

Canadian Open Mathematics Challenge 2022

Official Solutions



Canadian Open **Mathematics** Challenge

Canada's most prestigious math competition.



Canadian Mathematical Society

The COMC has three sections:

- A. Short answer questions worth 4 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.
- B. Short answer questions worth 6 marks each. A correct answer receives full marks. Partial marks may be awarded for work shown if a correct answer is not provided.
- C. Multi-part full solution questions worth 10 marks each. Solutions must be complete and clearly presented to receive full marks.

COMC exams from other years, with or without the solutions included, are free to download online. Please visit <https://comc.math.ca/2022/getting-ready/>

Part A

A1. John had a box of candies. On the first day he ate exactly half of the candies and gave one to his little sister. On the second day he ate exactly half of the remaining candies and gave one to his little sister. On the third day he ate exactly half of the remaining candies and gave one to his little sister, at which point no candies remained. How many candies were in the box at the start?

Solution 1: We work backwards. At the start of the third day, John had 2 candies. At the start of the second day, he had $2 \times (2 + 1) = 6$ candies. At the start of the first day, he had $2 \times (6 + 1) = 14$ candies, so there were 14 candies in the box at the start.

Solution 2: Assume that there were x candies in the box at the start. After the first day, there are $\frac{x}{2} - 1$ candies. After the second day, there are

$$\frac{\frac{x}{2} - 1}{2} - 1$$

candies, and after the last day, there are

$$\frac{\frac{\frac{x}{2} - 1}{2} - 1}{2} - 1 = 0$$

candies. Expanding the left-hand side, we see that this is equivalent to

$$\frac{x}{8} - \frac{7}{4} = 0,$$

so we have $x = 8 \times \frac{7}{4} = 14$.

Answer: 14.

A2. A palindrome is a whole number whose digits are the same when read from left to right as from right to left. For example, 565 and 7887 are palindromes. Find the smallest six-digit palindrome divisible by 12.

Solution: We need a number in the form \overline{abcba} that is divisible by both 3 and 4. Thus, we require that the sum of digits $2a + 2b + 2c$ be divisible by 3, and that the number formed by the last two digits, $10b + a$ be divisible by 4. Due to the second condition, it is impossible to have $a = 1$, so the smallest number must have $a \geq 2$. If $a = 2$, then we cannot have $b = 0$, again by the second condition, so we must have $b \geq 1$. The number 210012 satisfies both conditions, so it is the smallest six-digit palindrome divisible by 12.

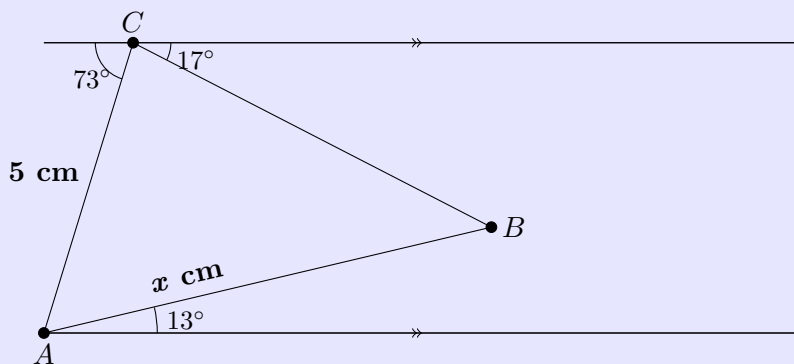
Answer: 210012.

- A3.** Initially, there are four red balls, seven green balls, eight blue balls, ten white balls, and eleven black balls on a table. Every minute, we may repaint one of the balls into any of the other four colours. What is the minimum number of minutes after which the number of balls of each of the five colours is the same?

Solution: We require there to be eight balls of each colour at the end. Thus, at least two white balls and three black balls must be painted over; they can be painted into four red balls and one green ball, for a total of five repaintings in five minutes.

Answer: $\boxed{5}$.

- A4.** In the diagram, triangle ABC lies between two parallel lines as shown. If segment AC has length 5 cm, what is the length (in cm) of segment AB ?



Solution: By the Supplementary Angle Theorem, the triangle has $\angle C = 180^\circ - (73^\circ + 17^\circ) = 90^\circ$. By the Alternating Angle Theorem, the triangle has $\angle A = 73^\circ - 13^\circ = 60^\circ$. The remaining internal angle of the triangle is $\angle B = 180^\circ - \angle A - \angle C = 30^\circ$.

