A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome VIII

PROBLEMS FOR MATHEMATICS LEAGUES III

Peter I. Booth

Memorial University of Newfoundland

John Grant McLoughlin

University of New Brunswick

Bruce L.R. Shawyer

Memorial University of Newfoundland

The ATOM series

The booklets in the series, **A Taste of Mathematics**, are published by the Canadian Mathematical Society (CMS). They are designed as enrichment materials for high school students with an interest in and aptitude for mathematics. Some booklets in the series will also cover the materials useful for mathematical competitions at national and international levels.

La collection ATOM

Publiés par la Société mathématique du Canada (SMC), les livrets de la collection Aime-t-on les mathématiques (ATOM) sont destinés au perfectionnement des étudiants du cycle secondaire qui manifestent un intérêt et des aptitudes pour les mathématiques. Certains livrets de la collection ATOM servent également de matériel de préparation aux concours de mathématiques sur l'échiquier national et international.

Net proceeds from the sale of the booklets in the ATOM series will be donated to the Mathematical Olympiads Fund of the CMS.

Le produit net de la vente des livrets de la collection ATOM sera versé au Fonds de la SMC pour les Olympiades mathématiques

A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome VIII

PROBLEMS FOR MATHEMATICS LEAGUES III

Peter I. Booth
Memorial University of Newfoundland
John Grant McLoughlin
University of New Brunswick
Bruce L.R. Shawyer
Memorial University of Newfoundland

Published by the Canadian Mathematical Society, Ottawa, Ontario
and produced by the CMS CRUX with MAYHEM Office, St. John's, NL, Canada

Publié par la Société mathématique du Canada, Ottawa (Ontario)
et produit par le Bureau CRUX with MAYHEM de la SMC, St. John's, NL, Canada

Printed in Canada by / imprimé au Canada par
The University of Toronto Press Incorporated

ISBN 0-919558-17-8

Table of Contents

Foreword

This volume is a follow up to our previous publications on *Problems for Mathematics Leagues*. It is the fourth book published by the authors based on their cooperation of devising problems for the Newfoundland and Labrador Senior Mathematics League over a period of more than 16 years.

Since the publication of the first ATOM volume, other mathematics leagues, based on our model, have sprung up in other parts of Canada. We are always pleased to assist other leagues, and are prepared to provide current games to help them get started.

While we have tried to make the text as correct as possible, some mathematical and typographical errors might remain, for which we accept full responsibility. We would be grateful to any reader drawing our attention to errors as well as to alternative solutions.

It is the hope of the Canadian Mathematical Society that this collection may find its way to high school students including those who may have the talent, ambition and mathematical expertise to represent Canada internationally. Those who find the problems too challenging at present can work their way up through other collections. For example:

1. The journal *Crux Mathematicorum with Mathematical Mayhem* (subscriptions can be obtained from the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5). Mayhem is also available online;
2. The book *The Canadian Mathematical Olympiad 1969–1993 / L'Olympiade mathématique du Canada*, which contains the problems and solutions of the first twenty five Olympiads held in Canada (published by the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
3. The book *Five Hundred Mathematical Challenges*, by E.J. Barbeau, M.S. Klamkin & W.O.J. Moser (published by the Mathematical Association of America, 1529 Eighteenth Street NW, Washington, DC 20036, USA);
4. The CMS website,

   `www.cms.math.ca`

   where all of the International Mathematical Talent Search problem sets are available.

The authors would like to thank Kyle Sampson for some suggestions of problems.

Peter I. Booth
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, NL
Canada A1C 5S7

John Grant McLoughlin
Faculty of Education
University of New Brunswick
Fredericton, NB
Canada E3B 5A3

Bruce L.R. Shawyer
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, NL
Canada A1C 5S7

The Authors

- **Peter Booth**

  Peter Booth is Honorary Research Professor of Mathematics in the Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland and Labrador, Canada.

  Peter has been involved in the Newfoundland and Labrador Teachers Association Senior Mathematics League for many years.

  His research is mostly in the area of Algebraic Topology, and in particular on applications of the Fibrewise, Pairwise and Freerange Exponential Laws to problems in Homotopy Theory. He has written extensively in those areas.

  He is co-author (with the other authors) of *Shaking Hands in Corner Brook* and *Problems for Mathematics Leagues*. He was winner of the Dean of Science Distinguished Scholar Medal for 1996/97, served on the Canadian Mathematical Society's Canadian Mathematical Olympiad Committee from 1996 to 1999, and was a member of the Board of Directors of the Canadian Mathematical Society from 1999 to 2003.

- **John Grant McLoughlin**

  John Grant McLoughlin is Professor in the Faculty of Education (with a cross appointment to Mathematics and Statistics) at the University of New Brunswick. John has also taught in British Columbia, Ontario, and Newfoundland and Labrador. He has created numerous original problems for contests and Math League games, in addition to (co)-authoring books such as *Calendar Problems from the Mathematics Teacher* and *Combinatorial Explorations*. John is the Book Review editor for the Canadian Mathematical Society's problem-solving journal *Crux Mathematicorum with Mathematical Mayhem*, and is a frequent contributor of problems and columns for *Mayhem*.

  He has served on the Education Committee of the CMS.

- **Bruce Shawyer**

  Bruce Shawyer is Professor Emeritus in the Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland and Labrador, Canada.

  His professional interests in mathematics research include approximation of the sums of slowly convergent series, the summability of series and integrals, and Euclidean geometry. In the area of mathematics education he is interested in mathematics enrichment, why certain mathematical ideas are not well-learned and the propagation of mathematics.

  He is founder of of the Newfoundland and Labrador Teachers Association Senior Mathematics League and of the mathematical challenge for junior high school students (now replaced by a Mathematics League) and former head coach of the Canadian team for the International Mathematical Olympiad.

History of the NLTA Senior Mathematics League

The Newfoundland and Labrador Senior Mathematics League began in 1987 as a competition amongst the high schools in the St. John's area. It has grown since then into a province-wide competition with many schools competing in local leagues in several districts all over the province. The same game takes place simultaneously in each place, with the schools competing at the district level. A provincial championship game (game 5 of each season) takes place towards the end of a school year, with the top schools from each district competing at a common site. Teachers from the participating schools attend and act as proctors.

There have even been leagues outside Newfoundland and Labrador, using our games. We are happy to provide master copies of materials for a modest fee.

The NLTA Senior Math League stresses cooperative problem solving. Each participating school sends a team of four students, who will work together on each problem. Students may, if they wish, submit individual answers. However, to reward cooperative work, a bonus mark is given for a correct team answer.

A typical contest consists of ten questions and a relay. Unlike most mathematics competitions where the contestant receives all of the questions at once, the team receives only the first problem at the start.

Each team is seated around a table, and is given two copies of the first problem. Thus, everyone can read the problem easily. There is a time limit announced for each problem (usually between three and ten minutes). When time is called, the team hands one sheet to a proctor for marking.

While the contestants' answers are being marked, a solution to the problem is presented, preferably by a student, usually using an overhead projector. A correct team answer gets five marks whereas an incorrect team answer gets zero marks. If the team members do not agree on the answer, each may submit an answer for one possible mark each. This results in much discussion and debate, especially if two team members arrive at different answers. Such debate is indeed encouraged. The next problem is now given to the teams, and the above sequence of events is repeated.

After ten problems, which usually increase in difficulty, and in time allotted, students do the relay question.

The relay question has four parts: the answer to part # 1 is an input to part # 2; the answer to part # 2 is an input to part # 3; and the answer to part # 3 is an input to part #4.

The relay question has twenty minutes allotted, and up to ten bonus points over and above the five normal points. One point is awarded if only part # 1 is correct; two points are awarded if only both parts # 1 and # 2 are correct; three points are awarded if only all of parts # 1, # 2 and # 3 are correct; and five points are awarded if all four parts are correct.

If time has not been called when the team hands the relay answer sheet to the proctor, the proctor will say either "CORRECT" or "WRONG". If the proctor answers "WRONG", the relay answer sheet is returned to the team. No indication of the place of any error is given.

Bonus points, up to ten, are awarded as follows: there are ten bonus points for a CORRECT answer sheet completed within the first six minutes; then the number of bonus points for a CORRECT answer sheet is reduced by one at the end of each subsequent minute until there are no more available.

The theoretical maximum score consists of five points for each question plus five for the relay and ten time bonus points, giving a grand total of 65. Scores above 60 are rarely achieved.

This bonus point scheme has been in operation for about 5 years, and is different from what was previously used and reported in our previous volumes.

After the scores have been tallied, the day's winners are announced unless there is a tie for first place. In this case, a tiebreaker question is used to determine the contest's winner. No points are awarded for winning a tiebreaker.

The rules for the tiebreaker are a little different. Essentially, the first team to solve the question wins. But to prevent silly guessing games, the following rule applies: a team that has offered an incorrect answer to a tiebreaker may not offer any other answer for a subsequent period of one minute.

In some locations, it has been customary for the host (institution or school) to award a plaque to the winning team. Scores are cumulative over the league year, and at the end of the season (we have four contests), the top three schools receive awards. These are often in the form of a plaque for the season. In the St. John's district, the top team is awarded a trophy, which it holds for one year. The top teams in the province are invited to a Provincial Championship, usually held in May.

There is usually a "nutrition break" after the fifth question. Students and proctors are provided with a soft drink, juice, tea or coffee and a donut or muffin. This allows for some social interactions amongst the participants.

Attending an NLTA Math League contest is a very rewarding experience. They usually take place on Saturday mornings. It is very gratifying to be in a room full of young men and women, usually in equal numbers, doing mathematics, enjoying mathematics, and having fun.

Season 2003–2004

Game #1

1. Find all real solutions of the equation $9^x = 35 - 4^x$.

   Solution. The left side, 9^x, increases steadily as x increases. On the other hand, the right side, $35 - 4^x$, decreases steadily as x increases. Thus, there is only one solution.

   Trial and error leads to the solution $\boxed{\frac{3}{2}}$.

2. Find all real solutions of the equation

   $$x^6 - x^5 - x^4 + 2x^3 - x^2 - x + 1 \;=\; 0\,.$$

   Solution. By substitution of $x = 1$ in the left side, we find a solution. Now,

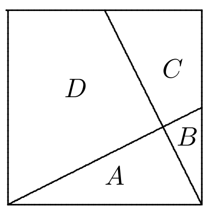
   $$x^6 - x^5 - x^4 + 2x^3 - x^2 - x + 1 \;=\; (x-1)(x^5 - x^3 + x^2 - 1)\,.$$

   But $x^5 - x^3 + x^2 - 1 \;=\; x^3(x^2-1) + (x^2-1) \;=\; (x^2-1)(x^3+1) \;=\; (x^2-1)(x+1)(x^2-x+1)$.

   Hence, all the real solutions are $\boxed{1 \text{ and } -1}$.

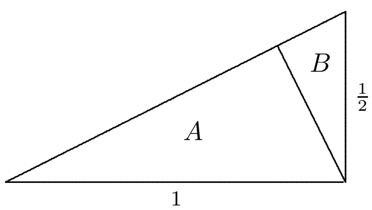
3. Two adjacent sides of a square of side 1 are bisected.

   The mid-points are joined to vertices and the areas of the interior regions are given by A, B, C and D as shown.



   Determine the exact value of D.

   Solution. By symmetry, $A + B = C + B$, so that $A = C$. Consider the triangle with A and B in it. Now consider the three triangles $A \cup B$, A and B as shown below.



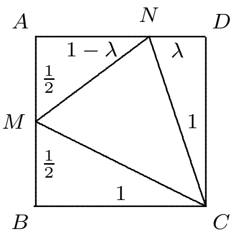
   The triangles A and B are similar, with the linear dimensions of A being twice the corresponding linear dimensions of B. Hence, $A = 2^2B = 4B$

   Therefore, $\frac{1}{4} = A + B = 5B$, so that $B = \frac{1}{20}$ and $C = A = \frac{1}{4} - \frac{1}{20} = \frac{1}{5}$.

   Finally, we have $D = 1 - A - B - C = 1 - \frac{1}{5} - \frac{1}{20} - \frac{1}{5} = \boxed{\frac{11}{20}}$.

4. On square $ABCD$, let M be the mid-point of AB and N a point on AD such that the area of $\triangle CMN$ is one third of the area of the square $ABCD$. Determine the fraction $\dfrac{AN}{ND}$.

Solution. Without loss of generality, let $AB = BC = CD = DA = 1$, so that $AM = MB = \frac{1}{2}$. Let $ND = \lambda$, so that $AN = 1 - \lambda$.



We clearly have:

Area $\triangle BCM = \frac{1}{4}$, Area $\triangle CDN = \frac{\lambda}{2}$ and Area $\triangle AMN = \frac{1-\lambda}{4}$.

Thus, Area $\triangle CMN = 1 - \frac{1}{4} - \frac{\lambda}{2} - \frac{1-\lambda}{4} = \frac{2-\lambda}{4} = \frac{1}{3}$ if and only if $\lambda = \frac{2}{3}$. Thus,

$$\frac{AN}{ND} = \frac{1-\lambda}{\lambda} = \frac{\frac{1}{3}}{\frac{2}{3}} = \boxed{\frac{1}{2}}.$$

5. Here is a rule for multiplying ordered pairs of real numbers:

$$(a, b)(x, y) = (ax - by, ay + bx)\,.$$

Express $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{256}$ in the form (x, y).

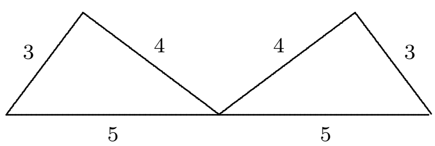
Solution. $(x, y)^2 = (x^2 - y^2, 2xy)$, so that $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^2 = (0, 1)$.

Thus, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^4 = (0, 1)^2 = (-1, 0)$,

and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^8 = (-1, 0)^2 = (1, 0)$.

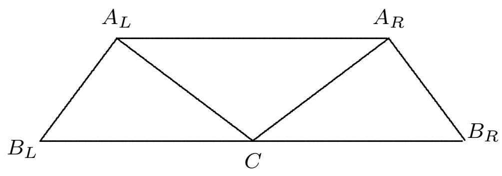
Now, all powers of $(1, 0)$ are $(1, 0)$, so that $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^{256} = \boxed{(1, 0)}$.

6. Two 3, 4, 5 triangles are placed on a straight line as shown:

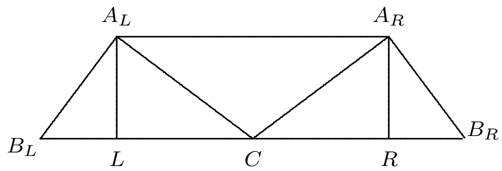


Calculate the exact value of the distance between the two points where the right angles are located.

Solution. We shall label the diagram.



Drop perpendiculars from A_L and A_R to B_LB_R, at L and R, respectively.



From similar triangles (for example, A_LCB_L, LCA_L and LA_LB_L), we see that $LC = CR = \dfrac{16}{5}$,

so that $A_LA_R = LR = \boxed{\dfrac{32}{5}}$.

