A Taste Of Mathematics

Aime-T-On les Mathématiques

Volume / Tome V
COMBINATORIAL EXPLORATIONS

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Dalhousie University

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University of New Brunswick

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Contents

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1 Preface

Mathematical games, puzzles, and contest problems invite a playful spirit of exploration. Quite often, these entertaining and challenging problems are drawn from the fascinating mathematical realm of **Combinatorics**. This book is designed to engage those who enjoy playing with mathematics.

Emphasis is placed upon the development of mathematical perspective. The three chapters are each characterized by a single problem which is stated at the outset of the chapter. An analysis of the problem including detailed comments about the problem-solving process follows. You are encouraged to grasp the mathematical concepts and processes underlying the problem prior to proceeding further with the chapter at hand. The remainder of a chapter builds upon the basic problem. Problem sets, extensions and novel twists offer means through which you can delve deeper into the mathematics. These three chapters each culminate with three important proposed investigations. The investigations are more demanding in that they require concentrated attention on a problem that may appear open-ended in its form. The value of the investigations will be recognized by those who attempt at least one of the investigations in each chapter. Doing mathematics entails making conjectures, encountering obstacles, revisiting assumptions, and gaining a sense of accomplishment through insights (even when the insight is that a particular direction is going nowhere!). The book is intended to bringing this sense of doing mathematics to the forefront. The investigations at the conclusions of these chapters are best suited to bring this sense into the experience of aspiring mathematicians.

As mentioned, the three problems that form the bases for the chapters are followed by detailed analyses of the solution processes. The problems also appear at the end of this **Preface**, to allow you to attempt the problems before proceeding to the detailed chapters. The order of the chapters is somewhat arbitrary for people with a basic working knowledge of combinatorics. If this field of mathematics is unfamiliar to you, then it is wise to proceed through the chapters in the order they appear. An effort is made to introduce appropriate terminology and notation, whenever appropriate. Therefore, a development of the topics is supported by a chapter by chapter approach. The **Introduction** is specifically devoted to the development of basic notation and terminology that is central to any discussion of combinatorics. Language such as “factorial”, “n choose k”, combinations, and permutations are fundamental to discussing this area of mathematics. Those familiar with the terms are advised to skim the **Introduction** while warming up with some simple problems. Others should review it more carefully, in addition to subsequently using it as a reference.

The authors welcome the opportunity to use combinatorics as the mathematical topic through which a sense of "doing mathematics" can be achieved. We feel this way for several reasons. The mathematics is accessible. A minimal amount of complicated math is involved in this book. Rather the mathematics is rich in that its simplicity lends itself to deep connections with basic ideas. Fundamental mathematics comes to life in this arena.

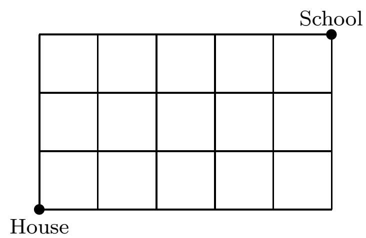
Also, the practical dimension of communicating mathematics through speech and writing pushes us to develop a language. No prior experience with this branch of mathematics nor its language is assumed at the outset. Hence, notation and terminology need to be learned through the experience. This is important since advanced mathematical study demands such fluency. Here it is offered through an avenue that promotes genuine understanding of the terminology.

The problems that ground the chapters are presented on the following page. You may wish to begin here or with the **Introduction**. Enjoy the journey!

The Core Problems

1. There were 10 women at a party, and everyone shook hands with each other person. How many handshakes took place?

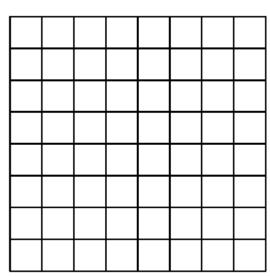
2. Mrs. Rogers leaves her house to go to school. This is a map of Mrs. Rogers' neighbourhood.

   To get to school, Mrs. Rogers must walk up for three blocks, and walk right for five blocks. She is not allowed to backtrack by moving down or left.

   How many different routes are there from her house to her school?

3. Consider the following 8 by 8 checkerboard:

   How many squares (of all sizes) appear on this checkerboard?

2 Introduction

We begin by introducing six girls, Anna, Heather, Moira, Lydia, Shirley, and Deirdre, and four boys, Andrew, Isaac, Sam, and Will. Collectively they shall be referred to as the "Group of Ten". A variety of contextual selections and arrangements will be employed to illustrate some of the basic components of combinatorics. Appropriate language and notation will be introduced along with the examples. The intent is to provide a common basis from which more in-depth development can take place in subsequent chapters.

Consider the following problem:

In how many distinct ways can the Group of Ten arrange themselves in a row of 10 seats at a theatre?

If you begin to list the arrangements you will find that the task is unreasonable. You should be motivated to a look for pattern or reason in a methodical manner, as shown:

Suppose that Anna is the first person to arrive. Anna could seat herself in any of 10 seats. The next person would have 9 choices, since Anna's seat is already taken. Then the next person would have 8 choices since two seats have been filled, and so on. Finally the tenth person would have only 1 choice, as all the other seats have been taken. These choices need to be *multiplied* together to determine the total number of seating arrangements, namely, $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

Do you see why we have to multiply the numbers, and not add them? To work with a simpler example, consider the situation where we wish to make a meal with six choices of pasta and five choices of sauces. For *each* of the six choices of pasta, we may select *any* of the five sauces. Therefore, there are $6 \times 5 = 30$ ways of choosing a pasta dinner with a sauce, and not $6 + 5 = 11$.

In this particular example with the Group of Ten, the ordering of the people is important to consider. That is, Anna, Heather, Moira, Lydia, Andrew, Will, Shirley, Isaac, Sam, Deirdre differs from Anna, Moira, Heather, Lydia, Andrew, Will, Shirley, Isaac, Sam, Deirdre. Each of these arrangements is called a **permutation**.

Suppose that there were 50 people to be arranged in a row of 50 seats. The number of ways would be $50 \times 49 \times 48 \times \cdots \times 3 \times 2 \times 1$. Such an expression seems to be awkward and cumbersome. The use of an exclamation mark, !, allows us to write such a product as 50!. The symbol "!" is known as **factorial**. In general, $n!$ equals $(n)(n-1)(n-2)\cdots(2)(1)$. Returning to the example, the number of arrangements would be 10!, which is equal to 3628800.

Consider a problem that requires us to select a subset of the Group of Ten.

In how many ways can a committee of three people be selected from the Group of Ten?

Again an exhaustive list could be developed but that would be an exhausting task. At first glance, it may seem reasonable to think that $10 \times 9 \times 8$ is the answer. However, such an answer treats the following selections as being distinct:

1) Heather, Moira, Andrew
2) Heather, Andrew, Moira
3) Moira, Andrew, Heather
4) Moira, Heather, Andrew
5) Andrew, Moira, Heather
6) Andrew, Heather, Moira

In fact, all of these selections would result in the same committee being formed. Since Andrew, Moira and Heather can order themselves in 6 ways (observe that $3! = 6$), it is necessary to divide $10 \times 9 \times 8$ by 6 to get the proper result. When order does not matter, we are working with **combinations**.

Fortunately, there exists a convenient form for expressing such mathematical results, as the one desired in this problem. Verbally one can state, "In how many ways can 3 people be chosen from a group of 10 people?" The mathematical term that describes this is "10 choose 3". We write this as $\binom{10}{3}$. But what does this mean mathematically?

We know that "10 choose 3" must equal $10 \times 9 \times 8$ divided by 6, as discussed above. The formula for calculating such a result is explained here:

$$\binom{10}{3} = \frac{10!}{3!(10-3)!}.$$

Mathematically, consider this as a quotient:

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)}$$

It can be seen that cancellation of common terms gives us $\frac{10 \times 9 \times 8}{3 \times 2 \times 1}$, as expected. Consider what would happen if we calculated $\binom{10}{7}$. Does it make sense that $\binom{10}{7}$ must equal $\binom{10}{3}$? Yes. Selecting 3 of 10 people to sit on a committee is equivalent to selecting 7 of 10 people to not sit on the committee.

In general, we express "n choose k" as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

As an interesting connection to Canada, we remark that the expression "n choose k" was first coined by Richard Guy from the University of Calgary, sometime during the 1950's. Those unfamiliar with this notation should manually verify some basic results such as $\binom{12}{10} = 66$, $\binom{80}{79} = 80$, and $\binom{12}{5} = 792$.

In summary, understanding the distinction between permutations and combinations is critical to solving problems. Permutations are arrangements that depend upon order, whereas combinations are arrangements that are independent of order. (WARNING: Do not get confused by combination locks. They really should be called *permutation locks*.)

The selection of a committee is typically an example of a combination. However, if the roles of the members (such as president, treasurer, secretary, etc.) are designated, the ordering element is introduced. Consider our Group of Ten again. Suppose that four of them are to form a committee. The committee could be chosen in $\binom{10}{4}$ ways.

Now suppose that the problem was modified slightly to read: In how many ways can a president, a vice-president, a secretary and a treasurer be selected from the Group of Ten?

Here, $\binom{10}{4}$ would be insufficient. This would not allow for the assignment of roles, as required. Rather it would account only for the selection of the four committee members. That is, a combination would be in place but not an ordering (or permutation). How many ways could the four committee members be assigned to the roles? The answer is 4!. We can see this by noting that this problem is equivalent to finding the number of assignments of 4 seats to 4 people in a theatre row, where the first seat goes to the president, the second seat to the vice-president, and so on.

This multi-step approach of selecting the members (a combination) and then assigning them to roles (a permutation) is one way to handle the selection process. The resulting number of possibilities is $\binom{10}{4} \times 4!$.

There is a more elegant way to solve the problem. Considering the four roles in order, we would have 10 choices for president, 9 for vice-president, 8 for secretary, and 7 for treasurer. Therefore, $10 \times 9 \times 8 \times 7$ ways of assigning the roles are possible. This latter approach treats the problem strictly as a permutation though it allows for the larger number of potential candidates at each stage. How do we reconcile that $\binom{10}{4} \times 4!$ and $10 \times 9 \times 8 \times 7$ are equivalent? Simplifying, we get:

$$\binom{10}{4} \times 4! \;=\; \frac{10!}{4! \cdot (10-4)!} \times 4! \;=\; \frac{10!}{4! \cdot 6!} \times 4! \;=\; \frac{10!}{6!} \;=\; 10 \times 9 \times 8 \times 7.$$

Frequently combinatorial problems lend themselves to seemingly incompatible expressions that actually represent equivalent values despite being drawn from different processes. The following problem involves the finishing order of a race. Selecting the top members to form the list of winners is the same as the strict selection of a combination. Once the positional placements are considered, the analogy to the committee roles is evident.

Attempt to use two approaches to answer this problem: In how many ways can the top three positions in a race be awarded among a group of 12 runners? (Assume there are no ties.) The numerical answer of 1320 should be obtained using either approach.

Returning to our Group of Ten, we shall draw attention to some subtleties with respect to partitioning. The subsequent examples require us to break the Group of Ten into subgroups.

In how many ways can a committee of three people be selected from the Group of Ten if the committee must have two girls and one boy? Here it is necessary to consider the six girls and four boys as separate groups. There are $\binom{6}{2}$ ways of selecting two of the six girls. There are $\binom{4}{1}$ ways of selecting one of the four boys. Thus, there are $\binom{6}{2} \times \binom{4}{1}$ possible committees that can be formed.

In how many ways can a committee of three people be selected from the Group of Ten if Lydia or Anna (but not both of them) is to be on the committee? We would choose either Lydia or Anna, which can be done in $\binom{2}{1}$ ways, by definition. Among the other eight people, we must select two of them. This can be done in $\binom{8}{2}$ ways. Thus, there are $\binom{2}{1} \times \binom{8}{2}$ possible committees that can be formed.

We conclude the **Introduction** by giving you two more problems:

1. In how many ways can a committee of 4 people be chosen from the Group of Ten such that there is at least one girl on the committee?

2. In how many ways could a committee of 5 people be chosen from the Group of Ten such that the committee includes two boys and three girls, including both Heather and Moira?

The answers to the two problems are 209 and 24, respectively.

3 Handshakes

3.1 The Handshake Problem

There were ten women at a party, and everyone shook hands with everyone else. How many handshakes took place?

Before we tackle the actual problem, consider a simpler question. What happens when we change the number ten to something smaller? Look at a few small cases, and attempt to find a pattern.

If we have one person at the party, no handshakes take place. If two people attend the party, then exactly one handshake occurs. Suppose three people attend the party (say Audrey, Brittany, and Celine). Then exactly three handshakes take place (Audrey with Brittany, Brittany with Celine, and Audrey with Celine). We find that six handshakes take place with four people, and ten handshakes take place with five people. Let us see what we have so far:

Number of People	Number of Handshakes
1	0
2	1
3	3
4	6
5	10

Look at the numbers in the second column. Can we see any pattern in the sequence 0, 1, 3, 6, 10? Yes, an elegant pattern appears when we look at the *differences* between consecutive terms of the sequence.

$$0 \underbrace{\quad}_{1} 1 \underbrace{\quad}_{2} 3 \underbrace{\quad}_{3} 6 \underbrace{\quad}_{4} 10$$

Assuming that this pattern continues, the first ten terms of the sequence should be:

$$0 \underbrace{\quad}_{1} 1 \underbrace{\quad}_{2} 3 \underbrace{\quad}_{3} 6 \underbrace{\quad}_{4} 10 \underbrace{\quad}_{5} 15 \underbrace{\quad}_{6} 21 \underbrace{\quad}_{7} 28 \underbrace{\quad}_{8} 36 \underbrace{\quad}_{9} 45$$

Since there are 10 people in our problem, we are interested in the tenth number of this sequence. Hence, the answer to our problem appears to be **45**.

Note that this is *not* a proof. Based on the pattern, we have a strong feeling that the answer is 45, but we need to argue this more rigorously. To see that 45 indeed is the answer, we can employ the following **Sledgehammer Approach**: *list* all 45 handshakes that take place!

We use the colloquial term Sledgehammer Approach to denote any type of strategy where a brute-force tactic is used. Quite often, the Sledgehammer Approach is a very poor problem-solving strategy, since it is so time-consuming and devoid of elegance. Nevertheless, it certainly offers one way to get the right answer.

Solution A:

Suppose the names of our ten people are Audrey, Brittany, Celine, Donna, Ellie, Flavie, Gabriella, Harriet, Iris, and Jo. Then here is the list of all 45 handshakes, using the initials as representations of the people. We can readily verify that we have not missed any cases.