In the right triangle with angles 30° - 60° - 90° , the hypotenuse is twice the side opposite to the 30° angle. Thus, the missing side has measure 10 cm.

Answer: $\boxed{x = 10}$.

Part B

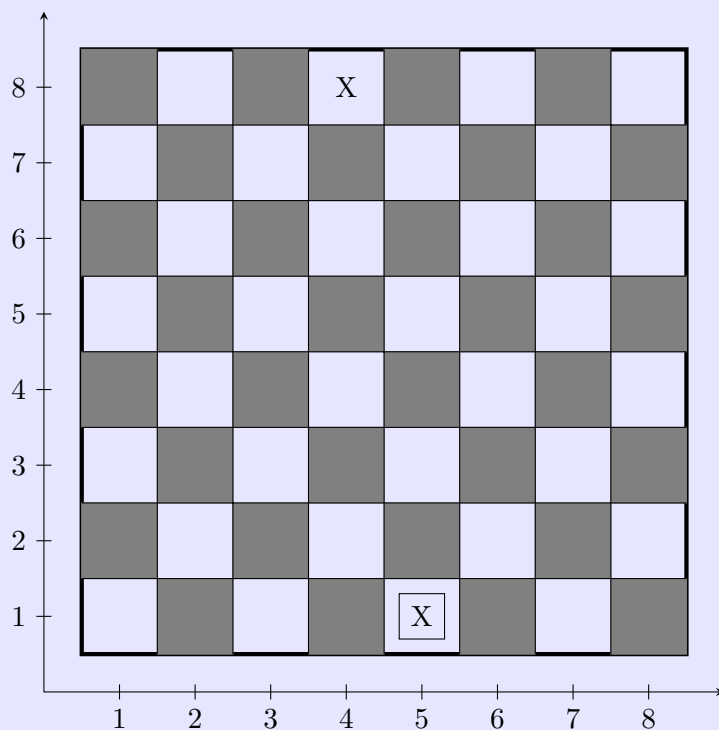
B1. The floor function of any real number a is the integer number denoted by $\lfloor a \rfloor$ such that $\lfloor a \rfloor \leq a$ and $\lfloor a \rfloor > a - 1$. For example, $\lfloor 5 \rfloor = 5$, $\lfloor \pi \rfloor = 3$ and $\lfloor -1.5 \rfloor = -2$. Find the difference between the largest integer solution of the equation $\lfloor x/3 \rfloor = 102$ and the smallest integer solution of the equation $\lfloor x/3 \rfloor = -102$.

Solution: For $\lfloor x/3 \rfloor = 102$, we need $103 > x/3 \geq 102$, so $309 > x \geq 306$. Thus, x can be 306, 307, or 308. The largest is 308.

For $\lfloor x/3 \rfloor = -102$, we need $-101 > x/3 \geq -102$, so $-303 > x \geq -306$. Thus, x can be -306 , -305 , or -304 . The smallest is -306 .

Answer: $308 - (-306) = \boxed{614}$.

B2. A stone general is a chess piece that moves one square diagonally upward on each move; that is, it may move from the coordinate (a, b) to either of the coordinates $(a - 1, b + 1)$ or $(a + 1, b + 1)$. How many ways are there for a stone general to move from $(5, 1)$ to $(4, 8)$ in seven moves on a standard 8 by 8 chessboard?



Solution 1: Since we are constrained to $8 - 1 = 7$ moves, and want to move from $a = 5$ to $a = 4$, we need to move northwest 4 times and northeast 3 times. Notice that we do not fall off the grid in any sequence of 4 northwest and 3 northeast moves because there are 4 cells to the left of the initial position and there are 3 cells to the right of it.