7. Find the sum of all the prime numbers between 20 and 120 which are one more than a multiple of 5.

   Solution. Clearly, the primes (being odd) must be one more than a multiple of 10.

   The possibilities are 21 (composite), 31 (prime), 41 (prime), 51 (composite), 61 (prime), 71 (prime), 81 (composite), 91 (composite), 101 (prime) and 111 (composite).

   The required sum is then $31 + 41 + 61 + 71 + 101 = \boxed{305}$.

8. This year (2003), 1 January occurred on a Wednesday. What is the next year on which 1 January will occur on a Wednesday?

   Solution. Don't forget leap years!!

2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Wed	Thu	Sat	Sun	Mon	Tue	Thu	Fri	Sat	Sun	Tue	Wed	Thu

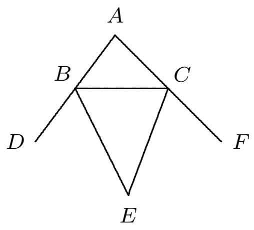
   The answer is $\boxed{2014}$.

9. A normal book is opened at random, and the product of the two visible page numbers is 4160. Determine the sum of the page numbers.

   Solution. We need to solve $n(n+1) = 4160$. Note that $n < \sqrt{n(n+1)} < n+1$ and $\sqrt{4160} \approx 64.498$, we see that $n = 64$. Thus, the page numbers are 64 and 65. Hence, the answer is $\boxed{129}$.

10. In $\triangle ABC$, we have $\angle BAC = 80°$. The exterior bisectors of $\angle ABC$ and of $\angle BCA$ meet at E. Determine the measure (in degrees) of $\angle BEC$.

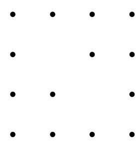
Solution.



Let $\angle DBE = \angle EBC = \alpha$ and $\angle BCE = \angle ECF = \beta$.
Then $\angle BEC = 180° - \alpha - \beta$.
In $\triangle ABC$, we then have $\angle ABC = 180° - 2\alpha$,
$\angle ACB = 180° - 2\beta$ and $\angle BAC = 80°$.
This shows that $\alpha + \beta = 130°$.
Hence, the answer is $180° - \alpha - \beta = \boxed{50}°$.

RELAY

Relay 1. Let X be the total number of rectangles (including squares) with sides parallel to the axes that can be made by joining up four of the dots in the diagram of points with:



Write X in Box #1 of the Relay Answer sheet.

Relay 2. Let $W = X + 4$. Let W' be the integer obtain by writing the digits of W in the reverse order. Let W'' be the integer obtained by concatenating W with W'. For example, if $W = 123$, then $W' = 321$ and $W'' = 123321$.

Let Y' be the smallest multiplier of (your) W'' in order that we have a perfect square. Let Y be the sum of the digits of Y'.

Write Y in Box #2 of the Relay Answer sheet.

Relay 3. A motorcyclist drives at Y kmph for one hour, then at $Y - 1$ kmph for two hours, then at $Y - 2$ kph for three hours, etc., until coming to a complete stop. How many kilometres has the motorcyclist travelled? Call this answer Z'. Let Z be the sum of the digits of $(Z' + 2)$.

Write Z in Box #3 of the Relay Answer sheet.

Relay 4. Pat's average mark after her first five courses at MUN was exactly $100 - Z$. Her worst mark was a 63 in English. What would have the English mark needed to be in order for her average to be exactly 85?

Write your answer in Box #4 of the Relay Answer Sheet, and hand it to your proctor.

Answers : $\boxed{\boxed{19}, \boxed{16}, \boxed{17}, \boxed{73}}$

Game #2

1. How many integer solutions are there such that $\dfrac{6}{n+2}$ is an integer?

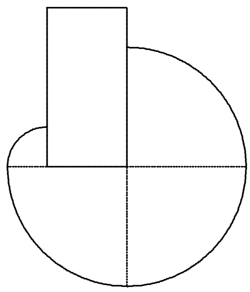
   Solution. We require that $n+2$ be an integer factor of 6.

   In other words, $n+2$ can take the values -1, -2, -3, -6, 1, 2, 3 and 6.

   The answer is then $\boxed{8}$.

2. Ivan lives in a house whose base has dimension 10 metres by 20 metres. His house is surrounded in all directions by a large grass field. Ivan owns a very cute dog. Ivan attaches one end of a 15 metre leash to his dog, and attaches the other end to a corner of his house. Assuming that there are no obstacles in the field, what is the exact area (in square metres) of the field that the dog can run around, with the leash on? [Exact answer required.]

   Solution. The area that the dog can reach is shown in the sketch:



   The area consists of $\frac{3}{4}$ of a circle of radius 15 plus $\frac{1}{4}$ of a circle of radius 5. This is

$$\frac{3}{4}\pi 15^2 + \frac{1}{4}\pi 5^2 \;=\; \boxed{175\pi} \text{ square metres}.$$

3. A wooden box with dimensions 5 cm by 6 cm by 7 cm has volume 210 cm^3.

   All six sides of this box are painted red.

   The box is then cut into 210 equal unit cubes (that is, cubes that measure 1 cm on each side).

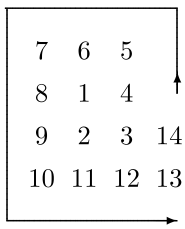
   Of these 210 unit cubes, how many of them have *exactly* two faces painted red?

   Solution. The cubes with exactly two faces painted red must come from the edges of the box excluding the corner cubes.

   On a side of length n, we would get $n-2$ such cubes from each of the four such edges.

   Thus, the answer is $4 \times (5+4+3) = \boxed{48}$.

4. A diagram is constructed in the form of a "rectangular spiral" as shown:



What number is in the bottom left hand corner when the diagram is a square with 101 numbers on each side?

Solution. A little extension of the diagram shows that the number in the bottom left hand corner of the odd sided square is the square of the number of numbers on the side.

The answer is $101^2 = \boxed{10201}$.

ALTERNATIVELY: we can use mathematical induction to show that when we have made a square of side $2n+1$, the number in the bottom left corner is $(2n+1)^2$.

This is true when $n = 0$ (only the number 1 is shown).

When we have a $(2k-1) \times (2k-1)$ square with $(2k-1)^2$ in the bottom left corner, to get a $(2k+1) \times (2k+1)$ square, we need to place an additional $4 \times (2k) = 8k$ numbers. This means that the number in the bottom left corner is now $(2k-1)^2 + 8k = (2k+1)^2$, and it follows, by the Principle of Mathematical Induction, that the answer is $101^2 = \boxed{10201}$.

5. Let a, b, c, d, e and f be integers.

We define $(a,b) \ast (c,d) = (ac, ad+b)$ and $(a,b) - (c,d) = (a-c, b-d)$.

Simplify to the form (p,q):

$$\Big(((a,b) \ast (c,d)) \ast (e,f) \Big) - \Big((a,b) \ast ((c,d) \ast (e,f)) \Big) .$$

Solution. We have $(a,b) \ast (c,d) = (ac, ad+b)$.

We need to calculate $(ac, ad+b) \ast (e,f)$. This is $(ace, acf+ad+b)$.

We have $(c,d) \ast (e,f) = (ce, cf+d)$.

We need to calculate $(a,b) \ast (ce, cf+d)$. This is $(ace, acf+ad+b)$.

The answer is then $\boxed{(0,0)}$.

6. Tamara looks at her watch at exactly 9:40 AM. She notices that the acute angle made by the hour and minute hands is x°. Determine the value of x.

   Solution. The minute hand is $\frac{2}{3}$ of the way around the clock dial.

   Hence, the hour hand must be $\frac{2}{3}$ of the way from 9 to 10.

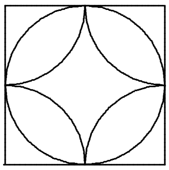
   Each hour determines an angle of $\frac{360}{12}^\circ = 30^\circ$.

   $\frac{2}{3}$ of this is 20°.

   The minute hand is actually pointing to 8.

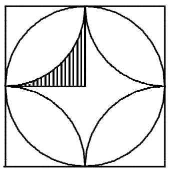
   Hence, the answer is $30^\circ + 20^\circ = \boxed{50}\,^\circ$.

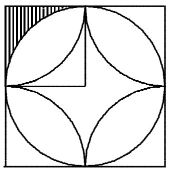
7. A circle is inscribed in a unit square. Four quarter circles are drawn inside the square with centres at the four vertices and radius $\frac{1}{2}$, making four curved spaces, and leaving a star shaped region.

   Determine the area (in square units) of the star shaped region. [Exact answer please.]

   Solution. Divide each curved space in half by the straight lines joining the mid-points of the sides. And divide the star shaped region into four equal parts with the lines joining the mid-points.

   The star shaped region is composed of four of the shaded regions. Each is congruent to a region outside the circle. Hence, the area of the star shaped region is equal to the area of the square, less the area of the circle.

   This area is $1 - \pi\left(\frac{1}{2}\right)^2 = \boxed{1 - \frac{\pi}{4}}$ square units.

8. Let $S = 2^{15} - 2^{14} - 2^{13} - 2^{12} - 2^{11} - 2^{10} - 2^9 - 2^8 - 2^7 - 2^6 - 2^5 - 2^4 - 2^3 - 2^2 - 2^1$.

   Determine the value of S.

   Solution.

   We need the sum of a geometric series: $\sum_{k=1}^{n} x^k = \frac{x^{n+1} - x}{x - 1}$.

   Thus, $\sum_{k=1}^{14} 2^k = \frac{2^{15} - 2}{2 - 1} = 2^{15} - 2$.

   The answer is then $2^{15} - \left(2^{15} - 2\right) = \boxed{2}$.

9. The sum of the digits of a set of three consecutive two digit integers is 42.

How many such sets of integers are there?

Solution. One way to get such a set is to have the integers $\begin{cases} \text{a (b-1)} \\ \text{a b} \\ \text{a (b+1)} \end{cases}$.

Thus, $3a + 3b = 42$, or $a + b = 14$.

From this, the possible sets are:

a	b	{a (b-1),	a b,	a (b+1)}
9	5	{9 4,	9 5,	9 6}
8	6	{8 5,	8 6,	8 7}
7	7	{7 6,	7 7,	7 8}
6	8	{6 7,	6 8,	6 9}

;

that is, 4 sets.

But we can also have $\begin{pmatrix} \text{a 8} \\ \text{a 9} \\ \text{(a+1) 0} \end{pmatrix}$ and $\begin{pmatrix} \text{a 9} \\ \text{(a+1) 0} \\ \text{(a+1) 1} \end{pmatrix}$.

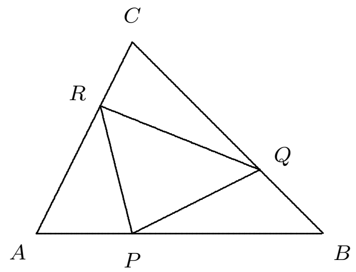
These give $\begin{cases} \text{3a+18} = 42, \\ \text{a} = 8 \\ \text{a set of } \{88,\ 89,\ 90\} \end{cases}$ and $\begin{cases} \text{3a+12} = 42, \\ \text{a} = 10 \\ \text{no set.} \end{cases}$

Thus, the number of sets is $\boxed{5}$.

10. In $\triangle ABC$, we have points P, Q and R on AB, BC and CA, respectively, such that $AP : PB = 1 : 2$, $BQ : QC = 1 : 2$ and $CR : RA = 1 : 2$.

Determine the ratio of the area of $\triangle PQR$ to the area of $\triangle ABC$.

Solution. Draw a diagram.



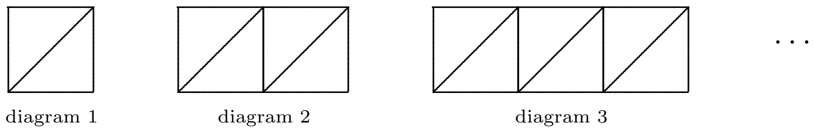
Examine $\triangle PBQ$. Its base is $\frac{2}{3}$ of the base of $\triangle ABC$. Its height is $\frac{1}{3}$ of the height of $\triangle ABC$. Hence, its area is $\frac{2}{9}$ of the area of $\triangle ABC$.

By symmetry, the same can be determined for $\triangle QCR$ and $\triangle RAP$.

Thus, the area of $\triangle PQR$ is $1 - 3 \times \frac{2}{9} = \frac{1}{3}$ of the area of $\triangle ABC$, so that the desired ratio is $\boxed{1 : 3}$.

RELAY

Relay 1. Toothpicks are arranged to form a sequence of diagrams as shown:



You have 97531 toothpicks available. Make the sequence as far as is possible. How many diagrams will be in the sequence produced? Call your answer W.

Write W in Box #1 of the Relay Answer sheet.

Relay 2. Let W' be the integer obtain by writing the digits of W in the reverse order. Let W'' be the integer obtained by contatenating W with W'. For example, if $W = 123$, then $W' = 321$ and $W'' = 123321$, or if $W = 100$, then $W' = 001$ and $W'' = 100001$.

Let X be the sum of the prime factors of (your) W''.

Write X in Box #2 of the Relay Answer sheet.

Relay 3. Start with the number $Y = X^2$.

A process is defined by:

Divide Y by the sum of its digits, and round down to the nearest integer.

Repeat this process until you get a fixed number. Call this number Z.

Write Z in Box #3 of the Relay Answer sheet.

Relay 4. Sheena Ham is bored by the easy problems in math class. She takes a piece of card (Z mm thick), cuts it in half and places one piece on top of the other.

She then cuts the pile in half and places one pile on top of the other.

She repeats this process.

How often must she do this to make a pile higher than Mount Everest? (The latest estimate of the height of Mount Everest is 8850 metres.)

Write your answer in Box #4 of the Relay Answer Sheet, and hand the sheet to your proctor.

Answers : 220, 223, 1, 24

Game #3

1. Determine the value of $(B - A)$, where A and B satisfy the equation below.

$$\frac{7x - 16}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}.$$

Solution. We have

$$\begin{aligned}\frac{7x - 16}{x^2 - 5x + 6} &= \frac{A}{x - 2} + \frac{B}{x - 3} \\ &= \frac{A(x - 3) + B(x - 2)}{x^2 - 5x + 6} = \frac{x(A + B) - (3A + 2B)}{x^2 - 5x + 6},\end{aligned}$$

so that $A + B = 7$ and $3A + 2B = 16$, giving $A = 2$ and $B = 5$.

Thus, $B - A = 5 - 2 = \boxed{3}$.

2. A right angled triangle with sides of integral length has a perimeter of P cm. and an area of P sq. cm. Find the smallest possible value of P.

Solution. Let the right triangle have legs a and b so that its hypotenuse is $\sqrt{a^2 + b^2}$. We then have

$$P = \frac{ab}{2} = a + b + \sqrt{a^2 + b^2}.$$

This leads to $a^2 + b^2 = (\frac{ab}{2} - a - b)^2$, which can be rearranged to give $\frac{ab(ab - 4a - 4b + 8)}{4} = 0$. Ignoring the part $\frac{ab}{4}$, we take $(ab - 4a - 4b + 8) + 8$, to get $(a - 4)(b - 4) = 8$. This solves to $b = \frac{4(a - 2)}{a - 4} = 4 + \frac{8}{a - 4}$.