(A,B)	(A,C)	(A,D)	(A,E)	(A,F)	(A,G)	(A,H)	(A,I)	(A,J)
(B,C)	(B,D)	(B,E)	(B,F)	(B,G)	(B,H)	(B,I)	(B,J)	(C,D)
(C,E)	(C,F)	(C,G)	(C,H)	(C,I)	(C,J)	(D,E)	(D,F)	(D,G)
(D,H)	(D,I)	(D,J)	(E,F)	(E,G)	(E,H)	(E,I)	(E,J)	(F,G)
(F,H)	(F,I)	(F,J)	(G,H)	(G,I)	(G,J)	(H,I)	(H,J)	(I,J)

Therefore, we have a total of 45 handshakes. ■

However, this solution has a major deficiency. What happens if we replace the number 10 with 1000? We certainly would not want to list out all the cases to answer the problem. Surely, there must be a more elegant way to solve this problem.

Indeed, here we provide several "better" solutions.

Solution B:

Assume that each woman will shake hands with each guest as soon as she arrives at the party. Suppose Audrey arrives first. She has no one to shake hands with. When Brittany arrives, she will shake hands with Audrey. Thus, one handshake takes place. When Celine arrives, she will shake hands with both Audrey and Brittany. This adds another two handshakes to our total. When Donna arrives, she will shake hands with Audrey, Brittany, and Celine, which adds three to our total. We continue this process, and eventually Jo will arrive at the party. She will shake hands with the other nine people, who are already present. Note that each pair of people will shake hands as soon as both are present. Thus, this process guarantees that everyone will shake hands with everyone else.

By looking at the problem this way, we see that the total number of handshakes is $0 + 1 + 2 + 3 + \cdots + 8 + 9$, which adds up to 45.

Therefore, there are 45 handshakes. ■

Solution C:

Each of the ten people shakes hands with everyone else. Thus, each person performs exactly nine handshakes. Thus, we have ten people, each shaking nine hands, and the total number of handshakes is $9 \times 10 = 90$. Right?

No! Each handshake is counted twice (that is, Audrey shaking hands with Brittany is the same as Brittany shaking hands with Audrey), so that 90 is *twice* the number of handshakes that took place. In other words, the correct answer to the problem is $\frac{9\times10}{2} = 45$ handshakes. ■

Solution D:

In the **Introduction**, we introduced the notation $\binom{n}{k}$. Here is the simplest solution.

The number of handshakes is equivalent to the number of two-person committees that can be formed from a group of 10 people (do you see why this is true?). Hence, the answer to the problem is $\binom{10}{2} = \frac{10!}{2!\cdot 8!} = \frac{10\times 9}{2\times 1} = 45$. ■

Compare Solutions B and C. In two completely separate ways, we arrived at the same final answer. We notice that

$$1 + 2 + 3 + \cdots + 9 \;=\; \frac{9\times 10}{2}.$$

Can we find a general formula? Replacing 9 by n, it appears that the following result is true:

$$1 + 2 + 3 + \cdots + n \;=\; \frac{n(n+1)}{2}.$$

Here is a quick proof that this identity is correct; consider a party with $(n+1)$ people. By the process described in Solution B, the total number of handshakes must be $1 + 2 + 3 + \cdots + n$. By the process described in Solution C, the total number of handshakes must be $\frac{n(n+1)}{2}$. Since these two expressions are equal, we have proved the desired identity. ■

Thus, we have used a "handshake" argument to derive a formula for the sum of the first n positive integers.

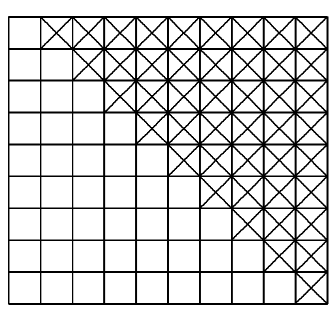
This formula is commonly introduced as an application of **Mathematical Induction**, which relies heavily on algebraic manipulation. The combinatorial interpretation of the result is more aesthetically pleasing. Mathematical Induction (see **Glossary**) is an excellent problem-solving tool, but the combinatorial "handshake" proof is easier to visualize and appreciate.

Here is yet another way to see that $1 + 2 + 3 + \cdots + 9 = \frac{9\times 10}{2}$.

We will call this the "geometric" solution.

Solution E:

Consider the following diagram, composed of 9 rows of 10 unit squares. We put a cross inside some of the squares, as shown:



Let $S = 1 + 2 + 3 + \cdots + 9$. Let us show that $S = \frac{9\times 10}{2}$.

Count the number of uncrossed squares: we have one in the first row, two in the second row, three in the third row, and so on. Therefore, the number of uncrossed squares is $1 + 2 + 3 + \cdots + 9 = S$. Similarly, the number of crossed squares is S, because we have nine crossed squares in the first row, eight in the second row, all the way down to one in the last row.

Thus, we have an equal number of crossed squares and uncrossed squares. In total, there are $S + S = 2S$ unit squares in the diagram. Since this diagram consists of 9×10 unit squares, we conclude that

$$1 + 2 + 3 + \cdots + 9 \;=\; S \;=\; \frac{9 \times 10}{2}. \qquad \blacksquare$$

This geometric argument may remind you of the approach made famous by the German mathematician Karl Gauss (1777-1855). When Gauss was a young child, his teacher asked the class to add up all the numbers from 1 to 100. While Gauss' classmates struggled for several minutes, it only took Gauss several seconds to determine the correct answer. Here is the essence of his brilliant solution.

Let $S = 1 + 2 + 3 + \cdots + 98 + 99 + 100$.

Then $S = 100 + 99 + 98 + \cdots + 3 + 2 + 1$, by writing the terms in the reverse order.

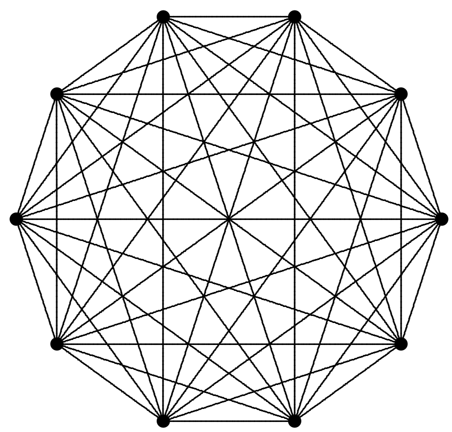
Adding the two rows term by term, we have

$$\begin{aligned} S + S &= (1+100) + (2+99) + \cdots + (99+2) + (100+1) \\ 2S &= \underbrace{101 + 101 + 101 + \cdots + 101 + 101 + 101}_{100 \text{ terms}} \\ 2S &= 100 \times 101 \\ S &= \frac{(100)(101)}{2}. \end{aligned}$$

Using this symmetric "pairing" argument and replacing 100 by n, we have another proof that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$. Do you see how this is identical to Solution E?

Let us revisit the original problem with the ten people.

Draw a (convex) polygon with 10 points, and construct line segments joining each pair of points.



We have just drawn a **graph**. The points are known as **vertices**, and the line segments are known as **edges**.

We can model our Handshake problem by drawing a graph with 10 vertices, and connecting each pair of vertices. The problem is now equivalent to counting the number of edges in our graph. Each of the 10 vertices is incident with 9 edges. Also, every edge is counted twice. Thus, the total number of edges must be $\frac{9\times 10}{2} = 45$. Recall that we used an identical argument earlier, in Solution C.

By converting a handshake problem into a graph (that is, a picture), we will be able to solve some very difficult problems, in simple and elegant ways. For example, consider the following problem.

3.2 Six People at the Party

Suppose there are six people at a party, some of whom shake hands with each other. Show that among the six, we can find either three people all of whom shook hands with each other, or three people, none of whom shook hands with each other.

At first glance, this problem looks impossible. Surely the Sledgehammer Approach fails us here, for there are too many cases to consider. However, there is an effective way to proceed.

Consider a graph with 6 vertices, representing the six people at the party. Draw edges joining each pair of vertices. Colour an edge *red* if those two people shook hands, and colour an edge *blue* if those two people did not shake hands. There will be $\frac{5\times 6}{2} = 15$ edges in total.

If there are three people, all of whom shook hands with each other, then those three vertices in our graph will form a triangle with all red edges. If there are three people, none of whom shook hands with each other, then those three vertices in our graph will form a triangle with all blue edges.

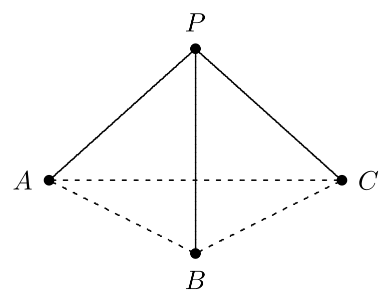
This way, we have translated the problem into a form that we can visualize:

Construct a graph with 6 vertices, and draw an edge joining each pair of vertices. Colour each of these 15 edges either red or blue.

Prove that no matter how the edges are coloured, there must exist a monochromatic triangle; that is, a triangle where all three sides have the same colour.

Let us solve this equivalent problem. Select an arbitrary vertex of the graph, and label it P. There is an edge from P to each of the other five vertices. Since we are colouring each of our five edges in one of two colours, at least three of our edges must be of the same colour. This is an application of the **Pigeonhole Principle** (see **Glossary**).

Consider three of these edges that are of the same colour, call them PA, PB, and PC. It does not really matter what this common colour is; therefore, suppose that it is red. (The argument is identical if we suppose this common colour is blue, except the colours become switched in our proof). We will prove that there is at least one monochromatic triangle in our graph.



If AB is coloured red, then $\triangle PAB$ is monochromatic (it is an all-red triangle). Similarly, if AC is coloured red, then $\triangle PAC$ is monochromatic, and if BC is coloured red, then $\triangle PBC$ is monochromatic. The only case left to consider is when AB, AC, and BC are *all* coloured blue. But then, $\triangle ABC$ is monochromatic (it is an all-blue triangle).

Therefore, we have proved that if there are six people at a party, then we can find three people, all of whom shook hands with each another, or three people, none of whom shook hands with each other. ∎

If we suppose that a handshake always occurs between two acquaintances (and does not occur between two strangers), then our result is equivalent to the following:

In any group of six people, there must be three mutual acquaintances or three mutual strangers.

This problem has spawned an entire field of mathematics, known as **Ramsey Theory**. Ramsey Theory is a very beautiful branch of research in combinatorics that has made many important connections to other areas of mathematics, such as algebra, geometry, and probability theory. This book contains a brief introduction to Ramsey Theory in our **Investigation** section at the end of this chapter.

We conclude the discussion of this chapter by presenting one of our favourite problems, which also involves handshakes.

3.3 Mr. and Mrs. Smith

Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?

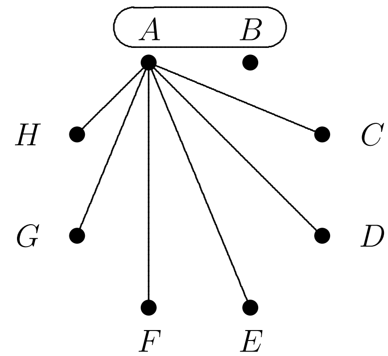
This problem is fascinating because it does not appear solvable. It is difficult to imagine that there is enough information here. However, we have all the information we need! Before reading any further, stop and attempt to solve this problem on your own.

As we did in the previous problem, let us model this problem graphically. Draw a graph with 8 vertices, and label the vertices A, B, C, D, E, F, G, and H. Suppose that A is married to B, C is married to D, E is married to F, and G is married to H.

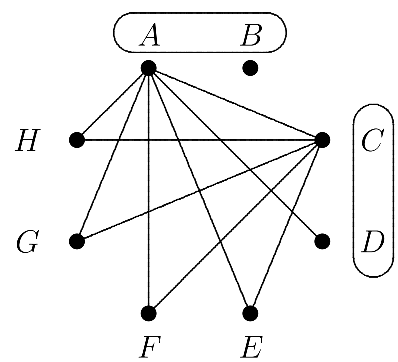
Again, each vertex represents a person at the party. In the previous problem, we connected each pair of vertices and coloured the edge either red or blue. In this problem, we will do something different. Instead of colouring edges, we will do the following: only join two vertices if those two people shook hands.

Since no one shakes their own hand, or the hand of their spouse, a person can shake at most six hands. Thus, every person at the party shook at least 0 hands and at most 6 hands. Therefore, when Mrs. Smith asks the other seven people how many hands they shook, there can be at most seven different answers. Thus, if no two people shook the same number of hands, that must imply that someone shook 0 hands, someone shook 1 hand, someone shook 2 hands, someone shook 3 hands, someone shook 4 hands, someone shook 5 hands, and someone shook 6 hands.

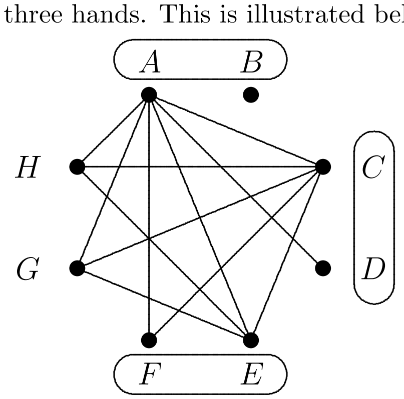
Consider the person who shook 6 hands. Let us assume this person is A. Thus, A shook hands with everyone at the party except for his (or her) spouse B. Represent this by drawing an edge from A to each of the other vertices in the graph, except for B. Thus, every person other than B has shaken at least one hand. Since someone at the party shook 0 hands, this implies that B *must* have been the person who shook 0 hands. This information is represented in the diagram below. In this diagram, the rectangular box around A and B signifies that this couple has finished performing all of their handshakes.