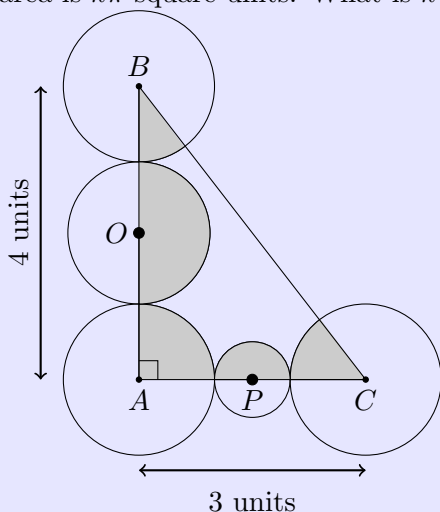
Thus, every sequence of 4 northwest and 3 northeast moves works, and the answer is $\binom{7}{3} = \binom{7}{4} = \boxed{35}$.

Solution 2: For each square (i, j) , let $P(i, j)$ be the number of ways to get to (i, j) from $(5, 1)$ using $j - 1$ moves. Then we have $P(i, j) = P(i - 1, j - 1) + P(i + 1, j - 1)$. Starting with $P(5, 1) = 1$ and zeroes in all other cells in the first row, we can compute $P(4, 8) = 35$, as shown below.

	20		35		34		14
5		15		20		14	
	5		10		10		4
1		4		6		4	
	1		3		3		1
		1		2		1	
			1		1		
0	0	0	0	1	0	0	0

Answer: $x = \boxed{35}$.

B3. In the diagram below, right triangle ABC has side lengths $AC = 3$ units, $AB = 4$ units, and $BC = 5$ units. Circles centred around the corners of the triangle all have the same radius, and the circle with centre O has area 4 times that of the circle with centre P . The shaded area is $k\pi$ square units. What is k ?



Solution: Since $A_{\odot} = \pi r_{\odot}^2$, the ratio of the areas $4 = \frac{A_{\odot}}{A_{\oplus}} = \frac{\pi r_{\odot}^2}{\pi r_{\oplus}^2}$ translates into a ratio $\frac{r_{\odot}}{r_{\oplus}} = \sqrt{4} = 2$, so $r_{\odot} = 2r_{\oplus}$. Let x be r_{\oplus} and y be the radius of the corner circles. Then $r_{\odot} = 2x$ and $y + 2x + y = 3$, $y + 4x + y = 4$. Subtracting $2x + 2y = 3$ from $4x + 2y = 4$ we have $x = 1/2$. Then $y = 1$. Thus, $r_{\odot} = 1$, and $r_{\oplus} = 1/2$.

The sum of internal angles in a triangle is π , so the shaded region is

$$A = \frac{\pi y^2}{2} + \frac{\pi r_{\odot}^2}{2} + \frac{\pi r_{\oplus}^2}{2} = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi (\frac{1}{2})^2}{2} = \frac{9}{8}\pi \implies \boxed{k = \frac{9}{8}}.$$

Answer: $\boxed{k = \frac{9}{8}}$.

B4. Determine all integers a for which $\frac{a}{1011-a}$ is an even integer.

Solution 1: Let $\frac{a}{1011-a} = 2k$ for some integer k and substitute $a = 1011 - b$. We have

$$2k = \frac{a}{1011-a} = \frac{1011-b}{b} = \frac{1011}{b} - 1.$$

We thus require that $\frac{1011}{b}$ be an odd integer. Since $1011 = 3 \times 337$, all factors of 1011 (positive and negative) are odd. We seek $b \in \{\pm 1, \pm 3, \pm 337, \pm 1011\}$ which correspond to solutions

$a \in \{1010, 1012, 1008, 1014, 674, 1348, 0, 2022\}$, respectively.

In other words, integers a for which $\frac{a}{1011-a}$ is an even integer from the set

$$\{1011 - b \text{ such that } b|1011\}.$$

Solution 2: As in the previous solution, let $\frac{a}{1011-a} = 2k$. Rearranging, we have

$$\frac{a}{1011-a} = 2k \iff a + 2ka = 2022k \iff a = \frac{2022k}{1+2k}. \quad (1)$$

Now, note that k and $1+2k$ are relatively prime, so we must have $1+2k \mid 2022$; that is, we seek odd divisors of 2022. Since $2022 = 2 \times 1011 = 2 \times 3 \times 337$, these are $1+2k = \pm 1, \pm 3, \pm 337$, or ± 1011 . Using equation (1),

$$\begin{aligned} 1+2k=1 &\implies a=0, \\ 1+2k=3 &\implies a=674, \\ 1+2k=337 &\implies a=1008, \\ 1+2k=1011 &\implies a=1010, \\ 1+2k=-1 &\implies a=2022, \\ 1+2k=-3 &\implies a=1348, \\ 1+2k=-337 &\implies a=1014, \\ 1+2k=-1011 &\implies a=1012, \end{aligned}$$