The smallest possible positive value to substitute is $a = 5$. This gives $b = 12$. The hypotenuse is then 13. The area and perimeter are 30.

The next smallest positive value to substitute is $a = 6$. This gives $b = 8$. The area and perimeter are 24.

In fact, the only other positive integers a that lead to integers b are $a = 8$ and $a = 12$, giving the same two triangles as above.

The answer is then $\boxed{24}$.

3. The mean of $5n$ consecutive integers starting with n is 10. Determine the value of n.

Solution. We have $n+(n+1)+(n+2)+\cdots+(n+(5n-1)) = 5n \times 10 = 50n$.

We use the fact that the mean of an arithmetic series is the mean of the first and last terms.

This gives $\frac{n+(n+(5n-1))}{2} = \frac{7n-1}{2} = 10$. This solves to $n = \boxed{3}$.

4. What is the final digit in 37^{2004}?

Solution. The powers of 7 end in 7, 9, 3, 1, 7, 9, 3, 1, ... The endings form a cycle of length 4.

Since 2004 is divisible by 4, the last digit of 37^{2004} is $\boxed{1}$.

5. A fraction such as $\frac{1}{2}$ will be referred to as a fraction in lowest terms, whereas, $\frac{2}{4}$ will not be so called.

How many distinct fractions $\dfrac{a}{b}$ in lowest terms exist such that $a + b = 40$ and $a < b$, where a and b are positive integers?

Solution. This question is: How many distinct lowest term fractions $\frac{a}{b}$ exist such that $a + b = 40$ and $a < b$, where a and b are positive integers?

Thus, we need consider only cases where both a and b are odd.

Listing these, we get: $\frac{1}{39}$, $\frac{3}{37}$, $\frac{5}{35}$, $\frac{7}{33}$, $\frac{9}{31}$, $\frac{11}{29}$, $\frac{13}{27}$, $\frac{15}{25}$, $\frac{17}{23}$, and $\frac{19}{21}$.

Now, we eliminate the non-lowest term fractions: $\frac{5}{35}$, and $\frac{15}{25}$.

This leaves: $\frac{1}{39}$, $\frac{3}{37}$, $\frac{7}{33}$, $\frac{9}{31}$, $\frac{11}{29}$, $\frac{13}{27}$, $\frac{17}{23}$ and $\frac{19}{21}$.

Hence, the answer is $\boxed{8}$.

6. A, B, C and D are scores from a test administered to 5 students. The scores on the test were A, B, B, C and D where $A < B < C < D$. If the score A is replaced with C, then the median would increase by 4 and the mean would increase by 3.

Determine the value of $(B - A)$.

Solution. The scores, in increasing order are

$$A,\ B,\ B,\ C,\ D.$$

If A is replaced by C, the scores in increasing order are

$$B,\ B,\ C,\ C,\ D.$$

Thus, $C = B + 4$.

The mean of the five scores is increased by 3. Thus, $C = A + 5 \times 3 = A + 15$.

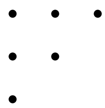
We then have $B + 4 = A + 15$, so that $(B - A) = \boxed{11}$.

7. What is the smallest positive integer that leaves a remainder of 3 when divided by 5, a remainder of 7 when divided by 9, and a remainder of 0 when divided by 8?

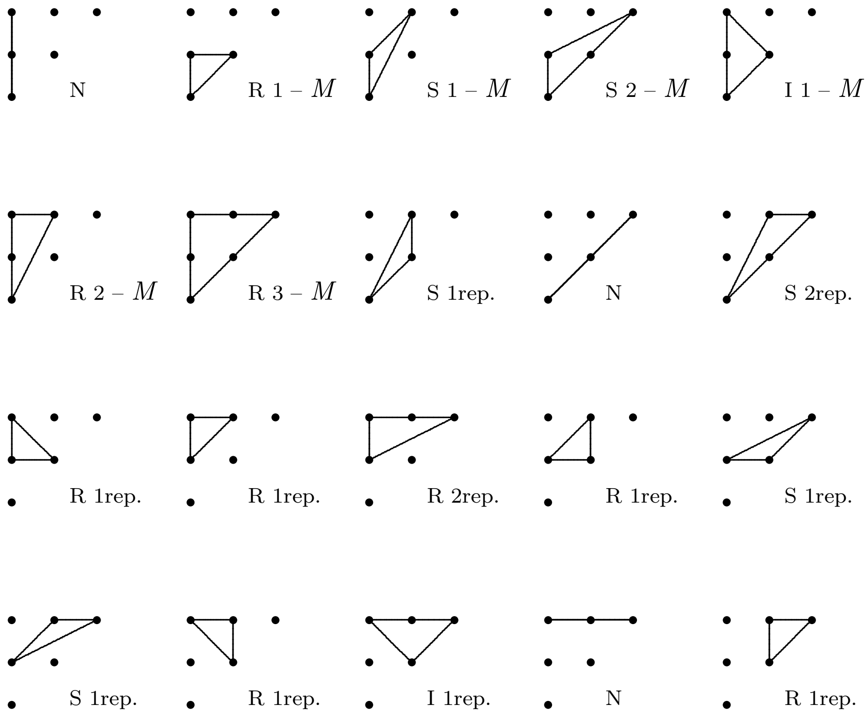
Solution. The required number n is 2 less than a multiple of 5 and 2 less than a multiple of 9. Therefore, n is 2 less than a multiple of 45, such as 43, 88, 133, $\cdots$.

Since n is a multiple of 8, the answer is $\boxed{88}$.

8. Determine the number of distinct non-congruent triangles that can be formed using three of the dots in the diagram as vertices:



Solution. We indicate all possible ways of choosing three dots as vertices.



6 distinct non-congruent triangles (three right, one isosceles, two scalene)

R	:	3 right triangles	M	:	first occurrence
I	:	1 isosceles triangle	rep.	:	repeat of triangle
S	:	2 scalene triangles	N	:	no triangle

The answer is then $\boxed{6}$.

9. Solve for p and q:

$$6^{(2p/11)+(q/11)+6} \;=\; 16^{(p/11)+5} \times 9^{7+(p/11)-(5q/11)}\,.$$

Express your answers as fractions in reduced form.

Solution. The right side is $2^{(4p/11)+20} \times 3^{(2p/11)-(10q/11)+14}$, giving the equations

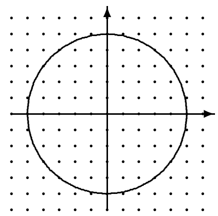
$$(2p/11)+(q/11)+6 \;=\; (4p/11)+20 \;=\; (2p/11)-(10q/11)+14\,.$$

These equations solve to $\boxed{p=-73 \text{ and } q=8}$.

10. A lattice point is a point in the plane for which both coordinates are integers. For example, $(-3,2)$ is a lattice point, whereas, $(5,3/7)$ is not. Exactly 97 lattice points are contained inside or on the circle defined by the equation $x^2+y^2=H$, including exactly 8 lattice points lying on the circumference of this circle. Determine the value of H.

[NO CALCULATORS ALLOWED]

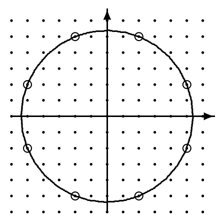
Solution. A short test shows that there are 81 lattice points inside or on a circle of radius 5 centred at the origin.



Thus, we must increase this number by 16.

The next nearest lattice points are at $(\pm1,\pm5)$ and $(\pm5,\pm1)$ — a total of 8, increasing the number to $81+8=89$ — not enough.

The next nearest lattice points are at $(\pm2,\pm5)$ and $(\pm5,\pm2)$ — a total of 8, increasing the number to $89+8=97$ — exact!



Thus, $H=2^2+5^2=\boxed{29}$.

RELAY

Relay 1. The diameter of a large semicircle is broken into J equal parts to construct smaller semicircles on its opposite side, as shown:

Determine J if the ratio of the circumference of the large semicircle to that of the circumference of a single small semicircle is $10 : 1$.

Write the value of J in Box #1 of the Relay Answer Sheet.

Relay 2. Given that K and $(K - J)$ are both two–digit prime numbers, determine the second largest possible value of K.

Write the value of K in Box #2 of the Relay Answer Sheet.

Relay 3. The measures, in degrees, of the interior angles of a pentagon form an arithmetic sequence. The measure of the smallest interior angle is K degrees and the measure of the largest interior angle is L degrees.

Write the value of L in Box #3 of the Relay Answer Sheet.

Relay 4. An operation $*$ is defined such that $A * B = A^2 - B^2$. Find all positive values of M for which $(5 * M) * M^2 = L + 42$.

Write these values of M in Box #4 of the Relay Answer Sheet, and hand it to your proctor.

Answers : $\boxed{\boxed{10}, \boxed{83}, \boxed{133}, \boxed{3}.}$

Game #4

1. Find all positive integers x and y that satisfy $11x + 13y = 100$,

   Solution. Note that $11x + 13y = 11(x + y) + 2y = 100$. Thus, $(x + y)$ must be an even positive integer. We try some values for $x + y$:

$x+y$	2	4	6	8	10
y	39	28	17	6	negative

   The only feasible solution is $\boxed{x = 2,\ y = 6}$.

   ALTERNATIVELY

   If $x + y \geq 10$, then $11x + 13y > 11x + 11y \geq 110 > 100$. Thus, $x + y \leq 9$.

   If $x + y \leq 7$, then $11x + 13y \leq 13x + 13y \leq 91 < 100$. Thus, $x + y \geq 8$.

   Now, $11x+13y = 11(x+y)+2y = 100$, so that $x+y$ is even. Thus, $x+y = 8$.

   Thus, $100 = 11x + 13y = 11x + 13(8 - x) = 104 - 2x$, giving $\boxed{x = 2,\ y = 6}$.

2. Andrew and Bruce live in different places, but agree to meet at Mars. Andrew has 75 km to drive, whereas Bruce only has 35 km to drive.

   Andrew drives at a steady 90 kmph. Bruce almost forgets to go, and when he remembers, he is worried about being late, so drives at a steady 105 kmph. They arrive at Mars at exactly the same moment.

   If Andrew set out at 9:00 am, determine the time when Bruce set out.

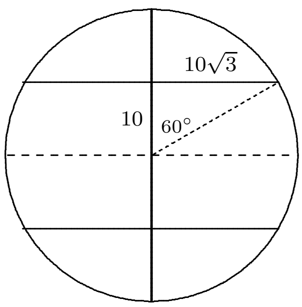
   Solution. Andrew drives 75 km at 90 kmph. This takes 50 minutes.

   Bruce drives 35 km at 105 kmph. This takes 20 minutes.

   Thus, Bruce sets out 30 minutes after Andrew set out — the time being $\boxed{\text{9:30 am}}$.

3. A circle has radius 20 units. A diameter is drawn, and at both points on the diameter that are half-way between the centre and the circumference, draw chords of the circle perpendicular to the diameter. Find the exact value of the area between these two chords inside the circle.

   Solution. Draw a diagram.



   By symmetry, we need four times the area of a triangle plus four times the area of a sector.

   Observe that the angle of the sector is $30° = \frac{\pi}{6}$, one twelfth of a complete circle radius 20 giving an area of $20^2\left(\frac{\pi}{6}\right)\frac{1}{2}$.

   The triangle is a right triangle with legs 10 and $10\sqrt{3}$, giving an area of $\frac{1}{2}(10)(10\sqrt{3}) = 50\sqrt{3}$.

   The total area is then $\boxed{\frac{400\pi}{3} + 200\sqrt{3}}$.

4. The expression $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$ can be written as $\dfrac{p}{x-1}-\dfrac{q}{x-2}+\dfrac{r}{x-3}$.
   Find the value of $p+q+r$.

   Solution. Write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{p}{x-1}-\frac{q}{x-2}+\frac{r}{x-3}\,.$$

   Multiply both sides by $(x-1)(x-2)(x-3)$ to get

$$x^2+x+1 = p(x-2)(x-3)-q(x-1)(x-3)+r(x-1)(x-2)\,. \qquad (1)$$

   Let $x=1$: we get $3=2p$ so that $p=\frac{3}{2}$. Let $x=2$: we get $7=q$. Let $x=3$: we get $13=2r$ so that $r=\frac{13}{2}$.

   Thus, the required quantity is $\frac{3}{2}+7+\frac{13}{2}=\boxed{15}$.

   ALTERNATIVELY, expand equation (1) and equate the coefficients to get three linear equations in the three unknowns p, q and r.

5. Find all quadruples of positive integers (a,b,c,d) that satisfy

$$\begin{aligned} a+b+c+d &= 16\,,\\ a+2b+3c+4d &= 50\,,\\ a+4b+9c+16d &= 170\,. \end{aligned}$$

   Solution. If we solve the four equations for a, b and c in terms of d, we get

$$a=8-d\,,\qquad b=3(d-6)\,,\qquad c=26-3d\,.$$

   From the first, we have $d<8$ and from the second, we have $d>6$. Thus, $d=7$.

   The answer is then $\boxed{(1,3,5,7)}$.

6. Using the date in the form $AB:CD:20XY$, (day:month:year) where there are always two digits used for the day or the month (for example, new year's day this year was $01:01:2004$), find how many palindromic dates there are in the 21^{st} century.

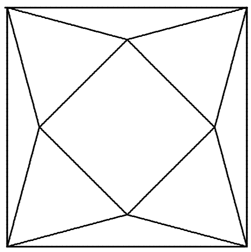
   Solution. Such a date must have the form $AB:02:20BA$; hence, it is a day in February.

   Now, "February has 28 days clear, and 29 in each leap year".

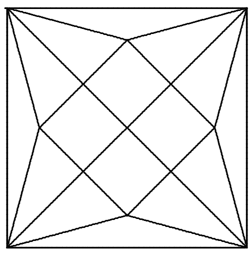
   We can start with $01:02:2010$ and proceed by increasing the day number by 1. The only problem would occur with 29. Is $29:02:2092$ a valid date? YES, since 4 divides 2092, meaning that the year 2092 is a leap year.

   Hence, the answer is $\boxed{29}$.

7. On the sides of a square of side length 1, draw equilateral triangles outwards. The four vertices of these triangles not on the given square form a larger square. Find the area of this square. [Exact answer please in simplest possible form.]



Solution. Draw the diagonals of the larger square.



Consider half of one of these diagonals: it consists of a part inside the smaller square equal to one half of a side plus the altitude of the equilateral triangle. This is of length $\frac{1}{2}+\frac{\sqrt{3}}{2}$. To get the side of the larger square, we multiply this by $\sqrt{2}$ to get $\sqrt{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)$.

The area of the larger square is then $\left(\sqrt{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\right)^2$, which simplifies to $\boxed{2+\sqrt{3}}$.

8. 28 cm of snow lay on the ground on Corner Brook on 1 February.