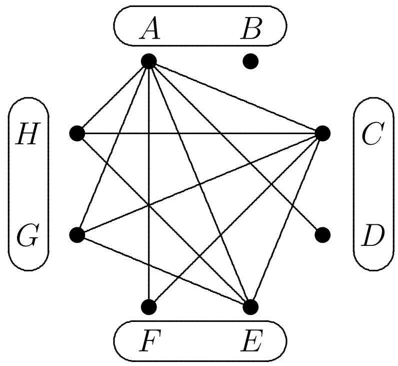
Now consider the person who shook 5 hands. Assume this person is C. Then C shook hands with everyone except D (her spouse) and B (since B shook hands with no one). Thus, draw an edge from C to each of E, F, G, and H. This shows that everyone (other than B and D) shook at least two hands. Therefore, it follows that D must have been the person who shook exactly 1 hand. This is illustrated below:



Now consider the person who shook 4 hands. Assume this person is E. We know that E shook hands with A and C, but not with B or D. Also, E does not shake hands with F (his spouse). Thus, E must have shaken hands with both G and H, because he shook four hands in total. Draw an edge from E to each of G and H. Now we see that F must be the person who shook 2 hands, as both G and H have shaken at least three hands. This is illustrated below:



Since G and H are married, they do not shake hands. Thus, both G and H shook three hands. We have now indicated all the handshakes that took place at this party.



Notice that each *couple* has shaken exactly six hands in total – that is not a coincidence (can you explain why?)

Now we need to determine which person represents Mr. Smith. If Mrs. Smith is one of the first six people (A to F), when she asked her question, two of the individuals would have replied that they shook exactly three hands. That is a contradiction because all seven replies were different. Therefore, Mrs. Smith *must* be either G or H.

Thus, Mrs. Smith shook three hands. Since Mr. Smith is her husband, we conclude that Mr. Smith also shook three hands. ∎

We now present some problems that build upon the ideas in this chapter.

3.4 Problem Set

1. Suppose that twenty people attended a party, and everyone shook hands with exactly three guests. How many handshakes took place?

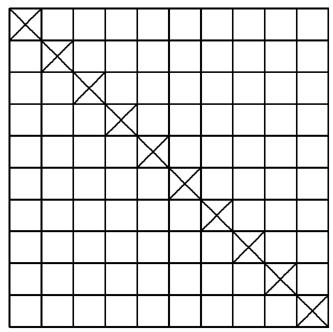
2. Suppose that five married couples attended a party, and everyone shook hands with everyone else other than their own spouse. How many handshakes took place?

3. Mr. and Mrs. Smith were at a party with ten other married couples. Various handshakes took place. No one shook hands with their spouse, and of course, no one shook their own hand! After all the introductions had been made, Mrs. Smith asked the other people how many hands they shook. Surprisingly, they all gave a different answer. How many hands did Mr. Smith shake?

4. At a party attended by n people, various handshakes took place. Just for fun, each person shouted out the number of hands they shook. If all of these numbers are added together, explain why this total cannot be an odd number.

5. At a party attended by n people, various handshakes took place. Just for fun, each person shouted out the number of hands they shook. Explain why there must have been at least two people who shouted out the same number.

6. Here is another solution to the Handshake problem posed at the beginning of the chapter. Take a 10 by 10 grid, and cross out all of the diagonal squares, as illustrated:

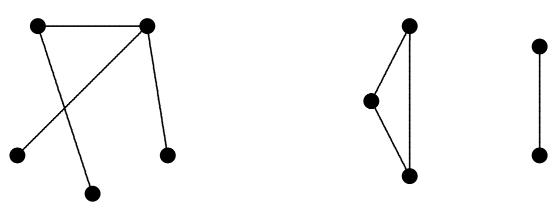


The number of unit squares remaining is $10^2 - 10 = 90$. Using this diagram, explain why the total number of handshakes is half of this number; that is, show that the number of handshakes is $\frac{10\times 10-10}{2} = 45$.

7. Using a geometric approach, prove that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$, for all positive integers n.

8. A small banquet took place in Moncton. At this banquet, all but one person arrived on time for the reception. During the reception, each person shook hands with everyone else. However, Karen strolled in twenty minutes late, so that she was only able to shake hands with some of the other guests. If there were exactly 73 handshakes in total, determine the number of hands Karen shook.

9. A graph is said to be *connected* if we can find a path joining any pair of vertices. For example, the graph on the left is connected; the graph on the right is not.



Let G be a graph with n vertices.

(a) Show that if G has less than $n - 1$ edges, then G is not connected.

(b) How many edges must G have in order to guarantee that it is connected?

10. Seventeen people are at a party. It turns out that for any two people present, exactly one of the following statements is true: "They have not met", "They like each other", or "They dislike each other". Prove that there must be three people, all of whom are either mutual strangers, mutual friends, or mutual enemies. Is this necessarily true if the party had only sixteen people?

3.5 Investigation 1: Bouts of Handshakes

There are n people in a room, and everyone shakes hands with everyone else. How *long* does this process take?

Let us phrase our question a bit more specifically. Suppose that our handshaking is organized into "bouts", and each person can only shake hands once every bout. Then the problem becomes: determine the least number of bouts required for everybody to shake hands with everybody else. Furthermore, what would an optimal schedule look like? For example, with 5 people (named A, B, C, D, and E), here is a possible schedule with 5 bouts.

Bout 1:	(A,B), (D,E)
Bout 2:	(A,C), (B,D)
Bout 3:	(A,D), (C,E)
Bout 4:	(A,E), (B,C)
Bout 5:	(B,E), (C,D)

(Can you show that five bouts are the best we can do?)

Let $T(n)$ denote the minimum number of bouts required for n people.

1. Work out schedules for small numbers. Show that $T(2) = 1$, $T(3) = 3$, $T(4) = 3$, and $T(5) = 5$.

2. Determine the values of $T(n)$ for $n = 6$, 7, 8, 9, 10.

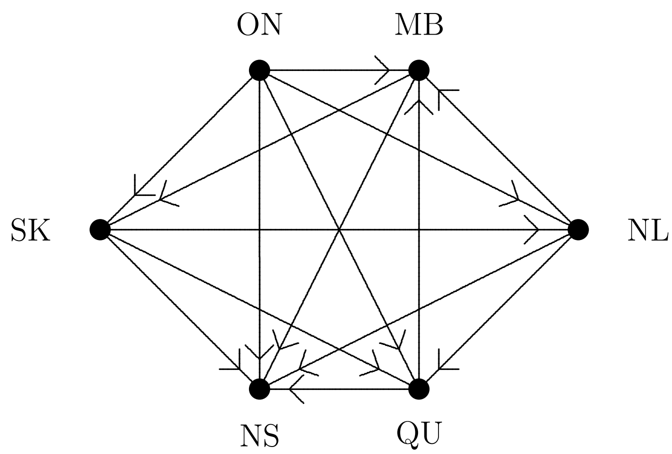
3. Find a general formula for $T(n)$. Can you prove it? (Hint: consider the odd and even cases separately).

For more information on this problem, we refer you to *Problems for Senior High School Math: In Process*, by Peter Taylor.

3.6 Investigation 2: Tournaments

Consider a curling bonspiel with 6 teams: Nova Scotia (NS), Quebec (QU), Manitoba (MB), Newfoundland & Labrador (NL), Saskatchewan (SK) and Ontario (ON). Each team plays each other team exactly once in this round-robin event. The winner of each game is awarded 1 point, whereas the loser is awarded 0 points. Curling matches do not end in ties.

We will draw a graph indicating the results of each game in the tournament:



For example, the arrow from Ontario (ON) to Manitoba (MB) means that Ontario lost to Manitoba. From the graph, we can determine the result of each game. Each team's score can be found by counting the number of arrows that point towards that team's vertex. The number of points for each team is indicated in the table:

Team Name	Score
NS	5
QU	3
MB	3
NL	2
SK	2
ON	0

Thus, the **score sequence** for this tournament is (0, 2, 2, 3, 3, 5). In a score sequence, we list the numbers in increasing order. Note that in any score sequence with six teams, the total number of points must be 15, because exactly 15 games must be played.

What score sequences are possible with 6 teams? For example, can you prove that a (0, 1, 2, 3, 4, 5) score sequence is possible, but a (0, 1, 1, 4, 4, 5) score sequence is impossible? In general, how can we show that a given score sequence is impossible?

Investigate this idea further. Find all impossible score sequences with 4 teams, 5 teams, and 6 teams. Do you notice any interesting patterns? What if we have 10 teams? Try to determine all possible score sequences.

Attempt to find and prove a general theorem for the possible score sequences in a tournament with n teams. This is a very difficult problem.

3.7 Investigation 3: Ramsey Theory

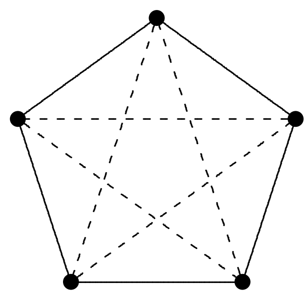
This is a guided investigation that will introduce the fundamentals of Ramsey Theory, a branch of combinatorial mathematics studied by researchers today.

Consider the following problem:

Determine the smallest positive integer n for which the following statement is true: if n people attend a party, then there must be three mutual acquaintances, or three mutual strangers.

Earlier in this chapter, we showed that if 6 people attend a party, then there must be three mutual acquaintances or three mutual strangers. Is 6 the *smallest* integer for which this statement is true?

To show that $n = 6$ indeed is the smallest possible answer, let us prove that the statement fails for $n = 5$. Consider the following graph on five vertices:



A solid edge connects two people who are acquaintances, and a dotted edge connects two people who are strangers.

We see that in this situation, there do not exist three mutual acquaintances (that is, a solid triangle joining three of the original vertices). Similarly, there do not exist three mutual strangers (that is, a dotted triangle joining three of the original vertices).

Since the statement fails for $n = 5$, it must fail for any $n < 5$. We can see this by removing any number of vertices from the above counterexample, to produce a graph with less than 5 vertices, so that it has no solid or dotted triangle. (In general, if the statement fails for $n = k$, then it must fail for all $n < k$).

This proves that $n = 6$ is the answer to our problem.

A more formal explanation is offered here. We define $R(x, y)$ to be the smallest integer n for which the following statement is true: if n people attend a party, then there *must* either be a group of x mutual acquaintances, or a group of y mutual strangers. Thus, in the above example, $R(3, 3) = 6$. By symmetry, we have $R(x, y) = R(y, x)$ for all x and y.

In 1930, Frank Ramsey showed that the number $R(x, y)$ must exist for all x and y. Even if x and y are one million, Ramsey's Theorem tells us that you can find n large enough so that in any group of n people, you can find a group of one million mutual strangers, or a group of one million mutual acquaintances.

Stated in its most general form, Ramsey Theory states that within any sufficiently large system, some regularity must always exist. In other words, "complete disorder is impossible". Ramsey Theory is the study of regularity in complex random structures. This branch of mathematics has laid the groundwork for many important areas of current combinatorial research, which have many applications to different areas of pure and applied mathematics.

In this investigation, we shall explore Ramsey numbers.

1. We will find the value of $R(3,4)$; that is, the smallest number of people that we need to guarantee the existence of a group of three mutual acquaintances or four mutual strangers. First we prove that $R(3,4) \leq 10$. Construct a graph with 10 vertices, and connect each pair of vertices with an edge. Colour each of these $\binom{10}{2} = 45$ edges red or blue, where a red edge joins two acquaintances, and a blue edge joins two strangers. We must prove that no matter how the edges are coloured, we must have three vertices that are all connected by red edges (a red triangle), or four vertices that are all connected by blue edges.

   Pick a vertex P. We have 9 edges joined to P. Explain why we must have one of the following two cases:

   (a) P is joined to at least four red edges.

   (b) P is joined to at least six blue edges.

   In each of these two cases, prove that no matter how the rest of the edges are coloured, we must have three vertices that are all connected by red edges, or four vertices that are all connected by blue edges. (Hint: for case (b), you will want to use the fact that $R(3,3) = 6$). This proves that $R(3,4) \leq 10$.

2. Now we will go a step further and prove that $R(3,4) \leq 9$. Consider what happens when we have a party with 9 people. We shall prove that if we have 9 people at a party, we must have three mutual acquaintances or four mutual strangers. Suppose that this is not the case. We will derive a contradiction.

   Consider some vertex P, which is joined to eight other edges. From the previous question, we have shown that if P is joined to at least four red edges or at least six blue edges, then we are done (that is, we have either three mutual acquaintances or four mutual strangers). Therefore, that only leaves the case in which P is joined to three red edges and five blue edges.

   Explain why we may assume that *all* the vertices are joined to exactly three red edges and five blue edges. Now derive a contradiction by showing that we cannot draw a graph with 9 vertices, where exactly three edges from each vertex are coloured red. (Hint: see Question 4 from the **Problem Set** in this chapter). Use this to conclude that $R(3,4) \leq 9$.

3. Arrange a party with 8 people so that we do *not* have three mutual acquaintances, or four mutual strangers. This will prove that $n = 9$ is the smallest integer such that in any group of n people, we must have either three mutual acquaintances or four mutual strangers. Thus, we conclude that $R(3,4) = 9$.

4. Prove that for any integers $s, t \geq 2$,

$$R(s,t) \ \leq \ R(s-1,t) + R(s,t-1).$$

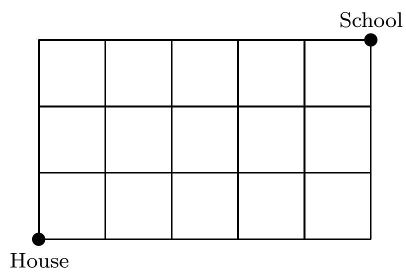
Very little is known about actual Ramsey numbers. So far, it is known that $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$,

$R(3,9) = 36$, $R(4,4) = 18$, and $R(4,5) = 25$. *No other Ramsey numbers are known*, and most of the known ones were found by computer within the past ten years. The best result for $R(5,5)$ is that it is somewhere between 43 and 49, but no one has been able to do any better than that!

4 Routes

4.1 The Route Problem

Mrs. Rogers leaves her house to go to school. This is a map of Mrs. Rogers' neighbourhood.



To get to school, Mrs. Rogers must walk up for three blocks, and walk right for five blocks. She is not allowed to backtrack by moving down or left.

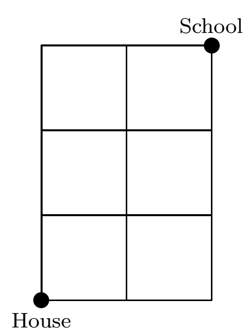
How many different routes are there from her house to her school?

We can use the Sledgehammer Approach to manually count all the routes, but that will be a lengthy and time-consuming task. Certainly, there must be a better way to tackle this problem! Indeed there is a better way.

Solution A:

Before we answer this question, let us look at the problem for a grid of smaller size, and see if we can discover some patterns.