Answer: $a \in \{1010, 1012, 1008, 1014, 674, 1348, 0, 2022\}$

Part C

C1.

a. Find all integer values of a such that equation $x^2 + ax + 1 = 0$ does not have real solutions in x .

b. Find all pairs of integers (a, b) such that both equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + bx + a = 0$$

have no real solutions in x .

c. How many ordered pairs (a, b) of positive integers satisfying $a \leq 8$ and $b \leq 8$ are there, such that each of the equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + bx + a = 0$$

has two unique real solutions in x ?

Solution:

a. To get no real solutions we need $a^2 - 4 < 0$, so $|a| < 2$. Thus, $\boxed{a = 0, \pm 1}$.

Answer: $\boxed{a = 0, \pm 1}$.

b. We need $a^2 - 4b < 0$ and $b^2 - 4a < 0$. Thus $a^2 < 4b$ and $b^2 < 4a$. a and b are thus both positive, and we can multiply the above inequalities to get

$$a^2b^2 < 16ab \iff ab < 16.$$

At least one of a and b is thus less than 4; if $a < 4$, then $b^2 < 4a < 16$, so $b < 4$ as well. Similarly, if $b < 4$, then $a < 4$ as well. By direct verification we find the pairs: $\boxed{(1, 1), (2, 2), (3, 3)}$.

Answer: $\boxed{(1, 1), (2, 2), (3, 3)}$.

c. To get two unique solutions, we need $a^2 - 4b > 0$ and $b^2 - 4a > 0$.

Since $a^2 > 4b$ and $b^2 > 4a$, we have $ab > 16$ by the same logic as above. If $a > 4$, then $b^2 > 4a > 16$, so $b > 4$ as well, so we must have that both a and b are greater than 4.

If $a = 5$ then $4b < 25$ and $b^2 > 20$, so $b = 5, 6$;

If $a = 6$ then $4b < 36$ and $b^2 > 24$, so $b = 5, 6, 7, 8$;

