j \)-th column of \( C_{n+1} \) are \( c_{jj} = c_{j+1,j} = 1 \), then the \((i, j)\)-entry of the product is equal to

\[
b_{ij} + b_{i,j+1} = \binom{n+1}{2i-j} + \binom{n+1}{2i-j-1} = \binom{n+2}{i-j}
\]

which is indeed the \((i, j)\)-entry of \( A_{n+1} \).
• For $j \neq n + 1$, the $(i,j)$-entry of the product is equal to

$$\sum_{k=1}^{n} (-1)^{n+1-k} \binom{n+1}{2i-k} \binom{n+1}{k-1} + (n+2) \binom{n+1}{2i-(n+1)}$$

Note that the coefficient of $x^{2i-1}$ of $(1-x)^{n+1}(1+x)^{n+1} = (1-x^2)^{n+1}$ is equal to 0. But it is also equal to

$$\sum_{r=0}^{n+1} \binom{n+1}{r} \binom{n+1}{2i-1-r} (-1)^r$$

$$= \sum_{k=1}^{n+2} \binom{n+1}{k-1} \binom{n+1}{2i-k} (-1)^{k-1}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{2i-k} (-1)^{k-1} + (n+1) \binom{n+1}{2i-(n+1)} (-1)^n + \binom{n+1}{2i-(n+2)} (-1)^{n+1}$$

Therefore

$$\sum_{k=1}^{n} (-1)^{n+1-k} \binom{n+1}{2i-k} \binom{n+1}{k-1} = \binom{n+1}{2i-(n+2)} - (n+1) \binom{n+1}{2i-(n+1)}$$

and

$$\sum_{k=1}^{n} (-1)^{n+1-k} \binom{n+1}{2i-k} \binom{n+1}{k-1} + (n+2) \binom{n+1}{2i-(n+1)}$$

$$= \binom{n+1}{2i-(n+2)} + \binom{n+1}{2i-(n+1)} = \binom{n+2}{2i-(n+1)}.$$

So indeed $A_{n+1} = B_{n+1} C_{n+1}$ and therefore by the inductive hypothesis we have

$$\det(A_{n+1}) = \det(B_{n+1}) \det(C_{n+1}) = \det(A_n) \det(C_{n+1}) = 2^{n(n+1)/2} \det(C_{n+1}).$$

Thus it is enough to show that $\det(C_{n+1}) = 2^{n+1}$.

We take cofactor expansion along the last column of $C_{n+1}$. The $(i,n+1)$-minor of $C_{n+1}$ is the determinant of a matrix of the form

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

where $X$ is an $(i-1) \times (i-1)$ lower triangular matrix with ones down the diagonal, and $Y$ is an $(n+1-i) \times (n+1-i)$ upper triangular matrix with ones down the diagonal. So every minor is equal to 1 and therefore

$$\det(C_{n+1}) = \sum_{i=1}^{n} (-1)^{n+1-i} \binom{n+1}{i-1} \binom{n+1}{n} + (n+2)$$

$$= 2^{n+1} - \binom{n+1}{n} - \binom{n+1}{n+1} - (n+2) = 2^{n+1}.$$
as required.

Note from the editor. The same problem with the author’s solution appeared recently in The Playground, Math Horizons, 2022, 30:1, 30-33. We strongly ask the proposers to not submit problems to multiple sources at the same time. Repetition wastes time and effort on behalf of the editorial boards who consider the same problems and duplicate resources result in poorer experience for the readers.

4747. Proposed by Stanescu Florin.

Determine all the functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
f(x^2f(x) + f(y)) = f(f(x^3)) + y
\]

for all \( x, y \in \mathbb{R} \).

We received 12 submissions and 7 of them were complete and correct. We present 2 solutions, beginning with the common parts of the 2 solutions.

Setting \( x = 0 \) in the given equation, we get

\[
f(f(y)) = f(f(0)) + y
\]  

Note that equation (1) implies that \( f \circ f \) is surjective and \( f \circ f \) is injective. Therefore, \( f \) must be bijective.