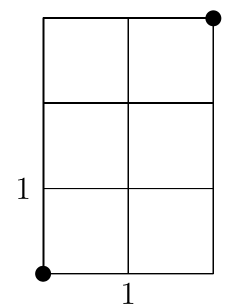
Suppose the diagram is as follows:



To make our notation easier, represent each point with coordinates. Therefore, let the house be located at $(0, 0)$ and the school at $(2, 3)$.

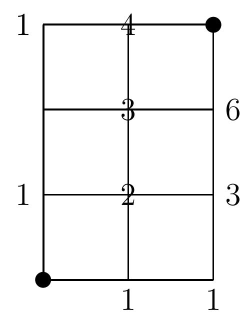
We want to find out how many routes there are from $(0, 0)$ to $(2, 3)$, if Mrs. Rogers must walk up or right on her way to school.

There is clearly only one way to get from $(0, 0)$ to $(1, 0)$: move right one block. Similarly, there is only one way to get from $(0, 0)$ to $(0, 1)$. This information is represented in our diagram by writing down the number of routes from Mrs. Rogers' house to each of those points.



This notation indicates that there is one route to $(1,0)$, and one route to $(0,1)$.

Investigate further. There are two ways to get to $(1,1)$, as Mrs. Rogers can walk up then right, or walk right then up. We quickly count the number of routes from the start to all of the other points, and we arrive at the following:



To answer the question, we must figure out how many routes there are from $(0,0)$ to $(2,3)$. But instead of counting the routes manually, let us look for a pattern.

First note that there is only one way to get to each point directly north or directly east of her starting point.

Notice that all other marked numbers have the property that they represent sums of two numbers: the number directly below it, and the number directly to its left. For example, the number 4 appears as the entry at point $(1,3)$. Notice that the entry at point $(0,3)$ is 1 and the entry at point $(1,2)$ is 3. Hence, $1+3=4$.

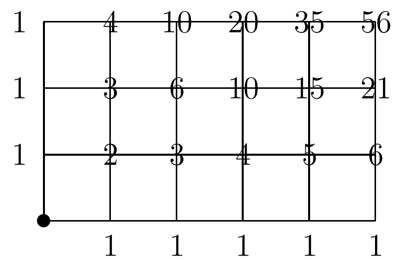
Using this pattern, it appears that the entry at point $(2,3)$ should be $4+6 = 10$. Indeed if you were to manually count the routes, you will see that there are ten ways to walk from $(0,0)$ to $(2,3)$.

But why does this work? We can justify this as follows. For any route from the house to the school, the penultimate point (that is, the second last point) in the path must be either $(2,2)$ or $(1,3)$. In other words, if Mrs. Rogers is to get to $(2,3)$, she must either walk up from $(2,2)$ or walk right from $(1,3)$. Since there are 6 ways to get to $(2,2)$ and 4 ways to get to $(1,3)$, it follows that there are $6+4=10$ different ways to get to $(2,3)$, as she can take any of the six routes to $(2,2)$ and move up to get to school, or take any of the four routes to $(1,3)$ and move right to get to school.

Therefore, we conclude that there are ten routes from $(0, 0)$ to $(2, 3)$. ∎

Revisiting the original question in which the school is located at the point $(5, 3)$, we can use the same technique as before, by counting the number of routes to each of the points in the neighbourhood. Note that Mrs. Rogers can only get to each of the points $(0, 1)$, $(0, 2)$, and $(0, 3)$ in exactly one way; that is, by moving straight up until she reaches that point. Similarly, Mrs. Rogers has only one route to each of the points on the x–axis, namely $(1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, and $(5, 0)$.

We can determine the missing numbers by following the pattern that we discovered earlier. That is, each entry is the sum of two entries: the entry directly below, and the entry directly to its left. The completed table is below.

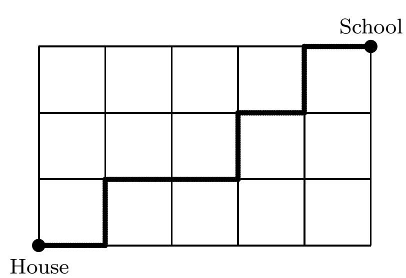


Therefore, we conclude that there are **56** routes from Mrs. Rogers' house to her school. ∎

This solution was relatively straightforward, but we will have some major difficulties if we employ such a technique for determining the number of routes from $(0, 0)$ to $(50, 100)$. Here is a more elegant solution to the problem:

Solution B:

Since Mrs. Rogers ends at $(5, 3)$, she must make a total of eight steps. In any path, she makes exactly five Right steps and three Up steps, and she may proceed in any order. Specifically, she may choose any three of her eight steps to be Up steps, and then the rest of her steps must be Right steps. For example, if she moves Up on her second, fifth, and seventh steps, then her route will be Right, Up, Right, Right, Up, Right, Up, Right. This route is indicated in the diagram below:



We recognize this "ordering" problem as one that uses combinations. There are $\binom{8}{3}$ ways that Mrs. Rogers can select three of her eight steps to be Up steps. Each such selection corresponds to a unique route that Mrs. Rogers can take from her house to school. Also, every such route is counted exactly once by this process. Thus, there are $\binom{8}{3} = 56$ different routes that Mrs. Rogers can take. ∎

In general, if Mrs. Rogers walks from $(0, 0)$ to (p, q), then she must make a total of p Right steps and q Up steps, for a total of $(p + q)$ steps. She can choose any p of the $(p + q)$ steps to be Right steps, and then the rest must be Up steps. Each such selection corresponds to a route. Thus, the total number of routes is $\binom{p+q}{p}$. Can you argue why the answer is also equal to $\binom{p+q}{q}$?

4.2 Pascal's Triangle and Pascal's Identity

Carefully examine the numbers on our grid. If you are familiar with **Pascal's Triangle**, you may have noticed that the numbers appearing in our grid also appear in Pascal's Triangle.

Pascal's Triangle is defined as follows:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

This triangle is named after the French mathematician Blaise Pascal (1623-1662), who conceived of it at the age of thirteen. However, several Chinese mathematicians discovered some properties of this triangle in the early 14^{th} century, long before Pascal was born!

To generate a row of Pascal's Triangle, look at the row immediately above it. Each element of the triangle is the sum of the two elements directly above it, which is quite similar to how the entries were created in Mrs. Rogers' grid.

By convention, the "first" row of the triangle is 1, 1, the second row is 1, 2, 1, the third row is 1, 3, 3, 1, and so on. The 1 at the very top of the triangle is known as Row 0. This may appear awkward, but you will see its value in a moment.

In the **Introduction**, we introduced the notation $\binom{n}{k}$. This is known as a **binomial coefficient**, for reasons we explain at the end of the chapter.

Before proceeding any further, calculate each of the binomial coefficients of the following table on the next page: (note: by convention, $\binom{0}{0} = 1$ and $0! = 1$).

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \vdots & & & & &
\end{array}
$$

Did you notice that this table corresponds exactly with Pascal's Triangle? But does this pattern continue indefinitely? It turns out it does. In other words, if we list the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n},$$

we will get precisely the n^{th} row of Pascal's Triangle. (That was why we labeled the rows beginning at Row 0).

Recall that each element of Pascal's Triangle is constructed by adding the two numbers directly above it. Thus, if our table of binomial coefficients is equivalent to Pascal's Triangle, then this table must have the property that each element is the sum of the two numbers directly above it. For example, we must have $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$. And if this property holds, then this will prove that the two tables are equivalent, because the "end" terms match up (that is, $\binom{n}{0} = \binom{n}{n} = 1$, for all $n \geq 1$).

Pick n and k arbitrarily, where $0 < k < n$. If we look at the entry $\binom{n}{k}$, the two entries directly above it are $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$.

It suffices to prove that for all n, k with $0 < k < n$, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This is known as **Pascal's Identity**. Here we provide three proofs of Pascal's Identity, in increasing level of elegance.

Solution A:

Our first solution is strictly algebraic.

By definition, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Substituting, we have

$$\begin{aligned}
\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)\cdot(n-k-1)!} \\
&\quad +\frac{(n-1)!}{k\cdot(k-1)!(n-k-1)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{k+(n-k)}{k(n-k)}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\cdot\frac{n}{k(n-k)} \\
&= \frac{n(n-1)!}{k(k-1)!\cdot(n-k)(n-k-1)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

■

Solution B:

In the **Introduction**, we solved a variety of problems that used the idea of forming committees. Using this idea, we provide a combinatorial proof of Pascal's Identity.

By definition, $\binom{n}{k}$ represents the number of ways we can select a k–member committee from a group of n people. Now consider one of those people. Suppose her name is Tara. There are two possibilities: either Tara is on the committee, or Tara is not on the committee. We will count the number of possible committees in each of these cases, and add up these numbers to determine the total number of possible committees. This total must be equal to $\binom{n}{k}$.

If Tara is on the committee, we must select $k-1$ more people for the committee, from the remaining group of $n-1$ people. This can be done in $\binom{n-1}{k-1}$ ways.

If Tara is not on the committee, we must select k people for the committee, from the remaining group of $n-1$ people. This can be done in $\binom{n-1}{k}$ ways.

Hence, there are $\binom{n-1}{k-1}+\binom{n-1}{k}$ possible k–member committees we can form from a group of n people. By definition, this number is also equal to $\binom{n}{k}$.

Therefore, we conclude that $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$. ■

Solution C:

This solution uses the Route context that we introduced at the beginning of this chapter. Suppose Mrs. Rogers wants to get from $(0,0)$ to the point $(k, n-k)$.

Since this path takes a total of $k+(n-k)=n$ steps, she can select any k of them to be Right steps. Thus, the total number of routes from $(0,0)$ to $(k, n-k)$ is $\binom{n}{k}$. Let us count this number in a different way.

To get to $(k, n-k)$, Mrs. Rogers must walk Right from $(k-1, n-k)$, or Up from $(k, n-k-1)$. Thus, let us count the number of possible routes from $(0,0)$ to each of $(k-1, n-k)$ and $(k, n-k-1)$, and add up these two expressions. By definition, this total must equal $\binom{n}{k}$.

There are $n-1$ steps in any path from $(0,0)$ to $(k-1, n-k)$, and so that the total number of routes is $\binom{n-1}{k-1}$, because Mrs. Rogers makes $k-1$ Right steps.

There are $n-1$ steps in any path from $(0,0)$ to $(k, n-k-1)$, and thus, the total number of routes is $\binom{n-1}{k}$, because Mrs. Rogers makes k Right steps.

Hence, we conclude that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. ■

To prove formulas such as Pascal's Identity, a direct combinatorial approach is often more instructive and elegant than awkward algebraic manipulation.

4.3 More Committees and Routes

As another example of the power and beauty of combinatorial solutions, consider the following problem, which does not have a clean algebraic proof:

Prove that for all positive integers n, the following identity holds:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

We present two proofs, one using committees, and the other using routes. Can you detect the subtle fact that these two proofs are absolutely identical, as one is a disguised form of the other?

Solution A:

Suppose we wish to choose a committee of n people from a group of n girls and n boys. By definition, this can be done in $\binom{2n}{n}$ ways.

We can also count the number of possible committees by considering all the cases: the number of committees with no girls, the number of committees with one girl, the number of committees with two girls and so on. We compute the number of possible committees in each case, then determine its sum. Since we have considered all the cases, this total must equal $\binom{2n}{n}$.

If there are no girls on the committee, then there must be n boys on the committee. By definition, there are $\binom{n}{0}$ ways of selecting the 0 girls and $\binom{n}{n}$ ways of selecting the n boys. Hence, the total number of committees in this case is $\binom{n}{0}\binom{n}{n}$.

If there is 1 girl on the committee, then there must be $n-1$ boys on the committee. There are $\binom{n}{1}$ ways of selecting the 1 girl and $\binom{n}{n-1}$ ways of selecting the $n-1$ boys. Hence, the total number of committees in this case is $\binom{n}{1}\binom{n}{n-1}$.

If there are 2 girls on the committee, then there must be $n-2$ boys on the committee. There are $\binom{n}{2}$ ways of selecting the 2 girls and $\binom{n}{n-2}$ ways of selecting the $n-2$ boys. Hence, the total number of committees in this case is $\binom{n}{2}\binom{n}{n-2}$.

In general, if there are k girls and $n-k$ boys on the committee, then there are $\binom{n}{k}\binom{n}{n-k}$ ways of choosing the committee. And this is true for each k from 0 to n.

Thus, the total number of possible committees of n people that can be formed from a group of n girls and n boys is:

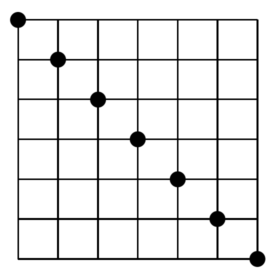
$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}.$$

By definition, this number equals $\binom{2n}{n}$, and so that we have proved that

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0} = \binom{2n}{n}. \qquad \blacksquare$$

Solution B:

Consider the number of routes from $(0,0)$ to (n,n). We mark each of the points $(0,n)$, $(1,n-1)$, $(2,n-2)$, $(3,n-3)$, …, $(n,0)$, and we refer to them as "checkpoints". This is illustrated below for the case $n=6$.



Note that in any route from $(0,0)$ to (n,n), we must hit *exactly* one of these checkpoints. Convince yourself that there is no route that misses a checkpoint, or passes through more than one checkpoint.

Thus, it remains to determine the number of routes passing through each of the checkpoints. We add up all these cases, and that gives us the total number of routes from $(0,0)$ to (n,n), which is equal to $\binom{2n}{n}$.

For example, if Mrs. Rogers passes through the checkpoint $(4,n-4)$, then the first part of her route will take her from $(0,0)$ to $(4,n-4)$, and the second part of her route will take her from $(4,n-4)$ to (n,n).

From our prior reasoning, because Mrs. Rogers must walk n steps to get to $(4,n-4)$, and she has 4 Right steps, the total number of routes from $(0,0)$ to $(4,n-4)$ is $\binom{n}{4}$.

Now, Mrs. Rogers must walk from $(4,n-4)$ to (n,n). There are n total steps in this route, with $n-4$ Right steps. Thus, the total number of routes from $(4,n-4)$ to (n,n) is $\binom{n}{n-4}$.

Therefore, for any route from $(0, 0)$ to (n, n), passing through this checkpoint, Mrs. Rogers can take any of the $\binom{n}{4}$ paths from the origin to the checkpoint, and any of the $\binom{n}{n-4}$ paths from the checkpoint to the destination. Thus, the total number of routes in this case is $\binom{n}{4}\binom{n}{n-4}$.