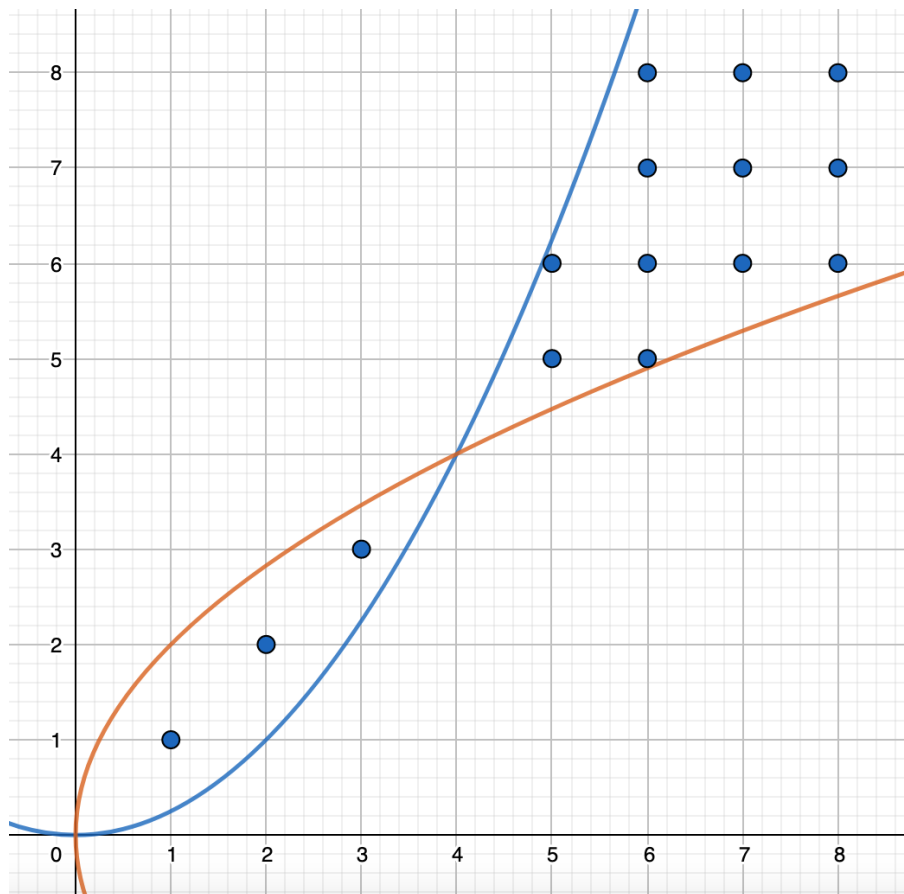
Similarly,

$a = 7, b = 6, 7, 8$;

$a = 8, b = 6, 7, 8$.

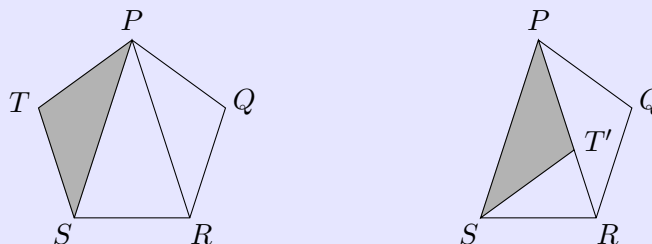
Answer: $\boxed{12}$.

A graphical solution for (b) and (c) is shown below. We plot curves $a^2 = 4b$ and $b^2 = 4a$ and identify lattice points in respective regions within $1 \leq a \leq 8$, $1 \leq b \leq 8$.



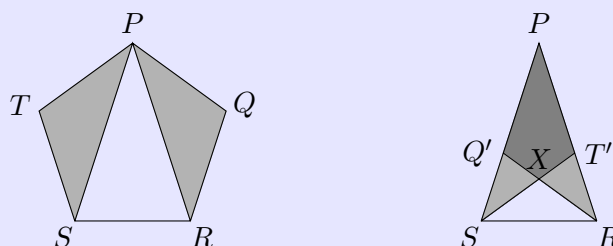
C2.

- Show that the two diagonals drawn from a vertex of a regular pentagon trisect the angle.
- Since the diagonals trisect the angle, if regular pentagon $PQRST$ is folded along the diagonal SP , the side TP will fall on the diagonal PR , as shown on the right. Here T' is the position of vertex T after the folding.



Find the ratio $\frac{PT'}{T'R}$. Express your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

- Regular pentagon $PQRST$ has an area of 1 square unit. The pentagon is folded along the diagonals SP and RP as shown on the right. Here, T' and Q' are the positions of vertices T and Q , respectively, after the foldings. The segments ST' and RQ' intersect at X .



Determine the area (in square units) of the uncovered triangle XSR . Express your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

Solution:

- The sum of the angles in a regular pentagon is $180^\circ \times (5 - 2) = 540^\circ$. Each angle in a regular pentagon is $540^\circ/5 = 108^\circ$.

Since all sides are equal, SPT and RPQ are isosceles triangles. Thus,
 $\angle SPT = (180^\circ - 108^\circ)/2 = 36^\circ$. Similarly, $\angle RPQ = (180^\circ - 108^\circ)/2 = 36^\circ$. Then,
 $\angle SPR = 108^\circ - 2 \times 36^\circ = 36^\circ$.

Thus, the diagonals SP and RP trisect the angle TPQ .

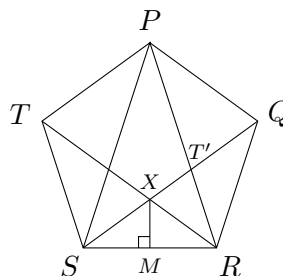
- b. Without loss of generality, assume that the pentagon has side length 1. Triangles PRS and $ST'R$ are similar isosceles triangles with angles $(36^\circ, 72^\circ, 72^\circ)$. Let the diagonal have length x . Then we have $\frac{SP}{SR} = \frac{SR}{T'R}$ and so $\frac{x}{1} = \frac{1}{x-1}$. Then $x^2 - x - 1 = 0$ and so $x = \frac{1+\sqrt{5}}{2}$, where we neglected the second root of the equation because it is negative.