Setting \( y = 0 \) in the given equation, we get

\[
f(x^2f(x) + f(0)) = f(f(x^3)).
\]

Since \( f \) is bijective, this reduces to \( x^2f(x) + f(0) = f(x^3) \), which implies that \( f(0) = 0 \) by setting \( x = 1 \). Then we must have

\[
x^2f(x) = f(x^3).
\]  

Moreover, since \( f(0) = 0 \), equation (1) reduces to

\[
f(f(y)) = y,
\]  

that is, \( f \) is an involution. Combining equation (2), equation (3), and the given equation, we obtain \( f(f(x^3) + f(y)) = x^3 + y \). Applying equation (3) again, we obtain \( f(x^3) + f(y) = x^3 + y \). This allows us to deduce that \( f \) satisfies the Cauchy equation

\[
f(x + y) = f(x) + f(y).
\]  

We have decomposed the given equation into equation (2), equation (3), and equation (4). It is easy to verify that \( f(x) = x \) and \( f(x) = -x \) are two solutions; to show that there are no other solutions, it suffices to show \( f(x) = xf(1) \) given equation (3). Next we present two different ways to show \( f(x) = xf(1) \) given equation (3). Both solutions rely on careful manipulations of equation (2).
Solution 1, by Kazuki Hamada and Mohamed Aminebit Ben Ajiba (done independently).

From equation (4), it is easy to deduce that \( f(qx) = qf(x) \) holds for all \( q \in \mathbb{Q}, x \in \mathbb{R} \). Using equation (2) and equation (4), we have:

\[
f((x + q)^3) = f(x^3) + 3q^2f(x) + 3qf(x^2) + q^3f(1),
\]

\[
(x + q)^2f(x + q) = (x^2 + 2qx + q^2)f(x) + qx^2f(1) + 2qx^2f(1) + q^3f(1).
\]

For each fixed \( x \), we treat the right-hand side of the above two equations as polynomials in \( q \). Given equation (2), these two polynomials are identical, which implies the quadratic coefficient of the two polynomials match with each other, that is, \( 2xf(1) + f(x) = 3f(x) \). It follows that \( f(x) = xf(1) \).

Solution 2, by the proposer.

We compute \( f((1 + x)^3 - (1 - x)^3) \) in two ways. On the one hand, by equation (2) and equation (4),

\[
f((1 + x)^3 - (1 - x)^3) = f(6x + 2x^3) = 6f(x) + 2f(x^3) = 6f(x) + 2x^2f(x).
\]

On the other hand,

\[
f((1 + x)^3 - (1 - x)^3) = (1 + x)^2f(1 + x) - (1 - x)^2f(1 - x) = 4xf(1) + (2 + 2x^2)f(x).
\]

Comparing the above two equations, we obtain \( f(x) = xf(1) \).

4748. Proposed by Mihaela Berindeanu.

Let \( \Gamma \) be the circumcircle of a given triangle \( ABC \), and define \( T \) to be the intersection of the tangents to \( \Gamma \) at \( B \) and at \( C \), \( S \) to be the second point where \( \Gamma \) intersects \( AT \), and \( \hat{A} \) to be the reflection of the vertex \( A \) in the line \( BC \). Prove that the line \( \hat{A}S \) intersects the side \( BC \) at its midpoint.

We received 9 solutions, all of which were correct. We present the solution by UCLan Cyprus Problem Solving Group.

It is well-known that \( T \) belongs on the \( A \)-symmedian of triangle \( ABC \). Hence so does \( S \).

We write \( A' \) instead of \( \hat{A} \). It is enough to show that the reflection of line \( A'S \) is the median of the triangle \( ABC \) through \( A \). Equivalently, letting \( M \) be the midpoint of \( BC \), it is enough to show that the reflection of \( S \) on \( BC \) belongs on \( AM \).

Since \( AS \) is the \( A \)-symmedian (with \( S \) on the circumcircle) then the triangles \( ABS \) and \( AMC \) are similar. We also have that \( AS \) is the \( S \)-symmedian of triangle \( SBC \) and therefore the triangles \( ABS \) and \( SMC \) are also similar.

Thus \( \angle AMC = \angle ABS = \angle SMC \). It now follows that the reflection of \( AS \) is on \( AM \) as required.

For an arbitrary, monotonically increasing, strictly concave, and twice differentiable function \( f(x) \) of a continuous variable \( x \), defined for \( h \leq x \leq k \), prove that

\[
\frac{1}{b-a} \int_{x=a}^{b} f(x) \, dx > \frac{1}{k-h} \int_{x=h}^{k} f(x) \, dx,
\]

if \( a + b > h + k \),

where \( h < a < b < k \) are arbitrary constants.