Using this argument, we find that the total number of routes passing through the checkpoint $(0, n)$ is $\binom{n}{0}\binom{n}{n}$, the total number of routes passing through the checkpoint $(1, n-1)$ is $\binom{n}{1}\binom{n}{n-1}$, and so on. Thus, the total number of routes passing through exactly one of the checkpoints is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}.$$

Since every route passes through exactly one of the checkpoints, this total is equal to the number of routes from $(0, 0)$ to (n, n). Since there are $\binom{2n}{n}$ such routes, we conclude that

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}. \qquad \blacksquare$$

4.4 The Hockey Stick Identity

Observe the numbers boxed in Pascal's Triangle.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & \boxed{1} & & & \\
 & & 1 & & 3 & & \boxed{3} & & 1 & & \\
 & 1 & & 4 & & \boxed{6} & & 4 & & 1 & \\
1 & & 5 & & 10 & & \boxed{10} & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

Notice the boxed number **10** is the sum of all the boxed numbers above it. We have $1 + 3 + 6 = 10$, or

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = \binom{5}{3}.$$

Can we generalize this pattern? For example, one can manually verify that the following identities hold:

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}.$$

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \binom{7}{3} + \binom{8}{3} = \binom{9}{4}.$$

This generalizes to the following formula, which holds for all n and k.

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+k}{n} = \binom{n+k+1}{n+1}.$$

This is known as the **Hockey Stick Identity**, as our numbers form a hockey stick when marked on Pascal's Triangle.

Here we present two proofs to the Hockey Stick Identity, the first using Pascal's Identity, and the second being a combinatorial proof counting the number of routes. Recall that Pascal's Identity states that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof A:

Before proceeding further, it may help to look at an example. Consider the following case, with $k = 3$ and $n = 6$. By repeated use of Pascal's Identity, we have

$$\begin{aligned}
\binom{6}{3} &= \binom{5}{2} + \binom{5}{3} \\
&= \binom{5}{2} + \left[\binom{4}{2} + \binom{4}{3}\right] \\
&= \binom{5}{2} + \binom{4}{2} + \binom{4}{3} \\
&= \binom{5}{2} + \binom{4}{2} + \left[\binom{3}{2} + \binom{3}{3}\right] \\
&= \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{3}{3} \\
&= \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}.
\end{aligned}$$

The last step follows since $\binom{3}{3} = \binom{2}{2} = 1$. We proceed with the actual proof:

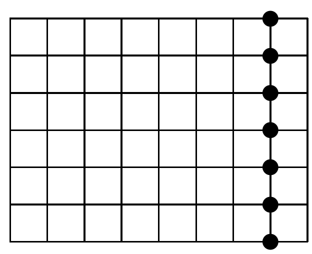
$$\begin{aligned}
\binom{n+k+1}{n+1} &= \binom{n+k}{n} + \binom{n+k}{n+1} \\
&= \binom{n+k}{n} + \left[\binom{n+k-1}{n} + \binom{n+k-1}{n+1}\right] \\
&= \binom{n+k}{n} + \binom{n+k-1}{n} + \binom{n+k-1}{n+1} \\
&= \binom{n+k}{n} + \binom{n+k-1}{n} \\
&\qquad + \left[\binom{n+k-2}{n} + \binom{n+k-2}{n+1}\right] \\
&= \binom{n+k}{n} + \binom{n+k-1}{n} + \binom{n+k-2}{n} + \binom{n+k-2}{n+1} \\
&\ \ \vdots
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n+k}{n}+\binom{n+k-1}{n}+\binom{n+k-2}{n}+\cdots \\
&\qquad +\binom{n+1}{n}+\binom{n+1}{n+1} \\
&= \binom{n+k}{n}+\binom{n+k-1}{n}+\binom{n+k-2}{n}+\cdots \\
&\qquad +\binom{n+1}{n}+\binom{n}{n}.
\end{aligned}
$$

Thus, we have proved the Hockey Stick Identity. ■

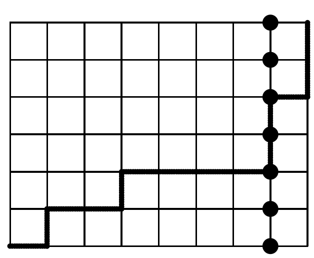
Proof B:

Consider the number of routes from $n(0,0)$ to $(n+1,k)$. We mark each of the points $(n,0)$, $(n,1)$, $(n,2)$, …, (n,k). This is illustrated below for the case $n=7$ and $k=6$.



Earlier we proved an identity using checkpoints. Here we will employ a similar strategy, but with one twist. In each route from $(0,0)$ to $(n+1,k)$, specify the checkpoint to be the *last* point on the path with x–coordinate equal to n.

For example, in the route indicated below, the checkpoint occurs at $(7,4)$, not at $(7,2)$ or $(7,3)$.



Clearly each route has a unique checkpoint, by this definition of checkpoint. Thus, to find the total number of routes, we need to determine the number of routes passing through each of the $k+1$ checkpoints. We add up all of these cases, and that gives us the total number of routes from $(0,0)$ to $(n+1,k)$. We know that this total equals $\binom{n+k+1}{n+1}$.

For example, if Mrs. Rogers passes through the checkpoint $(n,3)$, then the first part of her route will take her from $(0,0)$ to $(n,3)$, and the second part of her route will take her from $(n,3)$ to $(n+1,k)$.

There are $\binom{n+3}{n}$ routes from $(0,0)$ to $(n,3)$. Now, let us calculate the number of routes from $(n,3)$ to $(n+1,k)$. Here is the critical observation: since $(n,3)$ is a checkpoint, that must mean that Mrs. Rogers takes a Right step at $(n,3)$, otherwise $(n,3)$ would not be the checkpoint of the path.

Thus, if $(n,3)$ is a checkpoint, then the only route from $(n,3)$ to $(n+1,k)$ is a Right step, followed by Up steps all the way from $(n+1,3)$ to $(n+1,k)$. Thus, there is only one such route. Therefore, the total number of routes from $(0,0)$ to $(n+1,k)$ with $(n,3)$ as a checkpoint is $\binom{n+3}{n} \times 1 = \binom{n+3}{n}$.

Similarly, we can justify that the number of routes with $(n,0)$ as a checkpoint is $\binom{n}{n} \times 1$, the number of routes with $(n,1)$ as a checkpoint is $\binom{n+1}{n} \times 1$, and finally, the number of routes with (n,k) as a checkpoint is $\binom{n+k}{n} \times 1$.

Adding up all these cases, we arrive at the total number of routes that have one of those checkpoints. This sum is

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+k}{n}.$$

Since every route has a unique checkpoint, this sum is equal to the number of routes from $(0,0)$ to $(n+1,k)$. Since there are $\binom{n+k+1}{n+1}$ such routes, we conclude that

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+k}{n} = \binom{n+k+1}{n+1}. \qquad \blacksquare$$

Using the Hockey Stick Identity, we can make a beautiful connection to the **Handshakes** chapter. Recall that in the **Handshakes** chapter, we found a formula for the sum of the first n integers. We proved that

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Here we use the Hockey Stick Identity to come up with yet another proof of this formula:

$$\begin{aligned} 1+2+3+\cdots+n &= \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} \\ &= \binom{n+1}{2} \\ &= \frac{n(n+1)}{2}. \end{aligned} \qquad \blacksquare$$

4.5 A Formula for the Sum of Squares

Let us derive a formula for the sum of the first n squares, namely

$$1^2 + 2^2 + \cdots + n^2.$$

Unfortunately, this will not be as straightforward as the previous identity, as n^2 does not have a "nice" binomial coefficient representation. However, we do know that $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$. Does this help us?

We can be creative, and write n^2 as a combination of simple binomial coefficients. Since $n^2 - n = 2 \cdot \binom{n}{2}$, we have $n^2 = 2 \cdot \binom{n}{2} + n = 2 \cdot \binom{n}{2} + \binom{n}{1}$. Note that this identity is valid for $n = 1$, as $\binom{1}{2} = 0$, by definition. Thus, we have:

$$\begin{aligned}
1^2 &= 2 \cdot \binom{1}{2} + \binom{1}{1}, \\
2^2 &= 2 \cdot \binom{2}{2} + \binom{2}{1}, \\
3^2 &= 2 \cdot \binom{3}{2} + \binom{3}{1}, \\
&\vdots \\
n^2 &= 2 \cdot \binom{n}{2} + \binom{n}{1}.
\end{aligned}$$

Adding up these n equations and using the Hockey Stick Identity, we have:

$$\begin{aligned}
1^2 + 2^2 + \cdots + n^2 &= \left[2 \cdot 0 + \binom{1}{1}\right] + \left[2 \cdot \binom{2}{2} + \binom{2}{1}\right] + \cdots \\
&\quad + \left[2 \cdot \binom{n}{2} + \binom{n}{1}\right] \\
&= 2\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right] \\
&\quad + \left[\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1}\right] \\
&= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= 2 \cdot \frac{(n+1)(n)(n-1)}{3 \cdot 2 \cdot 1} + \frac{(n+1)n}{2} \\
&= n(n+1)\left(\frac{2(n-1)}{6} + \frac{1}{2}\right) \\
&= n(n+1)\left(\frac{2n-2}{6} + \frac{3}{6}\right) \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Thus, we have proved that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \qquad \blacksquare$$

4.6 The Binomial Theorem

Throughout this chapter, we have referred to the terms $\binom{n}{k}$ as **binomial coefficients**. Let us finally explain why we call these as such.

Begin by expanding the binomial $(x+y)^n$ for each $n \leq 5$. We have:

$$
\begin{aligned}
(x+y)^1 &= x+y, \\
(x+y)^2 &= x^2+2xy+y^2, \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3, \\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4, \\
(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.
\end{aligned}
$$

Look carefully at the coefficients. Notice that the coefficients of $(x+y)^n$ correspond exactly to the elements in the n^{th} row of Pascal's Triangle, namely

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{k}.$$

This is true for all n. This gives rise to the **Binomial Theorem**, which states that for all positive integers n, we have

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n}x^0 y^n.$$

A proof of the Binomial Theorem is a straightforward application of Pascal's Identity and Mathematical Induction. We invite you to fill in the details. We also refer you to Question 6 of the **Problem Set**, which asks you to derive a combinatorial proof of the Binomial Theorem.

There are several interesting applications of the Binomial Theorem. Here we describe two such applications.

Before proceeding further, add up all the elements in the fourth row of Pascal's Triangle. What is the sum? Now do the same thing with the fifth row. What is its sum? Conjecture a formula for the sum of the elements of the n^{th} row of Pascal's Triangle. Let us prove your conjecture.

Let $x = 1$ and $y = 1$ and substitute these values into the Binomial Theorem. We have

$$(1+1)^n = \binom{n}{0}1^n \cdot 1^0 + \binom{n}{1}1^{n-1} \cdot 1^1 + \binom{n}{2}1^{n-2} \cdot 1^2 + \cdots + \binom{n}{n}1^0 \cdot 1^n.$$

And this simplifies to

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \;=\; 2^n.$$

This application of the Binomial Theorem has given us a simple proof that the sum of the elements in the n^{th} row of Pascal's Triangle is 2^n. ∎

Now, let $x = 1$ and $y = -1$, and substitute these values into the Binomial Theorem. We have

$$\begin{aligned}(1-1)^n \;=\; & \binom{n}{0} 1^n \cdot (-1)^0 + \binom{n}{1} 1^{n-1} \cdot (-1)^1 \\ & + \binom{n}{2} 1^{n-2} \cdot (-1)^2 + \cdots \\ & + \binom{n}{n} 1^0 \cdot (-1)^n.\end{aligned}$$

This simplifies to

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} \;=\; 0.$$

Thus, using the Binomial Theorem, we have given a simple proof to the result that if we alternately add and subtract the elements in the n^{th} row of Pascal's Triangle, we always get a total of 0. We note that this result is obvious for odd values of n, as the positive and negative terms cancel in pairs, for example, $1 - 5 + 10 - 10 + 5 - 1 = 0$.

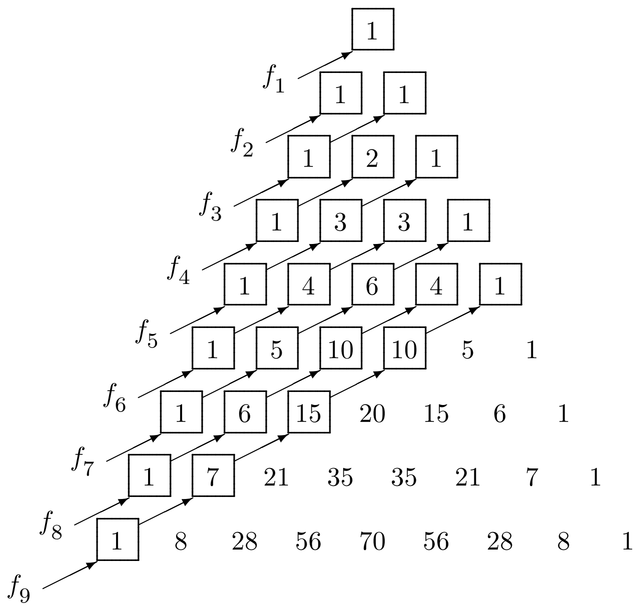
4.7 The Fibonacci Sequence

We close this chapter by mentioning the **Fibonacci sequence**, and describing its connection to Pascal's Triangle.

The Fibonacci sequence starts with the terms 1, 1, and each additional element in the sequence is the sum of the two preceding elements. Thus, the Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

The Fibonacci sequence has made numerous appearances in a wide variety of interesting contexts, such as in pineapples and sunflowers, in architectural designs like the Parthenon, and even in Beethoven's Fifth Symphony!

Starting with Pascal's Triangle, you can get the Fibonacci sequence!



For example, we have the following sums:

$$\begin{aligned}\binom{4}{0}+\binom{3}{1}+\binom{2}{2} &= 5,\\ \binom{5}{0}+\binom{4}{1}+\binom{3}{2} &= 8,\\ \binom{6}{0}+\binom{5}{1}+\binom{4}{2}+\binom{3}{3} &= 13.\end{aligned}$$

Can you develop a general identity and prove it?

4.8 Problem Set

1. Mrs. Rogers wants to travel from her house at $(0,0)$ to watch a hockey game at the arena, which is located at $(4,7)$. However, she first wants to drop by the Toronto Maple Leafs Store to purchase an authentic Maple Leafs jersey. The store is located at $(2,3)$. How many routes are there from her house to the arena, which pass through the store?