The required ratio is $\frac{PT'}{T'R} = \frac{1}{x-1} = x = \frac{1+\sqrt{5}}{2}$.

Answer: $\boxed{\frac{1+\sqrt{5}}{2}}$.

We can also make the following calculation that will be useful in (c). Notice that T' lies on the diagonal SQ , and so is X . Observe that

$$SX = QT' = SQ - ST' = \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2}.$$



Let M be the midpoint of SR , so $SM = 1/2$. This implies that

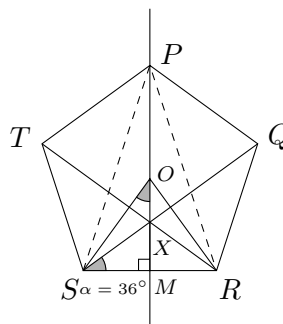
$$\cos(36^\circ) = \frac{SM}{SX} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}.$$

As well, from the right triangle SXM , we find $XM^2 = \frac{(\sqrt{5}-1)^2}{4} - \frac{1}{4} = \frac{5-2\sqrt{5}}{4}$ and so $XM = \frac{\sqrt{5-\sqrt{20}}}{2}$. Then $XM/SM = \sqrt{5-\sqrt{20}} = \tan(XSM) = \tan(36^\circ)$.

c. **Solution 1:**

Let O be the centre of the pentagon. From the right triangle SOM , we have $SM/OM = \tan(SOM)$, so $OM = SM/\tan(36^\circ)$.

The area $[PQRST] = 5[ROS] = \frac{5}{2}SR \cdot OM = 5SM \cdot OM = 5(SM)^2/\tan(36^\circ) = 1$.



Thus, $(SM)^2 = \frac{1}{5} \tan(36^\circ)$.

From the right triangle SXM (see (b)), we have $XM/SM = \tan(XSM)$, so $XM = SM \tan(36^\circ)$.

The area $[XSR] = 2[XXM] = SM \cdot XM = (SM)^2 \tan(36^\circ) = \frac{1}{5} \tan^2(36^\circ)$.

Since $\tan(36^\circ) = \sqrt{5 - \sqrt{20}}$, as found in (b), the area is $\boxed{\frac{5-2\sqrt{5}}{5}}$.

P.S. Another solution may avoid reference to $\tan(36^\circ)$ by using $\frac{XM}{SM} = \frac{SM}{OM}$.

Solution 2:

Let the side length of the pentagon be a . Then, from the isosceles 36° - 36° - 108° triangle PTS , where $PT = TS = a$, the diagonal $PS = 2a \cos(36^\circ)$. We express the area of STP and SPR in terms of a .

$$[SPT] = \frac{a^2}{2} \sin(108^\circ) = \frac{a^2}{2} \sin(72^\circ), \quad [SPR] = a^2 \cos(36^\circ) \sin(72^\circ).$$

Since $[SPR] + 2[SPT] = 1$, we have $a^2 \sin(72^\circ)(1 + \cos(36^\circ)) = 1$ and we obtain $a^2 = (\sin(72^\circ)(1 + \cos(36^\circ)))^{-1}$. We also use $\cos(36^\circ) = \frac{\sqrt{5}+1}{4}$ as found above, and so $1 + \cos(36^\circ) = \frac{\sqrt{5}+5}{4}$ and $(1 + \cos(36^\circ))^{-1} = \frac{5-\sqrt{5}}{5}$, to get

$$[SPT] = \frac{1}{2(1 + \cos(36^\circ))} = \frac{5 - \sqrt{5}}{10}, \quad [SPR] = \frac{\cos(36^\circ)}{1 + \cos(36^\circ)} = \frac{\sqrt{5}}{5}.$$

Now we notice that XTS and SPR are similar. Since $SX/SR = \frac{1-\sqrt{5}}{2}$, the coefficient of similarity for the triangles is the square of this ratio, namely $\frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$.

Finally,

$$[SXR] = [SRT] - [XTS] = \frac{5 - \sqrt{5}}{10} - \frac{3 - \sqrt{5}}{2} \times \frac{\sqrt{5}}{5} = \frac{5 - 2\sqrt{5}}{5}.$$

C3. Yana and Zahid are playing a game. Yana rolls her pair of fair six-sided dice and draws a rectangle whose length and width are the two numbers she rolled. Zahid rolls his pair of fair six-sided dice, and draws a square with side length according to the rule specified below.