We received 4 submissions each of which was correct and complete. We present the solution submitted by the UCLan Cyprus Problem Solving Group, modified by the editor.

Define \( F : [0, k-b] \to \mathbb{R} \) by

\[
F(x) = \frac{1}{b-a+2x} \int_{a-x}^{b+x} f(t) \, dt - \frac{1}{k-h} \int_{h}^{k} f(t) \, dt.
\]

Note that this is well-defined since for any \( x \in [0, k-b] \) we have \( h < a - x < b + x \leq k \). Now \( F \) is continuous on \([0, k-b]\) and differentiable on \((0, k-b)\) with

\[
F'(x) = \frac{f(b+x) + f(a-x)}{b-a+2x} - \frac{2}{(b-a+2x)^2} \int_{a-x}^{b+x} f(t) \, dt.
\]

Let \( g \) be the linear function with \( g(a-x) = f(a-x) \) and \( g(b+x) = f(b+x) \). Since \( f \) is strictly concave in \([a-x, b+x]\), then \( f(t) \geq g(t) \) for every \( t \in [a-x, b+x] \) with equality only at the endpoints. It follows that

\[
\int_{a-x}^{b+x} f(t) \, dt > \int_{a-x}^{b+x} g(t) \, dt = (b-a+2x) \cdot \frac{f(b+x) + f(a-x)}{2}
\]

Crux Mathematicorum, Vol. 48(10), December 2022
since the second integral is equal to the area of a trapezium of width \( b - a + 2x \) and parallel sides of heights \( f(a - x) \) and \( f(b + x) \) respectively. It follows that \( F'(x) < 0 \) and therefore \( F(0) > F(k - b) \). We need to show \( F(0) > 0 \), so it is enough to show that \( F(k - b) \geq 0 \) i.e.

\[
\frac{1}{k - h'} \int_{h'}^{k} f(t) \, dt \geq \frac{1}{k - h} \int_{h}^{k} f(t) \, dt
\]

where \( h' = a + b - k > h \).

Defining \( G : [h, k) \to \mathbb{R} \) by

\[
G(x) = \frac{1}{k - x} \int_{x}^{k} f(t) \, dt
\]

it is enough to show that \( G \) is an increasing function. Now \( G \) is continuous on \([h, k)\) with

\[
G'(x) = -\frac{f(x)}{k - x} + \frac{1}{(k - x)^2} \int_{x}^{k} f(t) \, dt.
\]

But since \( f \) is increasing, then

\[
\int_{x}^{k} f(t) \, dt \geq \int_{x}^{k} f(x) \, dt = (k - x)f(x)
\]

It follows that \( G'(x) \geq 0 \) and so \( G \) is increasing as required.

4750. **Proposed by Nguyen Viet Hung.**

Given a triangle \( ABC \) with orthocentre \( H \), let \( M \) be any point inside the triangle and let \( D, E, F \) be respectively projections of \( M \) onto the sides \( BC, CA, AB \). Denote by \( S, S_a, S_b, S_c \) the areas of triangles \( ABC, HBC, HCA, HAB \), respectively. Prove that

\[
(S_b + S_c)\overrightarrow{MD} + (S_c + S_a)\overrightarrow{ME} + (S_a + S_b)\overrightarrow{MF} = S \cdot \overrightarrow{MH}.
\]

All eight submissions were correct; we feature the solution by the UCLan Cyprus Problem Solving Group, with some details expanded on by the editor.

By insisting that the arbitrary point \( M \) be inside the given triangle and its projections lie on the sides, the proposer’s result applies only when the triangle is acute. But we shall see that there is no need for such restrictions as long as we allow signed areas; specifically, \( M \) can be any point in the plane of an arbitrary triangle \( ABC \) and the points \( D, E, F \) its projections on the lines \( BC, CA, AB \) when the value of any of the areas \( S_a, S_b, S_c \) is taken to be negative according as the corresponding triangle \( HBC, HCA, HAB \) has an orientation opposite to that of \( \triangle ABC \).