   Now, the snow melts at a rate of n^2 cm/hour when the temperature is n degrees Celsius.

   In Corner Brook, the temperature is always at an integer number of degrees.

   Suddenly, at 10:00 am, the temperature jumped from -5 degrees Celsius to 4 degrees Celsius. At the top of each of the following hours, the temperature dropped exactly 1 Celsius degree until freezing point was reached.

   Did all the snow melt? And if it did, determine the exact moment when there was no snow left in Corner Brook on 1 February.

Solution. In the hour from 10:00 am to 11:00 am, $4 \times 4 = 16$ cm of snow melted.

In the hour from 11:00 am to 12:00 noon, $3 \times 3 = 9$ cm of snow melted, giving a total of $16 + 9 = 25$ cm of snow melted.

There are then only 3 cm of snow left to melt.

In the hour after 12:00 noon, $2 \times 2 = 4$ cm of snow could melt, giving a total of $25 + 4 = 29$ cm of snow melted — too much! But 3 cm of snow will melt in $\frac{3}{4}$ of an hour at this temperature.

Thus, the snow had completely melted at $\boxed{\text{12:45 pm}}$.

9. The two parallel plates of a press are moving towards one another at 1 metre per minute.

   A not very intelligent insect starts on one plate, flies to the other, instantly reverses (being an ideal insect, this takes no time at all) and flies back to the first, repeating this process until the two plates meet (now you see why the insect is not very intelligent).

   If the insect started this flying at 10 metres per second when the two plates were 1 metre apart, calculate how far the insect flew in this manner.

   Solution. There are two ways to do this: calculate the distances of each of the infinitely many instances of back and forward flying, and add them up.

   But a little thought leads to the observation that the total time required for the two plates to meet is 1 minute.

   The insect flies at 10 metres per second for 60 seconds, and so, must have flown a total of $\boxed{\text{600 metres}}$.

10. If a, b, c and d are the roots of the equation $x^4 + 9x^3 + 12x^2 - 18x + 4 = 0$, determine the exact value of $\frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd}$.

    Solution. First, note that

$$\frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} = \frac{ab + ac + ad + bc + bd + cd}{abcd}.$$

We also have that

$$\begin{aligned} 0 &= x^4 + 9x^3 + 12x^2 - 18x + 4 \\ &= (x-a)(x-b)(x-c)(x-d) \\ &= x^4 - x^3(a+b+c+d) + x^2(ab+ac+ad+bc+bd+cd) \\ &\qquad -x(abc+bcd+cda+dab) + abcd\,. \end{aligned}$$

Thus,

$$ab + ac + ad + bc + bd + cd = 12 \qquad \text{and} \qquad abcd = 4\,.$$

The required answer is then $\frac{12}{4} = \boxed{3}$.

RELAY

Relay 1. How many two digit numbers contain at least one digit 5?

(A two digit number may not begin with the digit 0.)

Call your answer A, and write A in Box #1 of the Relay Answer Sheet.

Relay 2. Find the sum of all prime numbers less than or equal to $A - 2$.

Call your answer B, and write B in Box #2 of the Relay Answer Sheet.

Relay 3. Willow got B% in her first Biology test.

On her second test, she got $(B + C)\%$.

On her third test, she got $(B + 2C)\%$.

On her fourth test, she got $(B + 3C)\%$.

Her average on the tests was $(B + A)\%$.

Determine C, and write C in Box #3 of the Relay Answer Sheet.

Relay 4. Let $s_8 = A + B + C$. Divide s_8 by 8, and write the answer as a mixed fraction (without simplification).

Take the whole number part and concatenate it with the numerator.

Call this number s_7.

(For example, if it were to be the case that $s_8 = 100 = 8\left(12 + \frac{4}{8}\right)$, then $s_7 = 124$.)

Divide s_7 by 7, and in the similar way, create s_6.

Continue in the manner.

Determine s_1, and write your answer in Box #4 of the Relay Answer Sheet, and hand the sheet to your proctor.

Answers : $\boxed{\boxed{18}, \boxed{41}, \boxed{12}, \boxed{16701}}$

Championship Game

1. Andrew was born in the year 1441 and lived until Bernard was 7 years old, which was 23 years before the reformation in 1517. Bernard survived this remarkable event just 49 years. Charles was born 9 years after the death of Andrew and lived until Bernard was 36 years old.

   Find the sum of the number of years that the three men lived.

   Solution.

   (a) Andrew: 1441: so that $(1517 - 23) = 1494$. Years lived: 53;

   (b) Bernard: $(1494 - 7) = 1487$: so that $(1517 + 49) = 1566$. Years lived: 79;

   (c) Charles: $(1494 + 9) = 1503$: so that $(1487 + 36) = 1523$. Years lived: 20.

   Sum of years lived is $53 + 79 + 20 = \boxed{152}$.

2. Rebecca set out to cycle across Canada at 12:00 noon on 1 April 2004, her route being 3600 km long. At 12:00 noon, a few days later, Neil set out to catch up with her. Rebecca travelled 220 km per day. Neil travelled 396 km per day. As it happened, Neil caught up with Rebecca at exactly 12:00 noon on a day before they finished the cross country trip.

   On what day did Neil set out?

   Solution.

   In n days, Rebecca cycles $220n$ km.

   If Neil sets out d days later, he will have cycled $396(n-d)$ km when Rebecca has cycled $220n$ km, so that $396(n - d) = 220n$.

   This gives $d = \frac{4n}{9}$. Since the time that Neil catches up with Rebecca is an integer number of days, we have the possibilities: $n = 9$, 18, 27, ...

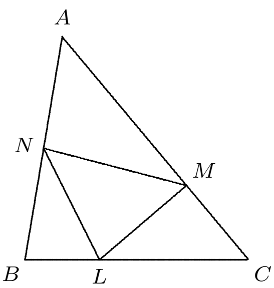
   In 9 days, Rebecca will have cycled 1980 km. In 18 days, she will have cycled 3760 km, which is longer than the journey.

   Thus, $n = 9$, giving $d = 4$. Therefore, Neil started on $\boxed{\text{5 April 2004}}$.

3. In $\triangle ABC$, L, M and N lie on BC, CA and AB, respectively, such that $BL : LC = 1 : 2$, $CM : MA = 1 : 2$ and N is the mid-point of AB.

   Determine the ratio of the area of $\triangle LMN$ to the area of $\triangle ABC$.

Solution. Draw a diagram:



Examine $\triangle BLN$: its base is $\frac{1}{3}$ of the base of $\triangle ABC$, and its height is $\frac{1}{2}$ of the height of $\triangle ABC$. Hence, its area is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ of the area of $\triangle ABC$.

Without loss of generality, let the area of $\triangle ABC$ be 1.

By similar arguments, the area of $\triangle CLM$ is $\frac{2}{9}$ and the area of $\triangle AMN$ is $\frac{1}{3}$.

Hence, the required answer (the area, in this case, of $\triangle LMN$) is $\boxed{5:18}$.

4. A operation $\diamond$ on ordered pairs of real numbers is defined by:

$$(a,b) \diamond (c,d) \; = \; (ad+bc, bd)\,.$$

In the form (a,b), find

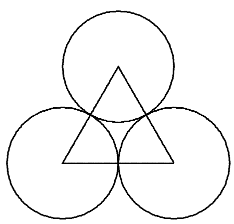
$$(1,2) \diamond (3,4) \diamond (5,6) \diamond (7,8)\,.$$

Solution. Note that this operation is just the same as adding fractions! $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$.

You are asked then to find $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{5}{6} + \dfrac{7}{8}$. This gives $\dfrac{71}{24}$, so that the answer is $\boxed{(71,24)}$.

5. Three equal circles are centred at the vertices of an equilateral triangle, with radii one half of the side length of the triangle.

   Determine the fraction of the area of the triangle not inside any of the three circles [exact answer please].



Solution. From the diagram, we see that we need the area of the triangle less three sectors of the circles with angles of 60°. The three sectors add up to a semicircle!

Let the side of the triangle be $2r$, so that the radius of each circle is r. The area of the semicircle is then $\pi r^2/2$.

The area of an equilateral triangle of side length $2r$ is $r^2\sqrt{3}$.

The answer is then $\dfrac{r^2\sqrt{3} - \pi r^2/2}{r^2\sqrt{3}} = \boxed{\dfrac{2\sqrt{3} - \pi}{2\sqrt{3}}}$ (or equivalent).

6. A team of three horses is hired to assist the contractor's horse in pulling a load of hay.

   The first and second hired horses are deemed to have done $\frac{2}{3}$ of the work; the second and third hired horses to have done $\frac{3}{8}$ of the work; and the first and third hired horses to have done $\frac{3}{10}$ of the work.

   The hired horses' owners are paid according to the fraction of the work they have done. What fraction of the work did the contractor's horse do? [Exact answer required.]

   Solution. Let the proportion of the work done by the three hired horses be x, y and z.

   We have

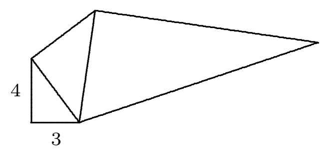
$$x + y = \frac{2}{3}, \qquad y + z = \frac{3}{8}, \qquad x + z = \frac{3}{10}.$$

   Adding gives $2(x + y + z) = \frac{2}{3} + \frac{3}{8} + \frac{3}{10} = \frac{322}{240}$, giving $x + y + z = \frac{161}{240}$.

   Thus, the contractor's horse has done $1 - \dfrac{161}{240} = \boxed{\dfrac{79}{240}}$ of the work.

7. A Heron Triangle is a right triangle whose sides are integers.

   Starting with the 3, 4, 5 Heron Triangle, construct a second Heron Triangle on the hypotenuse of the first Heron Triangle (as shown), and then a third Heron Triangle on the hypotenuse of the second Heron Triangle (as shown).



   Determine the length of the hypotenuse of the third Heron Triangle.

   Solution. The hypotenuse of the first Heron Triangle is 5.

   We now need a Heron Triangle with a leg of 5. There is only one: the 5, 12, 13 triangle. Hence, the length of the hypotenuse of the second Heron Triangle is 13.

We now need a Heron Triangle with a leg of 13. This can be determined in several ways. The worst is trial and error, which takes some time.

A preferable approach is to use the generating relationship for right triangles: the sides are given by $x^2 - y^2$, $2xy$, $x^2 + y^2$. (Check that this works! Become familiar with this if it is new to you.) Note that one leg of a Heron Triangle must be an even integer.

Thus, we have $13 = x^2 - y^2 = (x-y)(x+y)$. The only solution is $x = 7$, $y = 6$, resulting in the answer of $\boxed{85}$ for the hypotenuse $x^2 + y^2$ of the third Heron Triangle.

8. If the positive real numbers x, y and z are such that $xyz = 1$, express in the simplest possible form: $\dfrac{x+1}{xy+x+1} + \dfrac{y+1}{yz+y+1} + \dfrac{z+1}{zx+z+1}$.

Solution. Note that $\dfrac{x}{x} \times \dfrac{y+1}{yz+y+1} = \dfrac{xy+x}{xyz+xy+x} = \dfrac{xy+x}{xy+x+1}$, and that $\dfrac{xy}{xy} \times \dfrac{z+1}{zx+z+1} = \dfrac{xyz+xy}{x^2yz+xyz+xy} = \dfrac{1+xy}{x+1+xy} = \dfrac{1+xy}{xy+x+1}$.

Thus,

$$\begin{aligned} \frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{zx+z+1} &= \frac{(x+1)+(xy+x)+(1+xy)}{xy+x+1} \\ &= \frac{2(xy+x+1)}{xy+x+1} = \boxed{2}. \end{aligned}$$

9. If $a < b < c$ and

$$\begin{aligned} a+b+c &= 9\,, \\ ab+bc+ca &= 23\,, \\ abc &= 15\,, \end{aligned}$$

determine the three numbers a, b and c.

Solution. Consider

$$\begin{aligned} P(x) &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc \;=\; 0. \end{aligned}$$

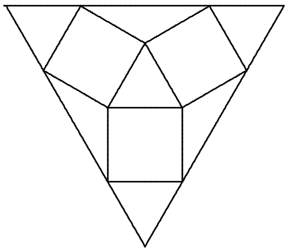
This is satisfied only when $x = a$, $x = b$ or $x = c$.

We have $P(x) = x^3 - 9x^2 + 23x - 15$. We need to factor this.

Note that $P(1) = 0$, giving $P(x) = (x-1)(x^2-8x+15) = (x-1)(x-3)(x-5)$.

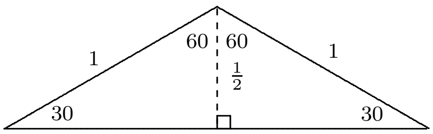
Hence, $\boxed{a = 1,\ b = 3 \text{ and } c = 5}$.

10. Three squares are drawn outwards on the sides of an equilateral triangle of side length 1. Lines are drawn on corresponding vertices of the square as shown forming an equilateral triangle. Calculate the length of a side of this triangle [exact answer please].



Solution. The triangles in the corners of the large equilateral triangle are themselves equilateral triangles of side length 1. We need then the length of the longest side of one of the three isosceles triangles with equal sides of length 1. The largest angle of such a triangle is

$$360 - 60 - 90 - 90 \ = \ 120 \text{ degrees.}$$



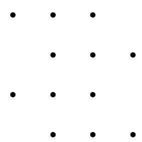
Here we have two 2, 1, $\sqrt{3}$ triangles — but half the side lengths! Hence, the length of the base of the whole triangle is $\sqrt{3}$.

Now, we must add in the lengths of the sides of the triangles at the vertices — each is of length 1.

Thus, the required length is $\boxed{2+\sqrt{3}}$.

RELAY

Relay 1. Let A be the maximum number of rectangles (including squares) that can be made by joining up the marked lattice points in the diagram below.



Write A in Box #1 of the Relay Answer Sheet.

Relay 2. A positive integer is said to be **perfect** if it is equal to the sum of its proper factors. For example, $6 = 1 + 2 + 3$ is perfect.

A positive integer is said to be **abundant** if it is less than the sum of its proper factors. For example, $12 < 1 + 2 + 3 + 4 + 6 = 16$ shows that 12 is abundant.

A positive integer is said to be **deficient** if it is greater than the sum of its proper factors. For example, $8 > 7 = 1 + 2 + 4$ shows that 8 is deficient.

Find the smallest perfect, abundant and deficient integers greater than A.

Call the sum of these three integers B, and write B in Box #2 of the Relay Answer Sheet.

Relay 3. A clock shows the correct time at 12 noon today. But the working of the clock is inaccurate, and for every true hour, the minute hand traverses B minutes.

Determine the number of hours before the minute hand is correctly at the "top of the hour" again.

Call your answer C, and write C in Box #3 of the Relay Answer Sheet.

Relay 4. Construct a sequence $\{X_n\}$ as follows:

Let X_1 be the integer obtained by starting with C, reversing the digits to obtain C', and then concatenating C and C'. For example, if it were true that $C = 73$, then $C' = 37$ and the concatentation is 7337. Or if it were true that $C = 20$, then $C' = 02$ and the concatenation is 2002.