2. There are 126 routes from $(0,0)$ to $(n,5)$. Determine the value of n.

3. Consider n lightbulbs in a room, numbered 1 to n. Determine the number of ways the lightbulbs can be turned on or off. By solving this problem in two different ways, prove that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n.$$

4. Let n be a positive integer. For a group of n people, show that the number of possible committees that can be formed with an odd number of people is equal to the number of possible committees that can be formed with an even number of people.

5. Starting from any letter on the outside and moving to an adjacent letter (including diagonally), find the number of ways of spelling "MATH" in the following array:

M	M	M	M	M	M	M
M	A	A	A	A	A	M
M	A	T	T	T	A	M
M	A	T	H	T	A	M
M	A	T	T	T	A	M
M	A	A	A	A	A	M
M	M	M	M	M	M	M

6. Find a combinatorial proof of the Binomial Theorem. Using either a committee or route argument, explain why the $x^{n-k}y^k$ coefficient of $(x+y)^n$ must be $\binom{n}{k}$.

7. (a) Expand $(x+y+z)^2$.
   (b) Expand $(x+y+z)^3$.
   (c) Develop a **Multinomial Theorem** by creating a formula for
$$(x_1 + x_2 + \cdots + x_k)^n.$$

8. In the third investigation of the **Handshakes** chapter, we introduced the Ramsey number $R(x, y)$. The final problem of the investigation was to prove that for any integers s, $t \geq 2$,
$$R(s,t) \;\leq\; R(s-1,t) + R(s,t-1).$$
Assuming this result, use the Hockey Stick Identity to prove that
$$R(s,t) \;\leq\; \binom{s+t-2}{s-1}.$$
(Hint: for all s, $t \geq 2$, we have $R(s, 2) = s$ and $R(2, t) = t$).

9. (a) For $n \geq 3$, determine the unique integers A, B, and C for which
$$n^3 \;=\; A\binom{n}{3} + B\binom{n}{2} + C\binom{n}{1}.$$
   (b) Using part (a) and the Hockey Stick Identity, prove that
$$1^3 + 2^3 + \cdots + n^3 \;=\; \frac{n^2(n+1)^2}{4}.$$

10. A certain student, having just finished a particularly hairy problem, stared blankly at an x_1y_2 which was written on her scrap paper. Following some doodling, the student wrote:

(Line 0)	x_1y_2,
(Line 1)	$x_1y_2y_3x_4$,
(Line 2)	$x_1y_2y_3x_4y_5x_6x_7y_8$,
(Line 3)	$x_1y_2y_3x_4y_5x_6x_7y_8, y_9x_{10}x_{11}y_{12}x_{13}y_{14}y_{15}x_{16}$.

On each line, the student copied the line above exactly, and then copied it again, changing x's to y's, y's to x's, and continuing the subscripts in order.

Notice that in Line 1, the sum of the x–subscripts equals the sum of the y–subscripts, namely $1 + 4 = 2 + 3$.

Notice that in Line 2, the sum of the squares of the x–subscripts equals the sum of the squares of the y–subscripts, namely

$$1^2 + 4^2 + 6^2 + 7^2 \ = \ 2^2 + 3^2 + 5^2 + 8^2.$$

The student wondered if there was a pattern here. She noticed that:

$$\begin{aligned}
1^0 &= 2^0, \\
1^1 + 4^1 &= 2^1 + 3^1, \\
1^2 + 4^2 + 6^2 + 7^2 &= 2^2 + 3^2 + 5^2 + 8^2, \\
1^3 + 4^3 + 6^3 + 7^3 + 10^3 + 11^3 + 13^3 + 16^3 & \\
&= 2^3 + 3^3 + 5^3 + 8^3 + 9^3 + 12^3 + 14^3 + 15^3.
\end{aligned}$$

This student was amazed! It appears that in Line n, the sum of the n^{th} powers of the x–subscripts always equals the sum of the n^{th} powers of the y–subscripts. Is this result correct? Prove or disprove this conjecture.

4.9 Investigation 1: Unfriendly Subsets

Consider all subsets of $\{1, 2, 3, \ldots, n\}$ with k elements. Such a subset is called *unfriendly* if no two of its elements are consecutive. For example, if we have $k = 3$ and $n = 7$, the subsets $\{1, 3, 7\}$ and $\{2, 4, 6\}$ are unfriendly, whereas $\{3, 6, 7\}$ is not.

Determine the number of unfriendly subsets for each of the following cases:

(i)	$k = 3$ and $n = 5$.	(ii)	$k = 3$ and $n = 6$.
(iii)	$k = 3$ and $n = 7$.	(iv)	$k = 3$ and $n = 8$.
(v)	$k = 3$ and $n = 9$.	(vi)	$k = 4$ and $n = 7$.
(vii)	$k = 4$ and $n = 8$.	(viii)	$k = 4$ and $n = 9$.

Consider other small cases, and try to develop a formula for the number of unfriendly subsets, in terms of n and k. (Hint: the formula is a nice binomial coefficient).

If you have a formula, attempt to develop a combinatorial proof that your answer is correct.

4.10 Investigation 2: Combinatorial Identities

This investigation considers more combinatorial identities, as sources for developing contexts involving routes and committees.

Here we describe three such examples and invite you to formulate more scenarios, which will give rise to new combinatorial identities.

1. Earlier in this chapter, we used the concept of *checkpoints* to develop two combinatorial identities.

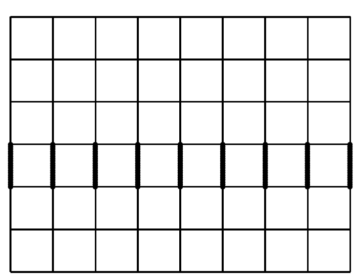
   Now, use a different set of checkpoints to develop a new identity. For example, consider the set of routes from $(0,0)$ to (n,n), where $n \geq 2$. Define three checkpoints at $(0,2)$, $(1,1)$, and $(2,0)$. Begin by convincing yourself that each route from $(0,0)$ to (n,n) must pass through exactly one of these checkpoints. Use this observation to prove that

   $$\binom{2n}{n} = \binom{2}{0}\binom{2n-2}{n} + \binom{2}{1}\binom{2n-2}{n-1} + \binom{2}{2}\binom{2n-2}{n-2}.$$

   Derive other identities using this idea of checkpoints.

2. Instead of making checkpoints, consider this notion of *checklines*.

   In this figure, find the number of routes from $(0,0)$ to $(8,6)$, where the dark lines indicate checklines. In other words, every path from $(0,0)$ to $(8,6)$ will pass through a unique checkline.

   Using the diagram, prove that

   $$\binom{14}{8} = \binom{2}{0}\binom{11}{8} + \binom{3}{1}\binom{10}{7} + \binom{4}{2}\binom{9}{6} + \cdots + \binom{10}{8}\binom{3}{0}.$$

Using checklines, what other identities can you derive? For example, try to use checklines to prove that

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 \;=\; \binom{n+2}{3}.$$

3. Play with variations of committees. For example, how many ways can you choose an r–person committee from a group of n people, if the committee must have a *group leader*? The answer is $r \cdot \binom{n}{r}$ since we can choose our committee $\binom{n}{r}$ ways, and any of those r people can be chosen as the leader of the group.

   Using this idea of a group leader, prove that

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} \;=\; n \cdot 2^{n-1}.$$

   What other identities can you derive using this idea of group leaders?

4.11 Investigation 3: Pascal's Odd Triangle

Count the number of *odd* elements that appear in each row of Pascal's Triangle. The results for small row numbers appear in the table below:

Row Number	**Number of Odd Elements in Row**
0	1
1	2
2	2
3	4
4	2
5	4
6	4
7	8
8	2
9	4

Notice that all numbers in the second column are powers of 2.

It appears that the following conjecture is true for any row of Pascal's Triangle: the number of odd elements is a power of 2.

Investigate this conjecture. To do this, try to prove an even stronger result, which relates the actual power of 2 with the **binary representation** of n (see **Glossary**).

Let $f(n)$ denote the number of 1's that appear in the binary representation of n. For example, $f(13) = 3$ because $13 = 1101_2$ and $f(18) = 2$ because $18 = 10010_2$.

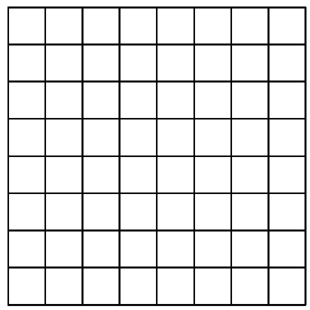
Let $g(n)$ denote the number of odd elements in the n^{th} row of Pascal's Triangle.

Make a list of $f(n)$ and $g(n)$ for each n up to 15. Do you notice any interesting patterns? What conjecture can you make? Can you prove it?

5 Checkerboards

5.1 The Checkerboard Problem

Consider the following 8 by 8 checkerboard:



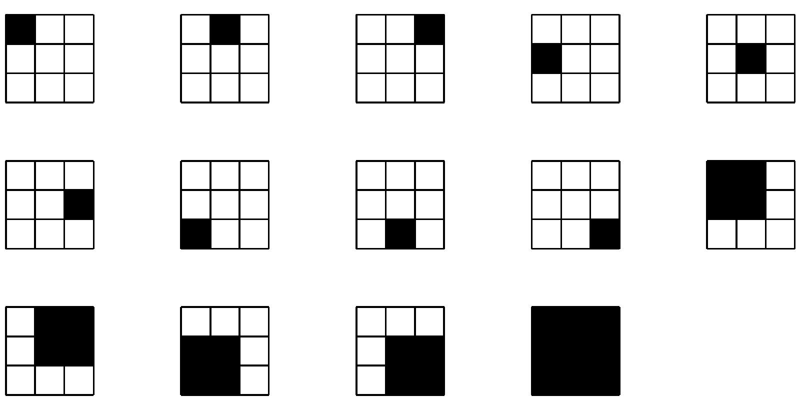
How many squares (of all sizes) appear on this checkerboard?

To get a feel for this problem, work with smaller sized checkerboards. Let our checkerboard be n by n, and count the number of squares that appear for small values of n.

For example, if $n = 0$, then there are no squares to be counted. And if $n = 1$, there is only one square to be counted.

If $n = 2$, then there are four 1 by 1 squares and one 2 by 2 square, or five squares in total.

If $n = 3$, then there are nine 1 by 1 squares, four 2 by 2 squares, and one 3 by 3 square. Thus, there are fourteen squares in total. These fourteen squares are illustrated below.



Work through the $n = 4$ case yourself, and ensure that you count thirty squares.

Summarizing this information in a table, we have the following:

Side Length n	Number of Squares
0	0
1	1
2	5
3	14
4	30

Consider the numbers in the second column. Is there a nice pattern in the sequence 0, 1, 5, 14, 30?

As we did in the **Handshake** problem, examine the differences between consecutive terms of the sequence, and see if we notice a pattern.

$$\mathbf{0}\underbrace{\qquad}_{1}\mathbf{1}\underbrace{\qquad}_{4}\mathbf{5}\underbrace{\qquad}_{9}\mathbf{14}\underbrace{\qquad}_{16}\mathbf{30}$$

This appears to be the sequence of perfect squares!

Assuming that this pattern continues, we suspect the following sequence would emerge:

$$\mathbf{0}\underbrace{\qquad}_{1}\mathbf{1}\underbrace{\qquad}_{4}\mathbf{5}\underbrace{\qquad}_{9}\mathbf{14}\underbrace{\qquad}_{16}\mathbf{30}\underbrace{\qquad}_{25}\mathbf{55}\underbrace{\qquad}_{36}\mathbf{91}\underbrace{\qquad}_{49}\mathbf{140}\underbrace{\qquad}_{64}\mathbf{204}$$

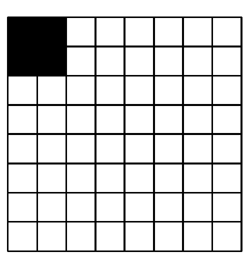
Since we are interested in an 8 by 8 checkerboard, it appears that the answer to our problem is $1^2 + 2^2 + 3^2 + \cdots + 8^2$, which equals **204**.

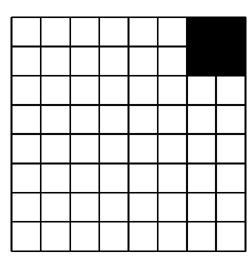
How can we *prove* that the correct answer is 204? We need to rigorously justify the number of squares of each size.

There are sixty-four squares that are 1 by 1, since there are 8^2 squares in our original checkerboard.

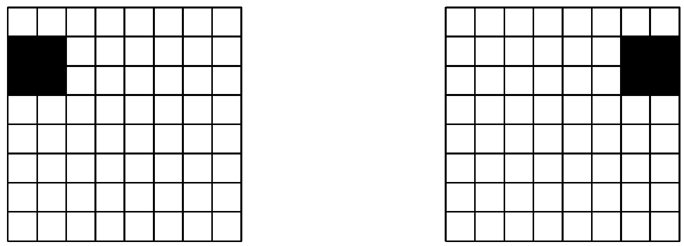
How many 2 by 2 squares are there? Instead of counting them manually, approach this more systematically. How many different 2 by 2 squares are there in the first two rows of our checkerboard?

There are seven such squares, with the "leftmost" and the "rightmost" squares illustrated in the diagram:

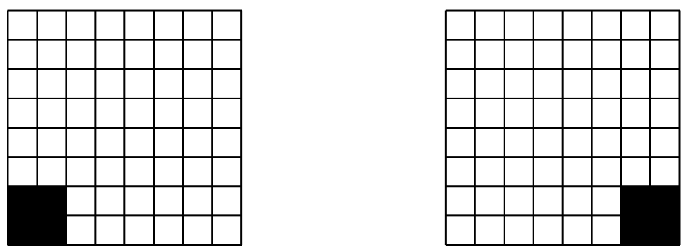




Similarly, there are seven 2 by 2 squares in the second and third rows, with the leftmost and rightmost illustrated:

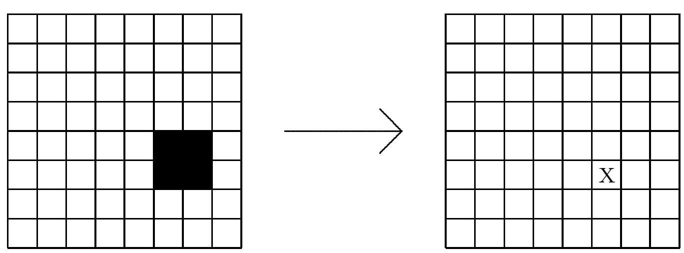


Continuing, we have seven 2 by 2 squares in each pair of two rows, with the leftmost and rightmost in the final pair illustrated:

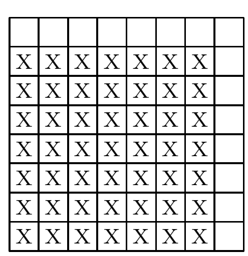


All in all, the number of 2 by 2 squares is $7 \times 7 = 7^2 = 49$.