First, we prove the result when \( ABC \) is a right triangle, say with \( \hat{A} = 90^\circ \). Then \( H = A \) so that \( S_a = S \) and \( S_b = S_c = 0 \); thus we are required to prove that
\[ \overrightarrow{ME} + \overrightarrow{MF} = \overrightarrow{MH}. \] This equality holds because \( MEAF \) is a rectangle (and in particular a parallelogram.) Indeed, we then have \( \overrightarrow{ME} + \overrightarrow{MF} = \overrightarrow{FA} + \overrightarrow{MF} = \overrightarrow{MA} = \overrightarrow{MH}. \)

Now assume that \( ABC \) is not a right triangle, and let \((x, y, z)\) be the barycentric coordinates of \( M \) normalized so that \( x + y + z = 1 \). We first see that the vector

\[ \mathbf{v} = (S_a + S_c)\overrightarrow{MD} + (S_c + S_b)\overrightarrow{ME} + (S_a + S_b)\overrightarrow{MF} - S \cdot \overrightarrow{MH} \]

has coordinates

\((a_1x + a_2y + a_3z + a_4, b_1x + b_2y + b_3z + b_4, c_1x + c_2y + c_3z + c_4)\)

for coefficients \(a_1, \ldots, c_4\) that depend only on the triangle and not on the point \( M\). Indeed, if \( H = (h_1, h_2, h_3) \), then \( \overrightarrow{AH} = (h_1 - 1, h_2, h_3) \) and since \( \overrightarrow{MD} \) is parallel to \( \overrightarrow{AH} \) with first coordinate \(-x\), then \( \overrightarrow{MD} = x(-1, \frac{b_2}{1-h_1}, \frac{b_3}{1-h_1}) \). (This is well-defined as \( ABC \) is not right-angled and therefore \( h_1 \neq 1 \)) We have similar expressions for \( \overrightarrow{ME} \) and \( \overrightarrow{MF} \), while \( \overrightarrow{MH} = (h_1 - x, h_2 - y, h_3 - z) \), and our claim is justified.

We next claim that it suffices to prove the result for three special cases, namely \( M = A, B, C \). Indeed, if it is true for \( M = A = (1, 0, 0) \) then we get \((a_1 + a_d, b_1 + b_4, c_1 + c_4) = 0 \) and so \( a_1 = -a_4, b_1 = -b_4, c_1 = -c_4 \). With the analogous results when \( M = B \) and \( M = C \) we see that the vector \( \mathbf{v} \) satisfies

\[ \mathbf{v} = (a_1(x + y + z - 1), b_1(x + y + z - 1), c_1(x + y + z - 1)) = \mathbf{0}, \]

as \( x + y + z = 1 \). The desired result then follows immediately.

Because these three cases are essentially the same, it suffices to prove that the result holds when \( M = A \). We then have \( E = F = A \) so we are required to prove that

\[ (S_a + S_c)\overrightarrow{AD} = S \cdot \overrightarrow{AH}. \]

To see this, note that \( H = \frac{1}{S}(S_a, S_b, S_c) \); therefore \( \overrightarrow{AH} = \frac{1}{S}(S_a - S, S_b, S_c) \) and

\[ S \cdot \overrightarrow{AH} = (S_a - S, S_b, S_c) = (-S_b - S_c, S_b, S_c) \]

\[ = (S_b + S_c)(-1, \frac{S_b}{S_b + S_c}, \frac{S_c}{S_b + S_c}) = (S_b + S_c)\overrightarrow{AD}, \]

where the last equality follows from the fact that \( \overrightarrow{AD} \) is parallel to \( \overrightarrow{AH} \) with first coordinate \( 0 - 1 = -1 \). The argument works analogously when \( M = B \) and \( M = C \), and the proof is complete.

Editor's comments. Tran Quang Hung observed that the result is really a theorem of affine geometry, and holds more generally with \( H \) replaced by an arbitrary fixed point \( P \): specifically,

Given two points \( P \) and \( M \) in the plane of an arbitrary triangle \( ABC \), let \( D, E, F \) be points on the lines \( BC, CA, AB \) for which \( MD, ME, MF \) are parallel to the respective cevians \( AP, BP, CP \). Denote by \( S, S_a, S_b, S_c \) the signed areas of triangles

\[ Crux Mathematicorum, \text{Vol. 48}(10), \text{December 2022} \]
ABC, PBC, PCA, PAB, respectively. Then
\[
(S_b + S_c)\overrightarrow{MD} + (S_c + S_a)\overrightarrow{ME} + (S_a + S_b)\overrightarrow{MF} = S \cdot \overrightarrow{MP}.
\]

He used this result in a lecture for the math team of the High School for Gifted Students of Vietnam National University in 2012. A version of his problem extended to a simplex in higher dimensions was accepted last year for the problem section of The American Mathematical Monthly.