Find the prime factors of X_1, find their sum, reverse the digits of the sum and concatenate as before to generate X_2. (N.B.: 1 is not a prime number.)

Repeat this process, generating X_3, X_4, etc.

At some stage you will discover that for some integers m and n, you have $X_m = X_n$. This means that the process is in a cycle.

Determine how many distinct numbers are in this cycle.

Write this number in Box #4 of the Relay Answer Sheet and hand the sheet to your proctor.

Answers : $\boxed{\boxed{14}, \boxed{61}, \boxed{60}, \boxed{3}.}$

Season 2004–2005

Game #1

1. Think of a number. Double it and subtract 1. Then square the result. Hold this new number.

   Take the original number and multiply it by the original number less 1. Multiply this quantity by 4 and subtract the result from the held number.

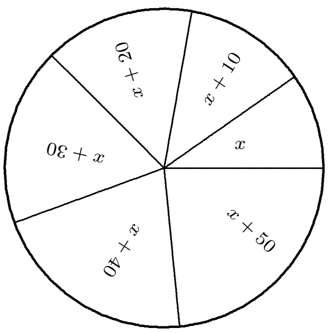
   What is the result?

   Solution. If your original number is x, we then get $2x$, and subsequently, $2x-1$. Squaring gives $(2x-1)^2 = 4x^2 - 4x + 1$.

   For the second part, we have $x \times (x-1) = x^2 - x$. Multiplying by 4 gives $4x^2 - 4x$, which, when subtracted from $4x^2 - 4x + 1$ leaves a result of $\boxed{1}$.

2. A circular stained glass window consists of six sectors. The central angles of the sectors form an arithmetic sequence, increasing by 10°. Calculate the central angle for the largest sector.

   Solution. Look at a diagram: Let the smallest angle be $x°$.

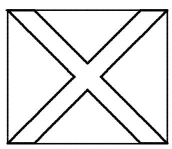


   We see that the sum of the angles is $6x + 150 = 360$, so that $x = 35$.

   Thus, the largest angle is

   $$35° + 50° = \boxed{85°}.$$

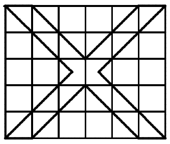
3. A large letter **X** is painted as shown in red on a 5 metre high, 6 metre long advertisement with white painted background.



   The horizontal and vertical breadths of the **X** are 1 metre.

   Red paint costs \$0.80 per square metre and white paint costs \$0.60 per square metre. Determine the cost of painting this advertisment.

Solution. We place a grid on the advertisment:



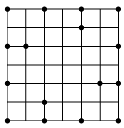
The letter **X** consists of 16 right triangles with legs of 1 and 1 (giving a total area of 8 square metres) and two other parts that are each $\frac{3}{4}$ of a square metre.

The total area of the letter X is then 9.5 square metres.

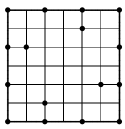
The area of the background is $5 \times 6 - 9.5 = 20.5$ square metres.

The total cost is $9.5 \times 0.80 + 20.5 \times 0.60 = \boxed{\$19.90}$.

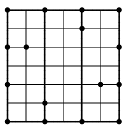
4. How many rectangles can be formed using a set of four dots on the grid as vertices?



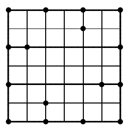
Solution.



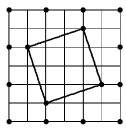
1



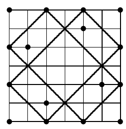
$3 + 2 = 5$



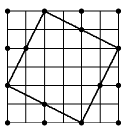
$3 + 2 = 5$



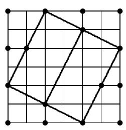
1



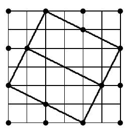
2



1



2



2

Total is $1 + 5 + 5 + 1 + 2 + 1 + 2 + 2 = \boxed{19}$.

5. Find all real solutions of $\sqrt{x-2} + x = 4$.

Solution. We must solve $\sqrt{x-2} = 4 - x$.

Squaring gives $x - 2 = 16 - 8x + x^2$ or $x^2 - 9x + 18 = (x-3)(x-6) = 0$.

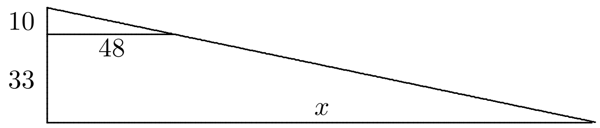
Possible solutions are $x = 6$ and $x = 3$.

Checking in the original equation shows that $x = 6$ is not a solution and that the only solution is $x = \boxed{3}$.

6. The top of a circular table, diameter 96 cm, is 33 cm above the floor of a large room. A candle, 10 cm tall, placed at the centre of the table top, is lit, and casts a shadow of the table top onto the floor.

   What is the horizontal distance (in cm) between the edge of the shadow cast by the candle and the centre of the table top?

   Solution. We draw a diagram:



   By similar triangles, we have $\frac{x}{48} = \frac{43}{10}$, giving

   $x = \frac{48 \times 43}{10} = \frac{2064}{10} = \boxed{206.4}$ cm.

7. An aircraft travels 240 kilometres from St. John's to Gander at 210 kmph, and then returns at 280 kmph. Determine the average speed in kmph for the complete round trip journey.

   Solution. The outward flight takes $\frac{240}{210}$ hours and the return trip takes $\frac{240}{280}$ hours. The total time is

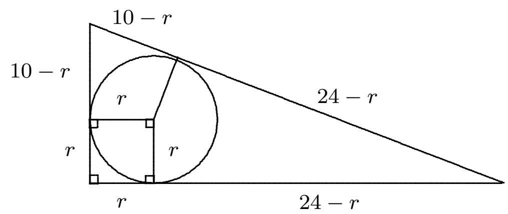
   $\frac{240}{210} + \frac{240}{280} = 240\left(\frac{1}{210} + \frac{1}{280}\right) = 240\left(\frac{1}{70}\right)\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{24}{7}\left(\frac{7}{12}\right) = 2$ hours.

   Hence, the average speed is $\frac{240 + 240}{2} = \boxed{240}$ kmph.

8. Determine the radius of the incircle (the circle that touches all three sides) of a triangle with sides 10, 24 and 26.

   Solution. The triangle is a right triangle. Note the square formed by two radii and parts of two legs.

   Since the lengths of both tangents from a point to a circle are equal, we have



   $(10 - r) + (24 - r) = 26$ or $r = \boxed{4}$.

9. Determine all real values of x such that $2x^2 - x - 36$ is the square of a prime number.

Solution. We require that $2x^2 - x - 36 = (x+4)(2x-9) = p^2$, where p is a prime number. There are several possibilities:

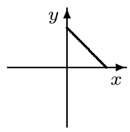
1.	$x+4=p^2$,	$2x-9=1$	$\Longrightarrow$	$x=5$	$\Longrightarrow$	$p=3$:	OK;
2.	$x+4=p$,	$2x-9=p$	$\Longrightarrow$	$x=13$	$\Longrightarrow$	$p=17$:	OK;
3.	$x+4=1$,	$2x-9=p^2$	$\Longrightarrow$	$x=-3$	$\Longrightarrow$	$p^2=-15$:	impossible;
4.	$x+4=-1$,	$2x-9=-p^2$	$\Longrightarrow$	$x=-5$	$\Longrightarrow$	$p^2=19$:	impossible;
5.	$x+4=-p$,	$2x-9=-p$	$\Longrightarrow$	$x=13$	$\Longrightarrow$	$p=-17$:	impossible;
6.	$x+4=-p^2$,	$2x-9=-1$	$\Longrightarrow$	$x=4$	$\Longrightarrow$	$p^2=-8$:	impossible.

The answer is then $\boxed{5 \text{ and } 13}$.

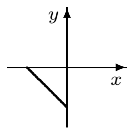
10. Determine the area of the set of points whose Cartesian coordinates satisfy the inequality:

$$|x| + |y| + |x+y| \le 2.$$

Solution. In the first quadrant, both $x \ge 0$ and $y \ge 0$, and the inequality becomes $x + y + (x+y) \le 2$, or $x + y \le 1$. This is the portion of the first quadrant bounded by the axes and the line $x + y = 1$.



In the third quadrant, $x \le 0$ and $y \le 0$, and the inequality becomes $-x - y + (-x - y) \le 2$, or $x + y \ge -1$. This is the portion of the third quadrant bounded by the axes and the line $x + y = -1$.

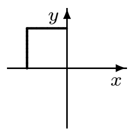


In the second quadrant, we have $x \le 0$ and $y \ge 0$. There are two cases to consider:

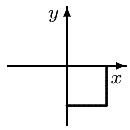
(a) $x + y \ge 0$: points above the line $y = x$. Then the inequality becomes $-x + y + (x+y) \le 2$; that is $y \le 1$.

(b) $x + y \le 0$; points below the line $y = x$. Then the inequality becomes $-x + y + (-x - y) \le 2$; that is $x \ge -1$.

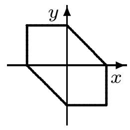
This gives a unit square in the second quadrant.



A similar argument holds in the fourth quadrant, giving



Putting these all together, we get



and it follows that the required area is $\boxed{3}$.

RELAY

Relay 1. The expression $\dfrac{14}{5x^2+16x+3}$ can be simplified to $\dfrac{P}{ax+b}+\dfrac{Q}{cx+d}$, where P, Q, a, b, c and d are integers.

Write $Z = a+b+c+d+P+Q$ in Box #1 of the Relay Answer sheet.

Relay 2. Find the sum of all the prime numbers that are less than or equal to Z.

Call your answer Y, and write Y in Box #2 of the Relay Answer sheet.

Relay 3. Chris has Z dollars and Pat has Y dollars. A group of their friends have an average of 200 dollars. After Chris and Pat join the group, the average is now 185 dollars.

How many friends are in the group (excluding Chris and Pat)? Call your answer X and write X in Box #3 of the Relay Answer sheet.

Relay 4. Consider the expression $xX + yY + zZ = 125$, where X, Y and Z are as above and x, y and z are positive integers.

Determine x, y and z, and let $W = x+y+z$.

Write W in Box #4 of the Relay Answer sheet, and hand the sheet to your proctor.

Answers : $\boxed{\boxed{14}, \boxed{41}, \boxed{21}, \boxed{6}.}$

Game #2

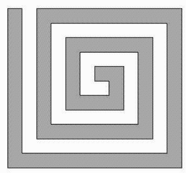
1. Given seven points in the plane, no three of which are collinear, determine the total number of straight lines that can be drawn through pairs of points.

   Solution. Choose any point and connect it to the six other points — this gives 6 lines.

   But we started with 7 points, so that there appear to be $6 \times 7 = 42$ lines.

   But each line is determined twice by this argument, so that there are in fact exactly $\frac{42}{2} = \boxed{21}$ lines.

2. The shaded path shown below is one unit wide. Determine the area of the path (in square units).



   Solution. $\boxed{77}$ square units.

3. Find all real solutions of the equation $11025x^6 - 1891x^4 + 83x^2 - 1 = 0$.

   Solution. Note that $11025 = 3^2\,5^2\,7^2$. We should therefore try $x = \pm\frac{1}{3}$, $x = \pm\frac{1}{5}$ and $x = \pm\frac{1}{7}$.

   And, lo and behold, they all work! Thus, the solutions are $\boxed{\pm\frac{1}{3},\ \pm\frac{1}{5} \text{ and } \pm\frac{1}{7}}$.

4. A group of students taking third year Geometry (PM3330) at MUN was split into three sections. The Dean has ruled that no class may have fewer than 20 students.

   There were 20 students in class G1, n students in class G2 and m students in class G3.

   They all wrote an identical mid-term examination, and the class averages were:

Section	Class Average
G1	80%
G2	70%
G3	60%

   The overall average was 69%.

   Determine the minimum number of students who were taking PM3330.

Solution. We have $69 = \dfrac{20 \times 80 + 70n + 60m}{20 + n + m}$.

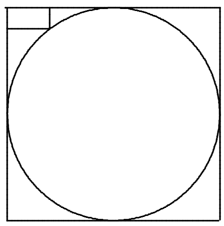
Solving for m gives $m = \dfrac{n}{9} + \dfrac{220}{9} = \dfrac{n}{9} + 24 + \dfrac{4}{9} = 24 + \dfrac{4+n}{9}$.

Now, $n \geq 20$ and m must be an integer, we have $n = 23, 32, \ldots$.

The smallest possible value of n is 23. This gives $m = 27$.

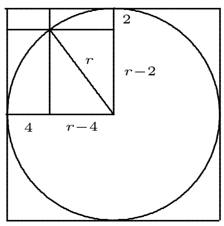
Hence, the least number of students taking PM3330 is $20 + 23 + 27 = \boxed{70}$.

5. A circle is inscribed in a square. A non-degenerate rectangle is drawn outside the circle with one vertex at a corner of the square, the opposite vertex on the circle, and sides parallel to the sides of the square.



If the sides of the rectangle are 2 cm and 4 cm, determine the radius of the circle in cm.

Solution. Draw the radii parallel to the sides of the rectangle, extend the sides of the rectangle to meet these radii and join the centre to the vertex of the rectangle on the circle. Let the radius of the circle be r.



From the right triangles in the diagram, we have

$$(r-4)^2 + (r-2)^2 = r^2,$$

or

$$r^2 - 12r + 20 = (r-10)(r-2) = 0.$$

But the solution $r = 2$ must be rejected since, then, $r - 2 = 0$, giving a degenerate rectangle.

Thus, the solution is $r = \boxed{10}$ cm.

6. Find the set of all real numbers that satisfy

$$\left(x^2 - 5x + 5\right)^{\left(x^2-9x+20\right)} = 1\,.$$

Solution. There are three cases to consider.

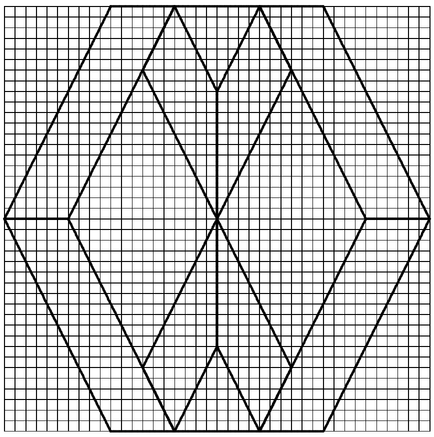
(a) The exponent has value zero: $x^2 - 9x + 20 = 0$; so that $x = 4$ or $x = 5$.

(b) The value of $x^2 - 5x + 5$ is 1: $x^2 - 5x + 5 = 1$, or $x^2 - 5x + 4 = 0$; so that $x = 1$ or $x = 4$.

(c) The value of $x^2 - 5x + 5$ is -1 and the exponent is an even positive integer: $x^2 - 5x + 5 = -1$, or $x^2 - 5x + 6 = 0$; so that $x = 2$ or $x = 3$; which give values of $x^2 - 9x + 20$ as 6 and 2, which are both even, as required.