We justify this in another way. For each 2 by 2 square, mark an “X” in the lower left corner. Note that each X corresponds to a unique 2 by 2 square. This is illustrated as shown.



Now it remains to determine the number of locations that we can place the X. Since our X represents the lower left corner of our 2 by 2 square, the X cannot go in the top or right row of our checkerboard. However, any other location is permissible.



Since there are $7 \times 7 = 49$ possible locations that we can place our X, we conclude that the number of 2 by 2 squares is 49.

Repeating this argument, we find that there are $6 \times 6 = 36$ possible locations for the lower left corner if we are counting 3 by 3 squares. Thus, the number of 3 by 3 squares is $6 \times 6 = 6^2 = 36$.

We can repeat this argument for each square size. We put this information into a table:

Side of Square	Number of Squares
1	8^2
2	7^2
3	6^2
4	5^2
5	4^2
6	3^2
7	2^2
8	1^2

The total number of squares on our checkerboard is

$$8^2 + 7^2 + \cdots + 2^2 + 1^2 \ = \ 204,$$

which confirms our earlier conjecture. The correct answer indeed is 204. ■

For brevity, we introduce **Sigma notation** (see **Glossary**). By definition,

$$\sum_{i=1}^{8} i^2 \ = \ 1^2 + 2^2 + \cdots + 8^2.$$

In other words, $\sum_{i=1}^{8} i^2$ represents the sum of all numbers of the form i^2, as i ranges from 1 to 8.

We have proved that the number of squares (of all sizes) on an 8 by 8 checkerboard is $\sum_{i=1}^{8} i^2$. What do you guess will be an expression for the number of squares (of all sizes) on an n by n checkerboard? We suspect the answer is $\sum_{i=1}^{n} i^2$.

Consider the table more carefully. Notice that on our 8 by 8 checkerboard, there are $(9 - i)^2$ squares of side length i. And this formula holds for each i.

Can we make a more general conjecture? If we had an n by n checkerboard, how many squares of side length i would there be? It appears that the formula should be $(n+1-i)^2$, since the $n = 8$ case gives us $(9-i)^2$ squares of side length i. This is indeed the case, and we invite you to justify this formally in the **Problem Set**.

Thus, on an n by n checkerboard, there are n^2 squares of side length 1, $(n - 1)^2$ squares of side length 2, $(n - 2)^2$ squares of side length 3, all the way down to 2^2 squares of side length $(n - 1)$ and 1^2 square of side length n.

Hence, the total number of squares (of all sizes) on an n by n checkerboard is $n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2$, which we can naturally rewrite as $1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$.

We have proved that there are $\sum_{i=1}^{n} i^2$ squares (of all sizes) on an n by n checkerboard, and that this formula holds for all n.

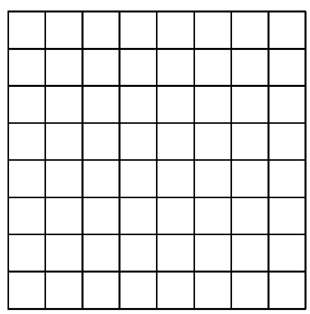
In the **Routes** chapter, we proved that

$$\sum_{i=1}^{n} i^2 \;=\; 1^2 + 2^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}.$$

Later in this chapter, we shall derive this identity using a different approach.

5.2 Counting Rectangles on a Checkerboard

Revisit our original 8 by 8 checkerboard, and consider a similar question.



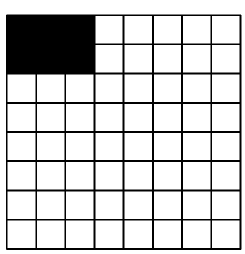
How many rectangles (of all sizes) appear on this checkerboard?

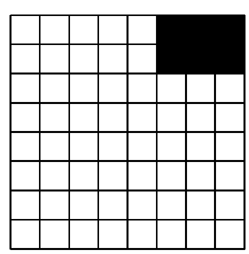
(Note: a square counts as a rectangle!)

Solution A:

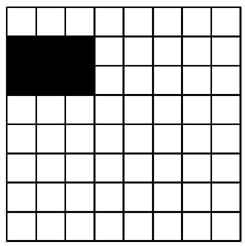
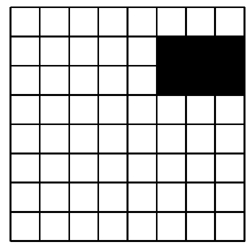
To illustrate the concept, we consider the number of 2 by 3 rectangles on our checkerboard (that is, 2 rows by 3 columns). In our notation, a 2 by 3 rectangle will be different from a 3 by 2 rectangle.

There are six such rectangles in the first two rows of the checkerboard, with the "leftmost" and the "rightmost" rectangles illustrated in the diagram:

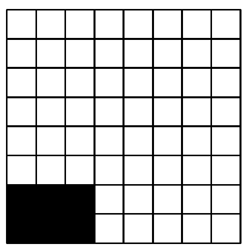
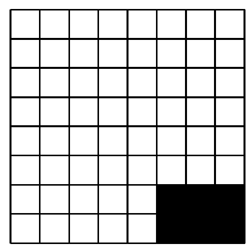




There are six such rectangles in the second and third rows, with the leftmost and rightmost illustrated:

Continuing, we have six such rectangles in each pair of two rows, with the leftmost and rightmost in the final pair illustrated:

Therefore, the total number of 2 by 3 rectangles on our checkerboard is 7×6.

We can also determine this number by marking each rectangle with an "X" in the lower left corner. Thus, the problem reduces to finding the number of places where we can place this X. We see that the X can be placed in each of the following locations:

X	X	X	X	X	X		
X	X	X	X	X	X		
X	X	X	X	X	X		
X	X	X	X	X	X		
X	X	X	X	X	X		
X	X	X	X	X	X		
X	X	X	X	X	X		

Since there are $7 \times 6 = 42$ possible locations that we can place our X, we conclude that this is the number of 2 by 3 rectangles on our checkerboard.

Similarly, if we were to count the total number of 5 by 7 rectangles in our checkerboard, we will find that there are 4×2 such rectangles on our checkerboard, using the same counting argument. Convince yourself that for all a and b, the total number of a by b rectangles on our 8 by 8 checkerboard is $(9-a) \times (9-b)$. Thus, to find the total number of rectangles on our checkerboard, we compute the value of $(9-a) \times (9-b)$ for each ordered pair (a, b) with $1 \le a,\ b \le 8$, and add these values.

We will represent this information in a table by writing down the 64 different ordered pairs (a, b), and for each, computing the value of $(9-a)\times(9-b)$. Although it may appear strange, we will write the ordered pairs backwards; that is, starting with $(8, 8)$ and ending with $(1, 1)$.

(a, b)	Number of Rectangles
$(8, 8)$	1×1
$(8, 7)$	1×2
$(8, 6)$	1×3
$(8, 5)$	1×4
$(8, 4)$	1×5
$(8, 3)$	1×6
$(8, 2)$	1×7
$(8, 1)$	1×8
$(7, 8)$	2×1
$(7, 7)$	2×2
$(7, 6)$	2×3
$(7, 5)$	2×4
$(7, 4)$	2×5
$(7, 3)$	2×6
$(7, 2)$	2×7
$(7, 1)$	2×8
$\vdots$	$\vdots$
$(1, 8)$	8×1
$(1, 7)$	8×2
$(1, 6)$	8×3
$(1, 5)$	8×4
$(1, 4)$	8×5
$(1, 3)$	8×6
$(1, 2)$	8×7
$(1, 1)$	8×8

Although it may be tempting to multiply each of the numbers in the second column before adding them up, let us search for patterns in the table. Notice that the sum of the first eight terms is

$$1 \times 1 + 1 \times 2 + 1 \times 3 + \cdots + 1 \times 8 \ = \ 1 \cdot (1 + 2 + 3 + \cdots + 8).$$

Similarly, the sum of the next eight terms is $2 \cdot (1+2+3+\cdots+8)$, the sum of the next eight terms is $3 \cdot (1 + 2 + 3 + \cdots + 8)$, and finally, the sum of the last eight terms is $8 \cdot (1 + 2 + 3 + \cdots + 8)$. Therefore, if we let S be the sum of the 64 products in the second column, then:

$$\begin{aligned} S \ &= \ 1 \cdot (1 + 2 + 3 + \cdots + 8) + 2 \cdot (1 + 2 + 3 + \cdots + 8) + \cdots \\ &\qquad\qquad + 8 \cdot (1 + 2 + 3 + \cdots + 8) \end{aligned}$$

$$
\begin{aligned}
&= (1+2+3+\cdots+8)\cdot(1+2+3+\cdots+8) \\
&= (1+2+3+\cdots+8)^2.
\end{aligned}
$$

Since $1+2+3+\cdots+8 = 36$, we have proved that $S = 36^2 = 1296$. Thus, there are 1296 rectangles (of all sizes) on an 8 by 8 checkerboard. ■

Using the same argument, we can show that the total number of rectangles (of all sizes) on an n by n checkerboard is

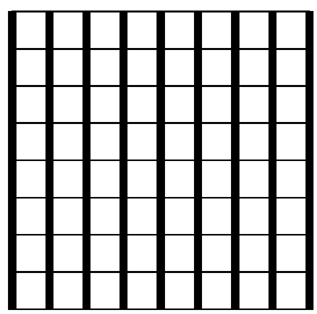
$$(1+2+3+\cdots+n)^2 = \left(\sum_{i=1}^{n} i\right)^2.$$

We invite you to provide the details.

Solution B:

Here is yet another method for finding the number of rectangles on an n by n checkerboard. We find this solution to be extremely satisfying.

A visual proof is presented for the case $n = 8$. We will prove that there are 1296 rectangles on an 8 by 8 checkerboard.

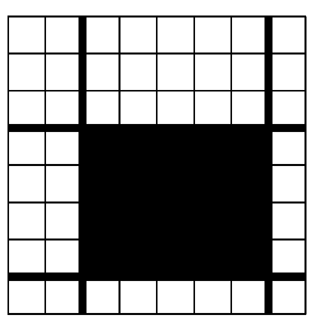


There are nine vertical lines on our checkerboard, which are indicated, as shown.

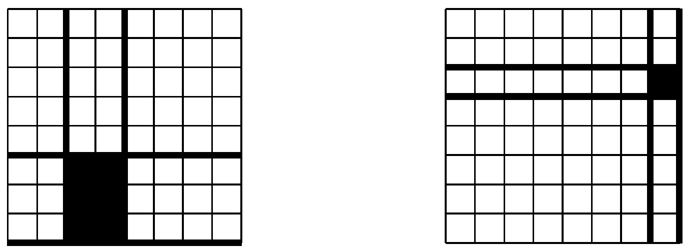
Now, choose any two of these vertical lines. There are $\binom{9}{2}$ ways of selecting two of these nine vertical lines.

Similarly, there are nine horizontal lines on our checkerboard. Choose any two of these lines. There are $\binom{9}{2}$ ways of selecting two of these nine horizontal lines.

Here is the key observation: each choice of two vertical lines and two horizontal lines determines a *unique* rectangle inside our checkerboard.



Here are two more examples of rectangles formed by choosing two horizontal and two vertical lines:



By this process, each rectangle is uniquely accounted for, so that this procedure of picking two horizontal lines and two vertical lines gives us all the rectangles on our checkerboard.

There are $\binom{9}{2}\binom{9}{2} = 36 \times 36 = 1296$ ways of selecting two vertical lines and two horizontal lines from our checkerboard, and thus, we conclude that there are 1296 total rectangles! ■

We can readily generalize this solution. If we have an n by n checkerboard, there are $(n+1)$ horizontal lines, and $(n+1)$ vertical lines. Therefore, the total number of rectangles (of all sizes) on an n by n checkerboard must be $\binom{n+1}{2}\binom{n+1}{2} = \binom{n+1}{2}^2$.

Earlier, we proved that the number of rectangles (of all sizes) on an n by n checkerboard is $(1+2+3+\cdots+n)^2$. Therefore, we have proved that

$$(1+2+3+\cdots+n)^2 \;=\; \binom{n+1}{2}^2.$$

Taking the square root of both sides, we have shown once again that

$$1+2+3+\cdots+n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$ ■

In the **Handshakes** chapter, we used a "handshake" argument to prove this identity. Later we presented several other solutions as well. In the **Routes** chapter, we used the Hockey-Stick Identity to derive another proof. This "rectangles on a checkerboard" argument offers yet another proof that the sum of the first n positive integers is $\frac{n(n+1)}{2}$. One of the rewarding aspects of studying combinatorics is discovering surprising ways in which different topics are interconnected.

5.3 Connection to the Sum of Cubes

Just for fun, compute the value of $(1+2+\cdots+n)^2$ for small values of n, and see if we notice any other interesting patterns. Since this expression is not defined for $n=0$, call the value 0 in this special case.

n	$(1+2+\cdots+n)^2$
0	0
1	1
2	9
3	36
4	100
5	225
6	441

When we look at the differences between successive terms, we get the following:

$$\mathbf{0}\underbrace{}_{1}\mathbf{1}\underbrace{}_{8}\mathbf{9}\underbrace{}_{27}\mathbf{36}\underbrace{}_{64}\mathbf{100}\underbrace{}_{125}\mathbf{225}\underbrace{}_{216}\mathbf{441}$$

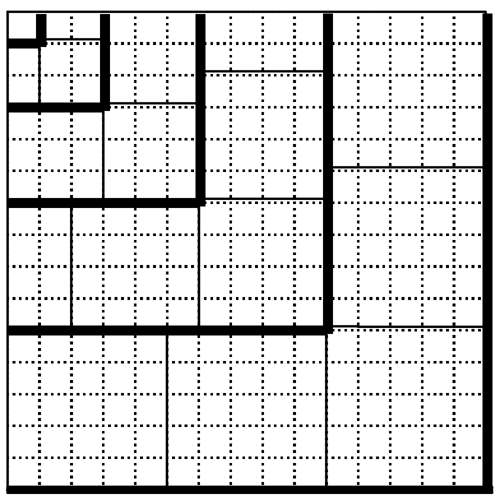
These differences are consecutive perfect cubes!