- Suppose that Zahid always uses the number from the first of his two dice as the side length of his square, and ignores the second. Whose shape has the larger average area, and by how much?
- Suppose now that Zahid draws a square with the side length equal to the minimum of his two dice results. What is the probability that Yana's and Zahid's shapes will have the same area?
- Suppose once again that Zahid draws a square with the side length equal to the minimum of his two dice results. Let $D = \text{Area}_{\text{Yana}} - \text{Area}_{\text{Zahid}}$ be the difference between the area of Yana's figure and the area of Zahid's figure. Find the expected value of D .

Solution:

- a. Zahid's average area is:

$$\frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{1 + 4 + 9 + 16 + 25 + 36}{6} = \frac{91}{6}.$$

Yana's average area is:

$$\frac{(1 + 2 + 3 + 4 + 5 + 6)^2}{36} = \frac{21^2}{6^2} = \frac{7^2}{2^2} = \frac{49}{4}.$$

Therefore, Zahid's average area is larger by $\frac{182 - 147}{12} = \boxed{35/12}$ (square units).

- b. All possible outcomes for two dice are:

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}.$$

The only way for them to have the same area is if (i) Zahid's minimum value is equal to that on Yana's two rolled dice or (ii) Zahid's minimum is 2 and Yana rolls a 1 and 4, in some order.

The probability that Zahid's minimum number is 1 is $P_z(1) = 11/36$. Similarly, $P_z(2) = 9/36$, $P_z(3) = 7/36$, $P_z(4) = 5/36$, $P_z(5) = 3/36$, $P_z(6) = 1/36$.

In general, the probability that Zahid's minimum number is n is $P_z(n) = \frac{13-2n}{36}$.

The probability for Yana to roll the same specific number on both dice is $1/36$. The probability to roll either (1, 4) or (4, 1) is $2/36$.

The total probability is $\frac{1}{36} \times (P_z(1) + \cdots + P_z(6)) + \frac{2}{36} \times P_z(2) = \frac{1}{36} + \frac{1}{72} = \boxed{1/24}$.

- c. As calculated in part (a), the expected value of Yana's area is $\frac{49}{4}$. From part (b), the probabilities that Zahid's minimum number is n is $\frac{13-2n}{36}$ for $n = 1, 2, \dots, 6$, and in these cases, his area is n^2 . Thus, the expected value of Zahid's area is

$$\sum_{n=1}^6 \frac{13-2n}{36} (n^2) = \frac{11}{36} + \frac{36}{36} + \frac{63}{36} + \frac{80}{36} + \frac{75}{36} + \frac{36}{36} = \frac{301}{36}.$$

The expected value of the difference of the areas is the difference of the expected values, which is

$$\frac{49}{4} - \frac{301}{36} = \frac{140}{36} = \frac{35}{9}.$$

Answer: $\boxed{35/9}$.

C4. An integer container (x, y, z) is a rectangular prism with positive integer side lengths x, y, z , where $x \leq y \leq z$. A stick has $x = y = 1$; a flat has $x = 1$ and $y > 1$; and a box has $x > 1$. There are 5 integer containers with volume 30: one stick $(1, 1, 30)$, three flats $(1, 2, 15)$, $(1, 3, 10)$, $(1, 5, 6)$ and one box $(2, 3, 5)$.

- How many sticks, flats and boxes are there among the integer containers with volume 36?
- How many flats and boxes are there among the integer containers with volume 210?
- Suppose $n = p_1^{e_1} \cdots p_k^{e_k}$ has k distinct prime factors p_1, p_2, \dots, p_k , each with integer exponent $e_1 \geq 1, e_2 \geq 1, \dots, e_k \geq 1$ and $k \geq 3$. How many boxes are there among the integer containers with volume n ? Express your answer in terms of e_1, e_2, \dots, e_k . How many boxes with volume $n = 8!$ are there?

Solution: There is always one stick for every volume. To count flats with volume n , we want pairs (y, z) with $1 < y \leq z$ and $yz = n$, so we can count divisors of n and discard the pair $(1, n)$. That is, if $d(n)$ is the number of divisors of n , the number of flats is $(d(n) - 2)/2$ if n is not a perfect square, and $(d(n) - 1)/2$ if n is a perfect square.