Thus, the solution set consists of $\boxed{1,\ 2,\ 3,\ 4 \text{ and } 5}$.

7. The Williams-Moore Partnership has decided to have a stained glass logo made (in the shape of a hexagon): it is shown on a 1 cm × 1 cm grid below.



The **M** is made in red glass at a cost of \$225.00 per square metre.

The **W** is made in blue glass at a cost of \$175.00 per square metre.

The other four parts are made of green glass at a cost of \$125.00 per square metre.

Determine the cost of the stained glass logo.

Solution. Note, each of the **M** and **W** have the same area.

Each consists of two parallelograms and two trapezoids.

The area of a parallelogram is (like a rectangle) base × width.

The base is 6 cm and the height is 20 cm. The area is then 120 sq. cm. We have two of these.

The area of each trapezoid is best calculated as the sum of two triangles and a parallelogram by drawing two lines parallel to the x–axis.

We have a triangle of base 6 cm and height 6 cm, giving an area of 18 sq. cm; a parallelogram of base 6 cm and height 2 cm, giving an area of 12 sq. cm; and a (right) triangle of base 6 cm and height 12 cm, giving an area of 36 sq. cm. We have this twice.

Thus the total area of each letter is $2(120 + 18 + 12 + 36) = 372$ sq. cm.

For the remainder, we have two triangles of base 8 cm and height 8 cm, giving each an area of 32 sq. cm.

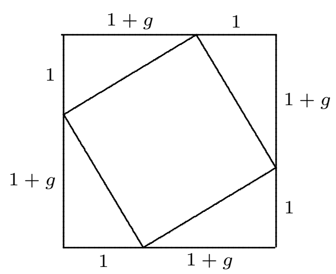
Also, two rhombi of diagonals 14 cm and 28 cm give an area of 196 sq. cm.

The total area of green glass is then $2(196 + 32) = 456$ sq. cm.

The cost of red glass is 2.25 cents per sq. cm; of blue glass, 1.75 cents per sq. cm; and of green glass, 1.25 cents per sq. cm.

This leads to a total cost of $\boxed{\$20.58}$ for the stained glass logo.

8. A square is placed symmetrically inside another square as shown:



There are four right triangles shown with legs of length 1 and $1 + g$ where g satisfies $g^2 + g = 1$.

Calculate the fraction of the area of the outer square that is occupied by the inner square. Express your answer in the form $p + q\sqrt{r}$, where p, q and r are integers.

Solution. Using Pythagoras' Theorem the area of the inner square is $(1 + g)^2 + 1^2 = g^2 + 2g + 2 = 3 + g$.

The area of the outer square is $(2 + g)^2 = g^2 + 4g + 4 = 5 + 3g$.

The required fraction is $\dfrac{3+g}{5+3g}$. Solving the equation for g gives $g = \frac{-1\pm\sqrt{5}}{2}$ (we must reject the negative root), so that $g = \frac{-1+\sqrt{5}}{2}$ or $2g = -1 + \sqrt{5}$.

Substituting gives the required fraction as

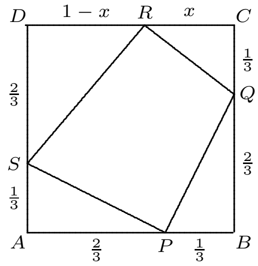
$$\begin{aligned}\frac{3+g}{5+3g} &= \frac{6+2g}{10+6g} = \frac{5+\sqrt{5}}{7+3\sqrt{5}} = \frac{(5+\sqrt{5})(7-3\sqrt{5})}{(7+3\sqrt{5})(7-3\sqrt{5})}\\ &= \frac{20-8\sqrt{5}}{49-45} = \boxed{5-2\sqrt{5}}.\end{aligned}$$

9. The points P, Q, R and S lie on the line segments AB, BC, CD and DA (sides of a square), respectively, such that

$$AP : PB \; = \; BQ : QC \; = \; DS : SA \; = \; 2 : 1\,.$$

Find the ratio $CR : RD$ such that the area of the quadrilateral $PQRS$ is one half of the area of the square $ABCD$.

Solution. Draw a diagram: without loss of generality, let the side of the square be 1. Let $CR = x$ so that $RD = 1 - x$.



If the area of the quadrilateral $PQRS$ is to be $\frac{1}{2}$, then the sum of the areas of the four triangles must also be $\frac{1}{2}$. Thus,

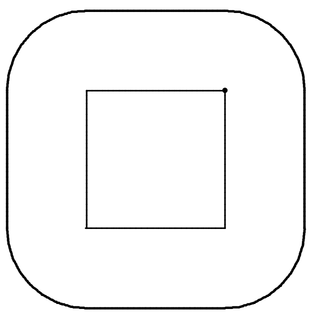
$$\frac{1}{2}\left(\frac{1}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot x+\frac{2}{3}\cdot(1-x)\right) \; = \; \frac{1}{2},$$

so that $x = \frac{1}{3}$. Thus, the answer is $CR : RD = \boxed{1 : 2}$.

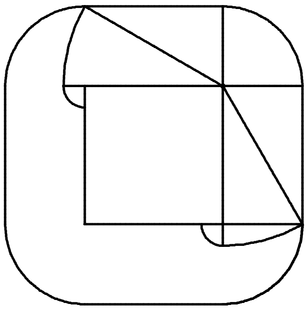
10. The new mathematics tower building at MUN has a square footprint of side $20\sqrt{3}$ metres. During construction, a fence was built around it at a distance of 20 metres from any point on the outside of the building.

To deter vandals, a guard dog was placed inside the fence on a lead 40 metres long, attached to one corner of the building.

Determine (correct to one decimal place) the percentage of the area around the building that the dog can reach.



Solution. We divide the area that the dog can reach into several parts.



Note that the area of a sector of a circle of radius r and angle θ is $\frac{\theta}{2}r^2$.

The parts of the area that the dog can reach are:

Item	Area	Number
A quarter circle, radius $20(2-\sqrt{3})$	$400(2-\sqrt{3})^2\pi/4$	2
One twelfth of a circle, radius 40	$40^2\pi/12$	2
A triangle with legs $20\sqrt{3}$ and 20	$200\sqrt{3}$	2
A quarter circle, radius 20	$400\pi/4$	1

Thus, the total area that the dog can reach is

$$\frac{5300\pi}{3} + 400\sqrt{3} - 800\pi\sqrt{3}.$$

The total area inside the fence and outside the building consists of four rectangles, sides $20\sqrt{3}$ and 20 plus four quarter circles, radius 20, giving a total area of $1600\sqrt{3} + 400\pi$.

The ratio gives $46.9187\ldots\%$, so that the answer is $\boxed{46.9}\%$.

RELAY

Relay 1. A square S and a rectangle A each have an area of 144 sq. cm. The ratio of the perimeter of the rectangle R to that of the square S is $5:4$. For the rectangle R, the ratio of the length to the width is $4:1$. If the length of the rectangle R is L cm., determine the value of L.

Write the value of L in Box #1 of the Relay Answer sheet.

Relay 2. The digits in the number 213 sum to $2+1+3$, or 6.

How many three digit numbers have digits that sum to L?

Call your answer H and write the value of H in Box #2 of the Relay Answer sheet.

Relay 3. The first three terms of a Fibonacci sequence are given as 21, 46, 67. Successive terms are found by adding the two preceding terms. For example, $67 = 21 + 46$.

Determine the H^{th} term of the sequence and call this term F.

Relay 4. Write the value of F in Box #3 of the Relay Answer sheet.

The letters A, B, C, and D represent different digits such that $F = DBA \times C$.

Determine the value of the product $CBA \times D$ and call this value M.

Write the value of M in Box #4 of the Relay Answer sheet, and hand the sheet to your proctor.

(Note: DBA and CBA represent three digit numbers.)

Answers : $\boxed{\boxed{24}, \boxed{10}, \boxed{2005}, \boxed{2004}.}$

Game #3

1. Find all triples of positive integers (x, y, z) that satisfy

$$\begin{aligned} x + y + z &= 10\,, \\ x + 2y + 3z &= 15\,. \end{aligned}$$

Solution. Subtracting gives $y + 2z = 5$.

There are only two solutions of this equation in positive integers: $y = 1$, $z = 2$ and $y = 3$, $z = 1$.

The answer is then $\boxed{(7,1,2) \text{ and } (6,3,1)}$.

2. On the planet Ecurb, populated by huwoman beings, they have only one arm with five fingers. Accordingly, they have adopted a number system based on 5. A huwoman on Ecurb stated that she had 1234 relatives. Express this in terms of the human base 10 number system.

Solution. We take

$$\begin{aligned} &(1 \times 5^3) + (2 \times 5^2) + (3 \times 5^1) + (4 \times 5^0) \\ &= \ 1 \times 125 + 2 \times 25 + 3 \times 5 + 4 \times 1 \\ &= \ 125 + 50 + 15 + 4 \ = \ \boxed{194}\,. \end{aligned}$$

3. Determine the number of ordered pairs (x, y) of positive integers with the property $x^2 - y^2 = 105$.

Solution. We have $x^2 - y^2 = (x + y)(x - y) = 105 = 1 \times 3 \times 5 \times 7$. Remember that $x + y > x - y$.

We have then the possibilities:

$$\left.\begin{aligned} x + y &= 105 \\ x - y &= 1 \end{aligned}\right\} x = 53\,,\ \ y = 52\,; \qquad \left.\begin{aligned} x + y &= 35 \\ x - y &= 3 \end{aligned}\right\} x = 19\,,\ \ y = 16\,;$$

$$\left.\begin{aligned} x + y &= 21 \\ x - y &= 5 \end{aligned}\right\} x = 13\,,\ \ y = 8\,; \qquad \left.\begin{aligned} x + y &= 15 \\ x - y &= 7 \end{aligned}\right\} x = 11\,,\ \ y = 4.$$

Hence, there are $\boxed{4}$ sets of ordered pairs.

4. By the symbol $\lfloor x \rfloor$, we mean the largest integer, less than or equal to x.

Solve for t, $\sum_{n=1}^{t} \left\lfloor \frac{n}{5} \right\rfloor \ = \ 42$.

Solution. Note that, for $1 \le n \le 4$, we have $\left\lfloor \frac{n}{5} \right\rfloor = 0$.

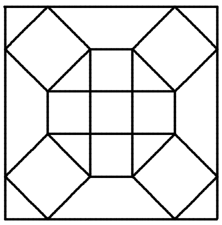
For $5 \le n \le 9$, we have $\left\lfloor \frac{n}{5} \right\rfloor = 1$, which, when summed, gives 5.

For $10 \leq n \leq 14$, we have $\left\lfloor \frac{n}{5} \right\rfloor = 2$, which, when summed, gives 10, and a total of 15.
For $15 \leq n \leq 19$, we have $\left\lfloor \frac{n}{5} \right\rfloor = 3$, which, when summed, gives 15, and a total of 30.
For $20 \leq n \leq 24$, we have $\left\lfloor \frac{n}{5} \right\rfloor = 4$, which, when summed, gives 20, and a total of 50, which is 8 too much.

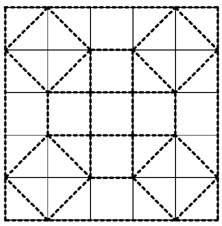
This means that we have taken 2 terms too many. Hence the solution is $t = \boxed{22}$.

5. Squares are drawn outwards on the sides of a square of side length 1. Vertices are then joined to form an octagon as shown. Squares are drawn outwards on the new sides of the octagon as shown. Lines are drawn through the outer points on these squares, forming a large square.

   Find the side length of the large square.



Solution. Extend the sides of the original square to the sides of the large square. Do the same to the sides of the octagon that are parallel to the sides of the original square.



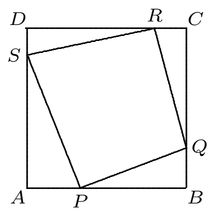
It is now clear that the side length of the large square is $\boxed{5}$.

6. The four sides of square $ABCD$ (AB, BC, CD, and DA) are divided at P, Q, R and S, respectively, in such a way that

$$\frac{AP}{PB} = \frac{1}{2}, \qquad \frac{BQ}{QC} = \frac{1}{3}, \qquad \frac{CR}{RD} = \frac{1}{4} \quad \text{and} \quad \frac{DS}{SA} = \frac{1}{5}.$$

   If the original square has side length 1, calculate the exact area of quadrilateral $PQRS$.

Solution. We draw a diagram



We have that $AP = \frac{1}{3}$, $PB = \frac{2}{3}$, $BQ = \frac{1}{4}$, $QC = \frac{3}{4}$, $CR = \frac{1}{5}$, $RD = \frac{4}{5}$, $DS = \frac{1}{6}$ and $SA = \frac{5}{6}$.

Thus, the areas of $\triangle APS$, $\triangle BPQ$, $\triangle CQR$ and $\triangle DRS$ are $\frac{1}{2} \times \frac{1}{3} \times \frac{5}{6} = \frac{5}{36}$, $\frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} = \frac{1}{12}$, $\frac{1}{2} \times \frac{1}{5} \times \frac{3}{4} = \frac{3}{40}$ and $\frac{1}{2} \times \frac{1}{6} \times \frac{4}{5} = \frac{1}{15}$, respectively.

Thus, the required area is $1 - \frac{5}{36} - \frac{1}{12} - \frac{3}{40} - \frac{1}{15} = \boxed{\frac{229}{360}}$.

7. The MUN Math and Stat Soc students took part in a special race at the Royal St. John's Regatta last year. They rowed their boat at 12 kmph and the course was 1600 metres long. The wind sped them up by 5 kmph in the first half of the course and slowed them down by the same amount during the second half. How long (correct to the nearest second) did it take them to row the course?

Solution. Suppose that the course is $2d$ km long, that the rowers' speed is v kmph and that wind speed is w kmph.

From the magic formula: **DIST** (**D**istance **I**s **S**peed times **T**ime), we have that the total time is

$$t = \frac{d}{v+w} + \frac{d}{v-w} = \frac{2vd}{v^2 - w^2}.$$

In our case, $d = 0.8$ (remember, keep your units straight), $v = 12$ and $w = 5$.

This gives $t = \frac{96}{595}$ hours, or $\boxed{\text{9 minutes, 41 seconds}}$, or $\boxed{\text{581 seconds}}$.

8. A four digit number, x, is of the form $ABAB$, and a three digit number, y, is of the form ABA.

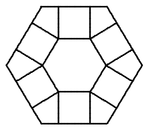
If $x - y = 1364$, determine A and B.

Solution. $x = 1000A + 100B + 10A + B$ and $y = 100A + 10B + A$, giving $x - y = 1000A + 100(B - A) + 10(A - B) + (B - A) = 1364$.

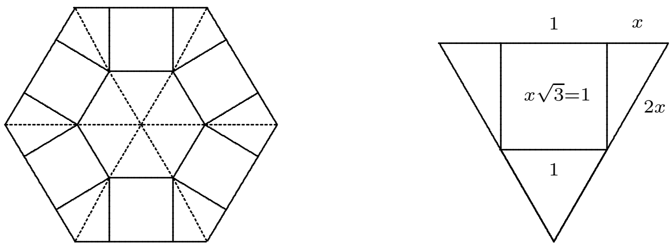
Thus, $A = 1$, and
$x - y = 1000 + 100(B - 1) + 10(1 - B) + (B - 1) = 909 + 91B = 1364$.

Hence, $\boxed{A = 1 \text{ and } B = 5}$.

9. Squares are drawn outwards on the sides of a regular hexagon with side length 1. The outer sides of the squares are extended to form a larger regular hexagon. Calculate the length of the side of the larger regular hexagon [exact answer please].



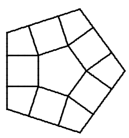
Solution. We need only consider one side of the new hexagon and join the vertices to the centre. The two triangles formed at each vertex are each one half of an equilateral triangle; that is, they are 2, 1, $\sqrt{3}$ triangles.



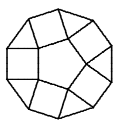
We see that $x = \frac{1}{\sqrt{3}}$, so that the length of the side of the larger regular hexagon is $1 + 2x = \boxed{1 + \frac{2}{\sqrt{3}}}$ or $\boxed{1 + \frac{2\sqrt{3}}{3}}$ or $\boxed{\frac{3+2\sqrt{3}}{3}}$.

10. Squares are drawn outwards on the sides of a regular pentagon of side length 1. Two polygons are formed thus:

 (a) extend the sides of the squares opposite to the sides of the pentagon to meet at the nearest points forming a (regular) pentagon P;

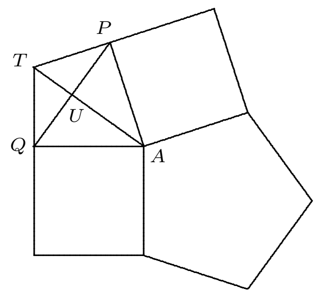


 (b) join the vertices of the sides of the squares opposite to the sides of the original pentagon to form a decagon D.



Find (correct to three decimal places) the difference between the perimeter of D and the perimeter of P.

Solution. Draw a diagram showing the situation at the vertices of two of the squares:



Since AT passes through the circumcentre of the regular pentagon, it follows, by symmetry, that $PQ \perp AT$. Let the point of intersection be U.

The interior angle of a regular pentagon is 108°; hence, $\angle PAQ = 72°$ and, by symmetry, $\angle PAT = 36°$.

Let $PT = x$ and $PU = y$. Since $PA = 1$, we have $PU = \sin(36°)$ and $PT = \tan(36°)$. The difference, $PT - PU$ is one tenth of the required quantity.

The quantity that we want is then $10\,(\tan(36°) - \sin(36°))$, which, by calculator is, correct to three decimal places, $\boxed{1.388}$.

RELAY

Relay 1. A sequence of **prime** numbers is called *good* if each term (after the first one) is obtained by adding one of the numbers 2, 3 or 4 to its predecessor. For example, 37, 41, 43, 47 is a good sequence of prime numbers.

Let A be the number of terms in the longest possible sequence of good prime numbers that starts with 2. NOTE: in counting the number of terms in a sequence, you must include the first term.

Determine the value of A and write that value in Box #1 of the Relay Answer sheet.

Relay 2. For which positive value of B does the equation

$$Ax^2 + Bx + 2 = 0$$

have two identical roots?

Write the value of B in Box #2 of the Relay Answer sheet.

Relay 3. Find the value of C that satisfies the equation

$$3 = \sqrt{41 - B\sqrt{22 - \sqrt{C}}}.$$

Write the value of C in Box #3 of the Relay Answer sheet.

Relay 4. An **arithmetic sequence** consists of members

$$a, \ a+d, \ a+2d, \ a+3d, \ \dots ,$$

where a and d are fixed numbers. For example, if $a = 2$ and $d = 7$, the corresponding arithmetic sequence is

$$2, \ 9, \ 16, \ 23, \ 30, \ \dots .$$

Find the number E such that

$$E + 26, \ C + 2E + 4, \ C + E + 2$$

are the fifteenth, sixteenth and seventeenth terms of an arithmetic sequence.

Write the value of E in Box #4 of the Relay Answer sheet, and hand the sheet to your proctor.

Answers : $\boxed{\boxed{8}, \boxed{8}, \boxed{36}, \boxed{-8}.}$

Game #4

1. An operation $\star$ is defined such that $A \star B$ equals the lowest common multiple of A and B. For example, $30 \star 12 = 60$.

   Evaluate $(5 \star 6) \star (8 \star 20)$.

   Solution.

   $5 \star 6 = 5 \star (2 \times 3) = (2 \times 3 \times 5) = 30$ and $8 \star 20 = 2^3 \star (2^2 \times 5) = (2^3 \times 5) = 40$. We now need $30 \star 40$.

   This is $(2 \times 3 \times 5) \star (2^3 \times 5) = (2^3 \times 3 \times 5) = \boxed{120}$.

2. Each of the digits 1, 2, 3, 4, 5, 6, and 7 must be used once to form a sum of three positive integers, as shown. (For example, possibilities would be $2536 + 12 + 7$ or $24 + 53 + 716$.)

   $$\underline{\qquad\quad} + \underline{\qquad\quad} + \underline{\qquad\quad}$$

   Determine the smallest total that can be obtained.

   Solution.

   We want to keep the digits in the leftmost places as small as possible.

   Thus, 5, 6 and 7 are in the units positions; 2, 3 and 4 are in the tens positions; leaving 1 in the hundreds positions.

   No matter how these are combined, the total is $\boxed{208}$.

3. The digits a, b, and c are distinct (unequal) non-zero digits. How many ordered triples (a, b, c) would complete the following sum correctly?

   $$\begin{array}{ccc} & b & c \\ + & a & b \\ \hline 1 & 0 & 0 \end{array}$$

   Solution.

   We must have $b + c = 10$ and $b + a = 9$. We make a table:

b	1	2	3	4	5	6	7	8	9
c	9	8	7	6	5	4	3	2	1
a	8	7	6	5	4	3	2	1	0

   The possibilities $(a, b, c) = (5, 5, 4)$ and $(9, 1, 0)$ are rejected (in the first, the integers are not distinct, and in the second, we have $a = 0$). Thus, there are $\boxed{7}$ triples.

4. [NO CALCULATORS]

   A fraction of the form a/b is equivalent to $2/7$. If $b - a = 2005$, determine the value of $(a + b)$.

   Solution.

   We start with $\frac{b}{a} = \frac{7}{2}$. This gives $7a = 2b = 2(a + 2005)$, so that $a = 802$.

   Then, $a + b = a + (a + 2005) = 802 + (802 + 2005) = \boxed{3609}$.

5. Define $f(x) = \frac{x}{x+2}$; $f^{(2)}(x) = f(f(x))$; $f^{(3)}(x) = f\left(f^{(2)}(x)\right)$; ..., $f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)$.

   Determine $f^{(n)}(x)$ in terms of x, n and some integers.

   Solution. $f^{(2)}(x) = f(f(x)) = f\left(\frac{x}{x+2}\right) = \frac{\frac{x}{x+2}}{\frac{x}{x+2} + 2} = \frac{x}{3x+4}$.

   With a similar calculation, we get

   $f^{(3)}(x) = \frac{x}{7x+8} = \frac{x}{(8-1)x+8} = \frac{x}{(2^3-1)x+2^3}$,

   $f^{(4)}(x) = \frac{x}{15x+16} = \frac{x}{(16-1)x+16} = \frac{x}{(2^4-1)x+2^4}$.

   This establishes a pattern leading to $f^{(n)}(x) = \boxed{\frac{x}{(2^n-1)x+2^n}}$.

6. A regular pair of dice is tossed. What is the probability that the total is one less than a prime number?

   Solution.

   The possible outcomes are 2 through 12, eleven possible numbers. But 12 can only be obtained in $\underline{1}$ way; 11 in $\underline{2}$ ways; 10 in $\underline{3}$ ways; 9 in $\underline{4}$ ways; 8 in $\underline{5}$ ways; 7 in $\underline{6}$ ways; 6 in $\underline{5}$ ways; 5 in $\underline{4}$ ways; 4 in $\underline{3}$ ways; 3 in $\underline{2}$ ways; and 2 in $\underline{1}$ way.

   The total number of outcomes is $1+2+3+4+5+6+5+4+3+2+1 = 36$.

   The acceptable results in this range are $2 = 3 - 1$, $4 = 5 - 1$, $6 = 7 - 1$, $10 = 11 - 1$ and $12 = 13 - 1$.

   Now, 2 comes in $\underline{1}$ way; 4 comes in $\underline{3}$ ways; 6 comes in $\underline{5}$ ways; 10 comes in $\underline{3}$ ways; and 12 comes in $\underline{1}$ way.

   The number of favourable outcomes is $1 + 3 + 5 + 3 + 1 = 13$.

   The answer is $\boxed{\frac{13}{36}}$.

7. Determine all triples (x, y, z) that are solutions to the equations

$$x + 2y + 4z = 0, \quad (1)$$
$$x + yz + 4 = 0, \quad (2)$$
$$2x - 2y - z = 0. \quad (3)$$

Solution.

First, verify that $z = 0$ does not lead to a solution.

Add (1) and (3) to get $3x + 3z = 0$, giving $x = -z$.

Substitute into (2) to get $-z + yz + 4 = 0$, giving $y = \frac{z-4}{z}$.

Finally, substitute into (1), to get

$$-z + 2\left(\frac{z-4}{z}\right) + 4z \;=\; 0.$$

This gives

$$\frac{3z^2 + 2z - 8}{z} \;=\; 0.$$

From $3z^2 + 2z - 8 = 0$, we get $z = \frac{4}{3}$ or $z = -2$, leading to the TWO triples: $(x, y, z) = \boxed{\left(-\frac{4}{3}, -2, \frac{4}{3}\right)}$ AND $\boxed{(2, 3, -2)}$.

8. Given that $\log 2 = A$, $\log 3 = B$, and $\log 5 = C$, express $\log(9.6)$ in terms of all of A, B, and C.

Solution.

$9.6 = \frac{96}{10} = \frac{2^5 \cdot 3}{2 \cdot 5} = \frac{2^4 \cdot 3}{5}$, so that

$\log(9.6) = \log\left(\frac{2^4 \cdot 3}{5}\right) = 4\log 2 + \log 3 - \log 5 = \boxed{4A + B - C}$.

Comment. Since $A + C = \log 2 + \log 5 = \log 10 = 1$, there are other possible correct answers.

9. A lattice point is a point (x, y) where x and y are both integers. For example, $(2, -9)$ is a lattice point, whereas, $(5, 1/3)$ is not.

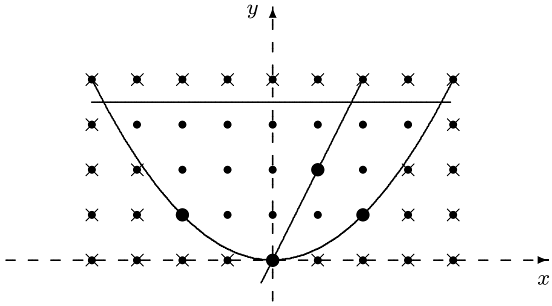
When the graphs of $4y = x^2$, $2x - y = 0$, and $y = 7/2$ are drawn, two bounded regions are formed.

How many more lattice points are in one region than are in the other region? (Note: do not count boundary points.)

The required answer is the absolute value of the difference between the numbers of lattice points.

Solution.

We draw a graph showing the curves and the lattice points:



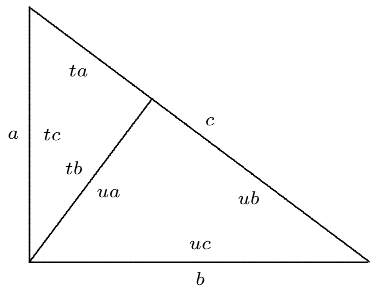
Of the lattice points shown, there are 4 lattice points on the boundary of the regions (shown as larger dots — ignore them), and those outside the regions are crossed out (ignore them).

There are 10 lattice points inside the left region (including 3 points on the y–axis) and 4 inside the right region.

Thus, the answer is $|10 - 4| = \boxed{6}$.

10. The two shorter sides of a right angled triangle are of lengths a and b, respectively. Determine the shortest distance, from the intersection of the two shorter sides to the hypotenuse. Give your answer in terms of a and b.

Solution. We draw a right triangle. The required distance is the length of the altitude from the right angle to the hypotenuse. The two smaller triangles formed are similar to the original triangle. Therefore, their sides are in proportion. See the diagram.



From the legs of the original triangle, we have $a = tc$ and $b = uc$, giving $t = \frac{a}{c}$ (all we need!).

The length of the altitude is then $tb = \frac{ab}{c} = \boxed{\frac{ab}{\sqrt{a^2 + b^2}}}$.

RELAY

Relay 1. The perimeter of a square of area 12.25 cm^2 equals the circumference of a circle with radius L/π cm.

Write the value of L in Box #1 of the Relay Answer sheet.

Relay 2. The coordinates of an isosceles right triangle are $A = (5, L)$, $B = (L, 9)$, and $C = (x, y)$. If AB is the hypotenuse, find the largest possible value of the product xy and call this value M.

Write the value of M in Box #2 of the Relay Answer sheet.

Relay 3. The mean scores of three classes are summarized here:

	Number of students in class	Mean score
Class #1	N	P
Class #2	$2N$	45
Class #3	$3N$	51

The overall mean score was M. Determine the value of P.

Write the value of P in Box #3 of the Relay Answer sheet.

Relay 4. Find the smallest positive integer Q for which the product $6 \times P \times Q$ is a perfect square.

Write the value of Q in Box #4 of the Relay Answer sheet, and hand the sheet to your proctor.

Answers : $\boxed{\boxed{7}, \boxed{49}, \boxed{51}, \boxed{34}.}$

Championship Game

1. A large container weighs 1 kilo. When filled with water, the total weight is now 100 kilos.

   After some time, some water has evaporated, resulting in the water consisting of 95% of the total weight.

   What weight of water has evaporated?

   Solution. Denote the number of kilos of water that have evaporated by x.

   We started with 99 kilos of water, so that we now have $(99 - x)$ kilos of water. This is 95% of the total weight, which is now $(100 - x)$ kilos.

   Thus, we need to solve

   $$99 - x \;=\; (0.95)(100 - x).$$

   The solution to this equation is $x = \boxed{80 \text{ kilos}}$.

2. A twidget and a sodjet weigh the same as three cohurts. A sodjet weighs the same as three hootics and one cohurt. Three twidjets weigh the same as one cohurt and one hootic. How many hootics weigh the same as a sodjet?

   Solution. Let t, s, c, h be the respective weights of the twidjet, sodjet, cohurt and hootic.

   We are given:

   (1) $t + s = 3c$; (2) $s = 3h + c$; (3) $3t = c + h$.