- There is one stick $(1, 1, 36)$ with volume 36, and four flats, by the divisor counting argument since $d(36) = 9$. Or, explicitly, the flats are $(1, 2, 18)$, $(1, 3, 12)$, $(1, 4, 9)$, and $(1, 6, 6)$.

To count boxes, we can directly find them all by case work: $(2, 2, 9)$, $(2, 3, 6)$, $(3, 3, 4)$. Alternatively, for each box, we can distribute the primes $2, 2, 3, 3$ to three sides, so that each side gets at least one prime factor. In this case, there must be one side that is a product of two primes (maybe not distinct), and there are three ways to do this.

Answer: $\boxed{1, 4, 3}$.

- b. We have $210 = 2 \cdot 3 \cdot 5 \cdot 7$, so that $d(210) = 2^4$, and hence there are $\boxed{7}$ flats. To count boxes, we note that one side length must be a product of two primes, while the other two are prime. There are $\binom{4}{2} = 6$ ways to choose two of the four, so there are $\boxed{6}$ boxes.

Explicitly, the boxes are $(2, 3, 35), (2, 5, 21), (2, 7, 15), (3, 5, 14), (3, 7, 10), (5, 6, 7)$.

- c. We first count the number of containers with volume n , C_n , followed by the number of flats, F_n . We then desire $C_n - F_n - 1$, as there is one stick of volume n .

There are three kinds of containers: ones where no two dimensions are equal, ones where two dimensions are equal, and ones where all three dimensions are equal. Call the number of such containers $C_{1,n}$, $C_{2,n}$, and $C_{3,n}$, respectively. Note that $C_{3,n} = 1$ when all e_i are multiples of 3, and $C_{3,n} = 0$ otherwise.

To count $C_{2,n}$, we first count the number of triplets (x, y, z) of positive integers with $xyz = n$ and $x = y$. We see that there are $\lfloor \frac{e_i}{2} \rfloor + 1$ ways to choose the number of factors of p_i in each of the three dimensions (we may choose the power of p_i in x as $0, 1, \dots, \lfloor e_i/2 \rfloor$). Each of these triplets corresponds to exactly one container, so there are

$$C_{2,n} + C_{3,n} = \prod_{i=1}^k \left(\left\lfloor \frac{e_i}{2} \right\rfloor + 1 \right)$$

containers where at least two dimensions are equal.

Finally, to count $C_{1,n}$, we first count the number of triplets (x, y, z) of positive integers with $xyz = n$. For each p_i , there are $\binom{e_i+2}{2}$ ways to distribute its divisors among x, y , and z . Here, every element of $C_{1,n}$ is counted six times, every element of $C_{2,n}$ is counted thrice, and every element of $C_{3,n}$ is counted once, whence

$$6C_{1,n} + 3C_{2,n} + C_{3,n} = \prod_{i=1}^k \binom{e_i + 2}{2}.$$

Combining all of this together (adding the last equation with thrice the first one), we have

$$C_n = C_{1,n} + C_{2,n} + C_{3,n} = \frac{1}{6} \left(\prod_{i=1}^k \binom{e_i + 2}{2} + 3 \prod_{i=1}^k \left(\left\lfloor \frac{e_i}{2} \right\rfloor + 1 \right) + \begin{cases} 2 & \text{if } 3|e_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases} \right)$$

Now we deal with flats. For every unordered pair of divisors of n , $(k, n/k)$, there is a flat with dimensions $1, k, n/k$ in some order, provided $k \neq 1, n$. The number of such unordered pairs is

$$F_n = \frac{1}{2} \left(\prod_{i=1}^k (e_i + 1) - 2 + \begin{cases} 1 & \text{if } 2|e_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases} \right).$$

Now we specialize to the case $n = 8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$.

We see that

$$C_n = \frac{1}{6} \left(\binom{9}{2} \binom{4}{2} \binom{3}{2} \binom{3}{2} + 3 \cdot 4 \cdot 2 \right) = 328,$$

and

$$F_n = \frac{1}{2}(8 \cdot 3 \cdot 2 \cdot 2 - 2) = 47,$$

so the number of boxes is

$$C_n - F_n - 1 = \boxed{280}.$$