   From these three equations, we get (4) $s = 2h + (h + c) = 2h + 3t$.

   Then, $t + s = 3c = 3(s - 3h) = 3s - 9h$, giving (5) $2s = 9h + t$.

   From $3\times(5)-(4)$, we get $5s = 25h$, or $s = 5h$.

   The answer is then $\boxed{5}$.

3. A unique number x can be expressed in the form

   $$x \;=\; \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}}.$$

   Determine x.

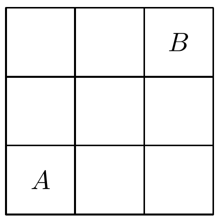
Solution. What we need to solve is the equation

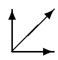
$$x = \sqrt{6+x}.$$

Squaring gives $x^2 = 6 + x$, or $0 = x^2 - x - 6 = (x-3)(x+2)$.

This gives $x = 3$ or $x = -2$ (which must be rejected since the right side is positive). Thus, the answer is $x = \boxed{3}$.

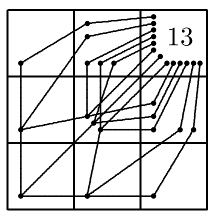
4. Moves are made on a 3×3 grid. The lower left square is the starting square A and the upper right square is the finishing square B.



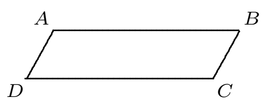
Moves to adjoining squares are permitted only in three directions 

Calculate the total number of paths to move from A to B.

Solution.

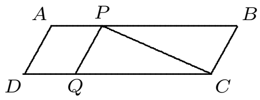


5. A parallelogram $ABCD$ has $AB = 70$ cm and height 18 cm.



P is a point on AB such that PC divides the area of the parallelogram in the ratio $4 : 3$. Determine the length AP.

Solution. Draw the lines PC and PQ, where $PQ \| AD$ and Q lies on CD.



Note that $\dfrac{[APQD]+[PCQ]}{[PCQ]} = \dfrac{4}{3}$, which implies that $3[APQD] = [PCQ]$. This gives $6[APQD] = 2[PCQ] = [PBCQ]$. Thus, $AP : PB = 1 : 6$.

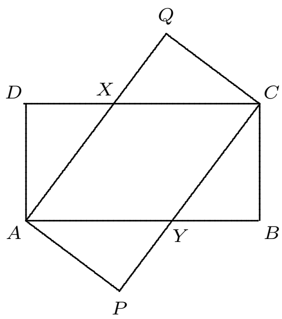
Therefore $AP = \boxed{10}$.

6. Find all real factors of the quintic $x^5 + 3x^4 + 9x^3 + 27x^2 + 81x + 243$.

   Solution.

$$\begin{aligned} x^5 + 3x^4 + 9x^3 + 27x^2 + 81x + 243 &= x^3(x^2+3x+9) + 27(x^2+3x+9) \\ &= (x^3+27)(x^2+3x+9) \\ &= \boxed{(x+3)(x^2-3x+9)(x^2+3x+9)}. \end{aligned}$$

7. Two congruent rectangles $ABCD$ and $AQCP$ have
   $AB = CD = AQ = PC = 8$ and $AD = BC = AP = CQ = 4$.
   If AQ and DC intersect at X and AB and PC intersect at Y, determine the area of quadrilateral $AXCY$.



Solution. Since, by symmetry, quadrilateral $AXCY$ is a rhombus of width 4, we need to find the length of XC.

The four right angled triangles outside the rhombus are all congruent. Hence $AX = XC = AY = YC$.

Let $DX = t$, then $XC = AX = 8 - t$ and $AD = 4$. Thus,

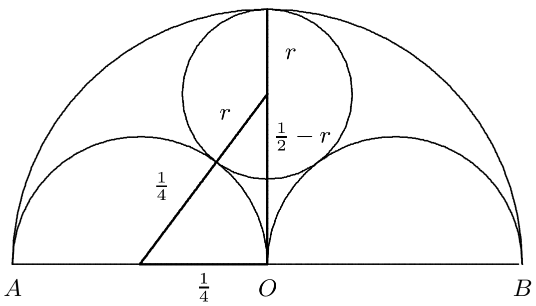
$$t^2 + 4^2 = t^2 + 16 = (8-t)^2 = t^2 - 16t + 64\,,$$

yielding $t = 3$, so that the side of the rhombus is $8 - t = 5$.

The area of $AXCY$ is $5 \times 4 = \boxed{20}$.

8. A semicircle of diameter $AB = 1$ has two equal semicircles inside it with diameters $AO = OB = \dfrac{1}{2}$. A circle of radius r is internally tangent to the first semicircle and externally tangent to the other two semicircles. Determine the value of r.

   Solution. Draw a diagram.



   We have a right triangle of hypotenuse $\left(r+\frac{1}{4}\right)$ and legs of $\left(\frac{1}{2}-r\right)$ and $\left(\frac{1}{4}\right)$. Pythagoras' Theorem gives us

$$\left(r+\frac{1}{4}\right)^2 = \left(\frac{1}{2}-r\right)^2 + \left(\frac{1}{4}\right)^2 ;$$

   that is, $r^2 + \frac{r}{2} + \frac{1}{16} = r^2 - r + \frac{5}{16}$, so that $r = \boxed{\frac{1}{6}}$.

9. At a M.U.N. reunion, I met fifteen former classmates (men and women). More than half were doctors, and the rest all practiced law. Of the former, more were females. More abundant still were women with a law degree. These statements are still true if you include me. If my friend, a noted lawyer, had left his wife and children home, how many male doctors were present?

   Solution. A chart can help with this question where the Male, Female and Totals are placed along the top row and the Doctor, Lawyer options are placed in the vertical columns.

   Here is a "verbal" solution.

   Notice that the first four lines do not include the speaker since they stated in line (1) that the speaker "met fifteen former classmates", thus, implying that the number of doctors is between 8 and 14, while the number of lawyers is between 1 and 7.

   Of the doctors, more were women but more women had law degrees so there must be at least 5 female doctors and 6 or more female lawyers.

   There is at least 1 male lawyer (the speaker's friend) so that, not including the speaker, there must be 6 lawyers that are female.

   This restricts the number of female doctors to 5 and the number of male doctors to $\boxed{3}$ in order to keep a total of 15.

10. Suppose that a, b and c are digits in the range 1 through 9.

The symbol xy means a two digit number and not the product of x and y.

Given two two digit numbers ab and bc and that a ratio of them is equal to b, determine all possible triples (a, b, c) with this property.

Solution. We are asked to find a, b, $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, such that either $\frac{10a+b}{10b+c} = b$ or $\frac{10b+c}{10a+b} = b$.

One method of solution is to first solve for a in terms of b and c. Algebraically, we have two cases: $a = b^2 + \frac{b(c-1)}{10}$ or $a = 1 - \frac{b}{10} + \frac{c}{10b}$.

If $a = b^2 + \frac{b(c-1)}{10}$, we have $b \in \{1, 2, 3\}$ and $b(c-1)$ is a multiple of 10.

This leads to $\begin{cases} \text{(i)} & b = 1, \quad c = 1, \quad a = 1; \\ \text{(ii)} & b = 2, \quad c = 1, \quad a = 4; \\ \text{(iii)} & b = 2, \quad c = 6, \quad a = 5; \\ \text{(iv)} & b = 3, \quad c = 1, \quad a = 9. \end{cases}$

If $a = 1 - \frac{b}{10} + \frac{c}{10b} = 1 + \frac{(c-b^2)}{10b}$, we have $a = 1$ and $b^2 = c$.

This leads to $\begin{cases} \text{(v)} & a = 1, \quad b = 1, \quad c = 1 \text{ (same as in (i) above)}; \\ \text{(vi)} & a = 1, \quad b = 2, \quad c = 4; \\ \text{(vii)} & a = 1, \quad b = 3, \quad c = 9. \end{cases}$

Thus, the answers are:

$\boxed{\boxed{(1,1,1)}, \boxed{(4,2,1)}, \boxed{(5,2,6)}, \boxed{(9,3,1)}, \boxed{(1,2,4)}, \boxed{(1,3,9)}}$.

RELAY

Relay 1. The point $(1, 2)$ is reflected in the line $y = x$. The resulting point is reflected in the x–axis. The new resulting point is reflected in the line $x = 5$. This leads to a point with coordinates (a, b).

Write the value of $M = a + b$ in Box #1 of the Relay Answer sheet.

Relay 2. Consider a right angled triangle AB_1B_2, with $AB_1 = M$, $B_1B_2 = 10$ and $\angle AB_1B_2 = 90°$.

We then add a second right angled triangle AB_2B_3 with $B_2B_3 = 10$ and $\angle AB_2B_3 = 90°$.

We then add a third right angled triangle AB_3B_4 with $B_3B_4 = 10$ and $\angle AB_3B_4 = 90°$. And so on. Given that the distance AB_N is $\sqrt{700 + M^2}$, determine the value of N.

Write the value of N in Box #2 of the Relay Answer sheet.

Relay 3. We define

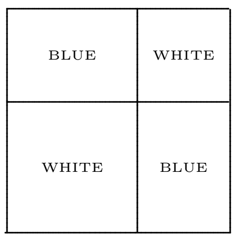
$$F(k) = \frac{1+k+k^2}{1+k} - \cfrac{1}{2 - \cfrac{1}{\left(1 - \frac{1}{1-\frac{1}{k}}\right)}},$$

$$F^{(2)}(k) = F(F(k)), \quad F^{(3)}(k) = F(F(F(k))), \quad \text{etc.}$$

Determine the value of $P = F^{(2005)}(N)$.

Write the value of P in Box #3 of the Relay Answer sheet.

Relay 4. The floor of a square room is to be tiled with square tiles (whose sides have length 1 unit) as illustrated below:



Both white sections are squares. The larger white section has sides that are P units longer than the sides of the smaller white section. If 232 white tiles are needed in all, how many blue tiles are required?

Write your answer in Box #4 of the Relay Answer sheet, and hand the sheet to your proctor.

Answers : $\boxed{\boxed{7}, \boxed{8}, \boxed{8}, \boxed{168}}$.

Tiebreakers

Tiebreakers are used in exceptional circumstances:

- In a regular game, only when there is more than one team tied for top spot.

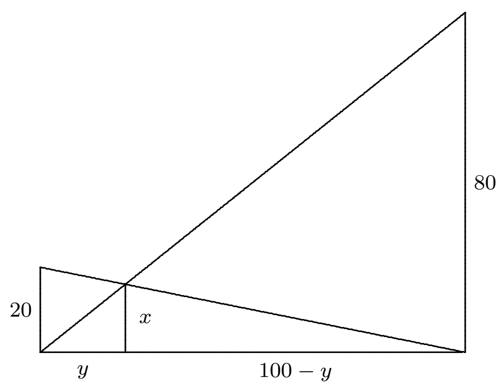
- In a championship game, when it is necessary to break ties for places one, two and three.

Here is a selection of problems used as tiebreakers.

1. Two radio masts are 20 metres and 80 metres tall. They are placed at a distance of 100 metres apart. One of the supporting guy wires for each mast is fixed to the base of the other.

   If the guy wires are perfectly straight, at what height (in metres) do they meet?

   Solution. Let the desired height be x metres. Then we have



We have similar triangles giving

$$\frac{100}{80} = \frac{y}{x} \qquad \text{or} \qquad x = \frac{4y}{5},$$

and

$$\frac{100}{20} = \frac{100-y}{x} \qquad \text{or} \qquad x = \frac{100-y}{5}.$$

Thus, $\frac{4y}{5} = \frac{100-y}{5}$, giving $y = 20$. Hence, $x = \boxed{16}$ metres.

2. Find all solutions of the equation

$$(x + 987654321)^2 = (x - 987654321)^2 + 987654321.$$

Solution. Writing $a = 987654321$, we have $(x + a)^2 = (x - a)^2 + a$, or

$(x + a)^2 - (x - a)^2 = 4ax = a$, so that, $x = \boxed{\frac{1}{4}}$.

3. [NO CALCULATORS]

The equation

$$x^2 - 20x + 21 = 0$$

has roots α and β. Determine the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

Solution. We can write the equation as

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - 20x + 21 = 0.$$

Thus, $\alpha + \beta = 20$ and $\alpha\beta = 21$.

Now, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$.

Also, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$, or $(20)^2 = \alpha^2 + \beta^2 + 2 \times 21$, giving $\alpha^2 + \beta^2 = 400 - 42 = 358$,

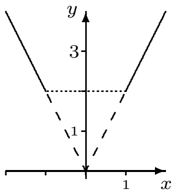
Thus, we have $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \boxed{\frac{358}{441}}$.

4. [NO (GRAPHICS) CALCULATORS]

Find the solution set of the inequality: $|x - 1| + |x + 1| > 2|x|$.

Solution.

Here is a graph:



The graph of $|x - 1| + |x + 1|$ is the solid line plus the dotted line.

The graph of $2|x|$ is the solid line plus the dashed line.

It is helpful to rewrite the inequality as $|x - 1| + |x + 1| - 2|x| > 0$.

For $x \leq -1$ and for $x \geq 1$, the value of $|x - 1| + |x + 1| - 2|x|$ is 0.

Hence, there is no such satisfactory value of x in these intervals.

Otherwise, $|x - 1| + |x + 1| - 2|x|$ is positive.

The answer is $\boxed{x \in (-1, 1)}$ or $\boxed{-1 < x < 1}$.

ATOM

A Taste Of Mathematics / Aime-T-On les Mathématiques

1. Edward J. Barbeau *Mathematical Olympiads' Correspondence Program (1995-1996)*
2. Bruce L.R. Shawyer *Algebra — Intermediate Methods*
3. Peter I. Booth, John Grant McLoughlin, and Bruce L.R. Shawyer *Problems for Mathematics Leagues*
4. Edward J. Barbeau, and Bruce L.R. Shawyer *Inequalities*
5. Richard Hoshino, and John Grant McLoughlin *Combinatorial Explorations*
6. Peter I. Booth, John Grant McLoughlin, and Bruce L.R. Shawyer *Problems for Mathematics Leagues — II*
7. Jim Totten *Problems of the Week*

Cost per volume (including shipping and handling charges):
Regular Rate \$12.00 — CMS Member's Rate \$9.00 (excluding taxes).
For non-Canadian shipping addresses the rates quoted must be paid in US funds.

Coût par volume (frais y compris d'expédition et de manutention) :
Taux régulier 12,00 \$ — Tarif d'membre de la SMC 9,00 \$ (avant taxes).
Pour une adresse à l'étranger, paiement requis en devises américaines.

For more information and to order:
Pour plus d'informations et pour compléter une commande :

www.cms.math.ca/Publications/books

or contact:
ou contacter :

CMS/SMC Publications
577 King Edward, Ottawa (Ontario) Canada K1N 6N5

Email: publications@cms.math.ca
Courriel : publications@smc.math.ca

Canadian Mathematical Society / Société mathématique du Canada
577 King Edward, POB/CP 450-A
Ottawa, Ontario, Canada K1N 6N5
www.cms.math.ca