Note that $1 = 1^3$, $9 = 1^3 + 2^3$, $36 = 1^3 + 2^3 + 3^3$, $100 = 1^3 + 2^3 + 3^3 + 4^3$, and so on. Since every new term adds the next perfect cube, it appears that

$$1^3 + 2^3 + \cdots + n^3 \;=\; (1 + 2 + \cdots + n)^2.$$

In other words, we conjecture that the sum of the first n cubes is equal to the square of the sum of the first n positive integers!

Here, we provide a geometric proof of the identity $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for the case $n = 5$. Consider this diagram.



This diagram is a large checkerboard, consisting of $(1 + 2 + 3 + 4 + 5)$ unit squares on each side. Thus, the total number of unit squares in this figure is $(1 + 2 + 3 + 4 + 5)^2$.

Count the number of unit squares another way. Partition the checkerboard into five reverse L–shaped pieces, as illustrated by the dark lines in the diagram. Count the number of unit squares in each reverse L–shaped piece.

The first piece consists of 1 square. Observe that $1^3 = 1$.

The second piece consists of a 2 by 2 square, as well as two rectangles that can be joined to form another 2 by 2 square. Thus, the total number of unit squares in this piece is $2 \times 2 + 2 \times 2 = 2 \cdot 2^2 = 2^3$.

The third piece consists of three 3 by 3 squares. Therefore, the total number of unit squares in this piece is $3 \times 3 \times 3 = 3^3$.

The fourth piece consists of three 4 by 4 squares, as well as two rectangles that can be joined to form another 4 by 4 square. Thus, the total number of unit squares in this piece is $4 \times 4 \times 4 = 4^3$.

The fifth piece consists of five 5 by 5 squares. Therefore, the total number of unit squares in this piece is $5 \times 5 \times 5 = 5^3$.

Adding these numbers up, we find that the total number of unit squares in this diagram is $1^3 + 2^3 + 3^3 + 4^3 + 5^3$.

Therefore, we have given a geometric proof that

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ = \ (1 + 2 + 3 + 4 + 5)^2. \qquad \blacksquare$$

By extending this diagram, we can use the same reasoning to prove that, for any integer n, we have $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$. To do this, we only need to prove that the n^{th} reverse L–shaped piece has a total of n^3 unit squares.

The key observation is the following: the n^{th} reverse L–shaped piece is formed by taking a square of side length $(1 + 2 + \cdots + n)$ and removing a square of side length $(1 + 2 + \cdots + (n-1))$, from the top-left corner.

For example, the number of squares in the fifth reverse L–shaped piece is

$$\begin{aligned} &(1 + 2 + 3 + 4 + 5)^2 - (1 + 2 + 3 + 4)^2 \\ &\quad = \ 15^2 - 10^2 \ = \ (15 + 10)(15 - 10) \ = \ 25 \cdot 5 \ = \ 125 \ = \ 5^3. \end{aligned}$$

The number of squares in the n^{th} reverse L–shaped piece is equal to $a^2 - b^2$, where $a = 1 + 2 + \cdots + n$, and $b = 1 + 2 + \cdots + (n-1)$.

Since $a = \dfrac{n(n+1)}{2} = \dfrac{n^2 + n}{2}$, and $b = \dfrac{(n-1)n}{2} = \dfrac{n^2 - n}{2}$, we have

$$a + b \ = \ \frac{n^2 + n}{2} + \frac{n^2 - n}{2} \ = \ n^2, \quad \text{and} \quad a - b \ = \ \frac{n^2 + n}{2} - \frac{n^2 - n}{2} \ = \ n.$$

Thus, the number of squares that appear in the n^{th} reverse L–shaped piece is $a^2 - b^2 = (a + b)(a - b) = n^2 \cdot n = n^3$.

Therefore, we have algebraically verified that our geometric analysis is valid, and thus, we conclude that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$. $\blacksquare$

Earlier in this chapter, as well as in the other two chapters, we proved that

$$\sum_{i=1}^{n} i \ = \ \frac{n(n+1)}{2}.$$

We just proved that

$$\sum_{i=1}^{n} i^3 \ = \ \left(\sum_{i=1}^{n} i \right)^2.$$

Thus, it follows that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}.$$

We have derived an explicit formula for the sum of the first n cubes.

5.4 Telescoping Series

Recall that we derived a formula for $\sum_{i=1}^{n} i^2$ in the **Routes** chapter. Here, we will look at an approach that does not require the use of binomial coefficients. A technique known as a **Telescoping Series** shall be employed.

Start with the identity $-k^3+(k+1)^3 = 3k^2+3k+1$. Note that this identity follows from $(k+1)^3 = k^3+3k^2+3k+1$, which follows from the Binomial Theorem that was derived in the **Routes** chapter.

This identity is true for all values of k. Specifically, this equation holds for each of $k = 1, 2, 3, \ldots, n$. Thus, we have:

$$\begin{aligned} -1^3+2^3 &= 3\times 1^2+3\times 1+1, \\ -2^3+3^3 &= 3\times 2^2+3\times 2+1, \\ -3^3+4^3 &= 3\times 3^2+3\times 3+1, \\ &\vdots \\ -(n-1)^3+n^3 &= 3\times (n-1)^2+3\times (n-1)+1, \\ -n^3+(n+1)^3 &= 3\times n^2+3\times n+1. \end{aligned}$$

Adding up the left side of these n equations, we get:

$$(-1^3+2^3)+(-2^3+3^3)+(-3^3+4^3)+\cdots+(-(n-1)^3+n^3)+(-n^3+(n+1)^3)$$

Look carefully at this expression. Can we simplify it? Notice that we have a $+2^3$ term and a -2^3 term. These two terms will cancel. Similarly, the $+3^3$ term cancels with the -3^3 term. Each term, except the first and last will disappear, and thus, this cumbersome expression simplifies nicely to $-1^3+(n+1)^3$.

This is known as a Telescoping Series, because the sum collapses (just like an old-fashioned telescope), into just its first and last term.

Now add up the right side of our n equations, and simplify, as shown.

$$\begin{aligned} & 3\cdot(1^2+2^2+\cdots+n^2)+3\cdot(1+2+\cdots+n)+(1+1+\cdots+1) \\ = \; & 3\left(\sum_{i=1}^{n} i^2\right)+3\cdot\frac{n(n+1)}{2}+n. \end{aligned}$$

Comparing the two equivalent expressions allows us to solve for $\sum_{i=1}^{n} i^2$:

$$-1+(n+1)^3 = 3\left(\sum_{i=1}^{n} i^2\right)+3\cdot\frac{n(n+1)}{2}+n,$$

$$\begin{aligned}
-1+(n^3+3n^2+3n+1) &= 3\left(\sum_{i=1}^{n} i^2\right) + 3\cdot\frac{n^2+n}{2} + n,\\
n^3+3n^2+3n &= 3\left(\sum_{i=1}^{n} i^2\right) + \frac{3n^2+3n}{2} + n,\\
3\left(\sum_{i=1}^{n} i^2\right) &= n^3+3n^2+3n - \frac{3n^2+3n}{2} - n,\\
3\left(\sum_{i=1}^{n} i^2\right) &= n^3 + \frac{3n^2}{2} + \frac{n}{2},\\
3\left(\sum_{i=1}^{n} i^2\right) &= \frac{2n^3+3n^2+n}{2},\\
\sum_{i=1}^{n} i^2 &= \frac{2n^3+3n^2+n}{6},\\
\sum_{i=1}^{n} i^2 &= \frac{n(2n^2+3n+1)}{6},\\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Therefore, we have another proof that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \qquad \blacksquare$$

We have now derived formulas for $1^k + 2^k + \cdots + n^k$, for $k = 1, 2, 3$. Can we find a formula for larger values of k? For example, is there a nice formula for the sum of the first n fourth-powers? Jakob Bernoulli (1654-1705) answered this question by discovering a method to find an explicit formula for $1^k + 2^k + \cdots + n^k$, for any given k. These are known as Bernoulli Polynomials, and his work has led to numerous breakthroughs in number theory and analysis. For more information, we refer you to *A Mathematical Mosaic: Patterns and Problem Solving*, by Ravi Vakil. Finding efficient methods and algorithms to compute the coefficients of large Bernoulli polynomials is an active area of current mathematical research. The Bernoulli polynomial plays a critical role in the formulation of the Riemann Hypothesis, the most difficult unsolved problem in mathematics today.

5.5 Problem Set

1. How many squares (of all sizes) appear on an 10 by 10 checkerboard?

2. How many rectangles of area 3 appear on an 8 by 8 checkerboard?

3. Explain why there are $(n + 1 - i)^2$ squares of side length i on an n by n checkerboard.

4. There are 36 rectangles of area 4 that appear on an n by n checkerboard. Determine the value of n.

   (Hint: do not forget that we must include 2 by 2 "rectangles" too!)

5. On a circle, n points are selected, and all the chords joining them in pairs are drawn. Assuming that no three of the chords are concurrent (except at the endpoints), how many points of intersection are there?

   (Hint: the answer is a very simple binomial coefficient!)

6. In this chapter, we have developed a formula for the expression $1^2 + 2^2 + \cdots + n^2$, using a Telescoping Series. Use this technique to prove that $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

   (Hint: start with the identity $-k^4 + (k+1)^4 = 4k^3 + 6k^2 + 4k + 1$).

7. (a) Show that, for all positive integers n,

   $$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

   (b) Determine a simple formula for the expression

   $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

   (c) Determine a simple formula for the expression

   $$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{n(n+2)}$$

8. Evaluate the sum

   $$\sum_{n=1}^{2004} (-1)^n \cdot \frac{n^2+n+1}{n!}$$

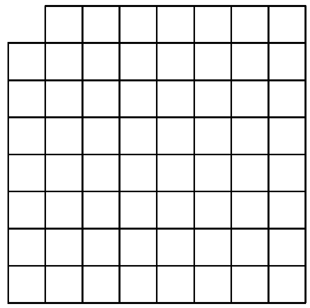
9. Prove that there are $2 \cdot \binom{n+2}{3}$ squares on an n by $(n+1)$ checkerboard. Is there a direct combinatorial proof of this formula?

10. Let m and k be positive integers. Let $f(m, km)$ be the number of rectangles on an m by km checkerboard. Determine all ordered pairs (m, k) for which $f(m, km)$ is a perfect square.

5.6 Investigation 1: Modified Checkerboards

In our original problem, we counted the number of squares and rectangles that appear on an 8 by 8 checkerboard.

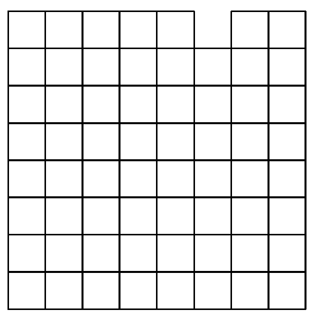
Extending this idea, consider what would happen if we modified the shape of our checkerboard. For example, suppose that a corner square is removed from the board:



How many squares appear on this modified checkerboard? How many rectangles appear?

Investigate this problem and see if you can come up with a general formula that calculates the number of squares and rectangles that appear on an n by n checkerboard if exactly one corner square is removed.

Explore this idea further. What happens if we remove a different unit square from the board? For example, determine the number of squares and rectangles that appear in the following diagram:



Investigate this idea, and see if you can extend this to other situations, such as removing two or more unit squares from this checkerboard.

5.7 Investigation 2: Divisors and Cubes

Make a table with three columns. In the first column, list all the divisors of 24, starting with 1 and ending with 24.

Beside each divisor, write down the number of divisors of that divisor. Write down this information in the second column. (For example, 12 is a divisor of 24, and the number 12 has six divisors.)

In the third column, write down the cubes of the numbers in the second column. Finally, add up all the numbers in the second and third columns. We arrive at the following table:

Divisors of 24	**Number of Divisors**	**(Number of Divisors)**^{3}
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
8	4	64
12	6	216
24	8	512
SUM	30	900

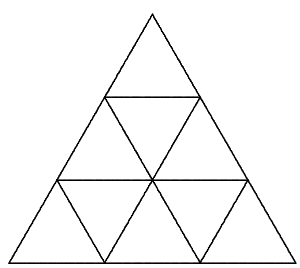
Notice that $900 = 30^2$. Is this a coincidence?

Replace 24 by each of the following numbers: 5, 8, 9, 16, 18, 30, 36, 72, and create the three columns. What happens? There is a general theorem that arises, that holds for every integer n. Make a conjecture, and try to prove it.

For more information on this problem, we refer you to *Problems for Senior High School Math: In Process*, by Peter Taylor.

5.8 Investigation 3: Equilateral Triangles

Consider the following equilateral triangle of side length 3.



In this figure, there are exactly 13 triangles: nine triangles of side length 1, three triangles of side length 2, and one triangle of side length 3. Note that all the triangles are equilateral.

Let $t(n)$ denote the number of triangles (of all sizes) that appear on an equilateral triangle of side length n. Thus, $t(3) = 13$.

1. Determine the values of $t(1)$ and $t(2)$.
2. Show that $t(4) = 27$. (Note: we have one equilateral triangle of side length 2 that points downwards.)
3. Try to find a general formula for $t(n)$. Can you prove it?

6 Glossary

Here are some mathematical terms and techniques that have been used in this book.

Binary Representation: The binary representation of n refers to the number n written in base 2. For example, $57 = 2^5 + 2^4 + 2^3 + 2^0$, so that the binary representation of 57 is 111001, and we write this as $57 = 111001_2$.

Mathematical Induction: This is a method of proof that is commonly used to prove that a statement is true for all positive integers n. For example, Mathematical Induction can be used to prove the identity

$$1 + 3 + 5 + \cdots + (2n-1) \; = \; n^2.$$

To prove such an identity using Mathematical Induction, you do the following:

1. Prove the statement is true for $n = 1$.
2. Prove that *if* the statement is true for $n = k$, *then* the statement must also be true for $n = k + 1$.

Modular Arithmetic: Suppose a and b are integers. We say that a is *congruent* to b *modulo* m if a and b both give the same remainder when divided by m. We write this as $a \equiv b \pmod{m}$.

For example, $5 \equiv 1 \pmod{4}$, $17 \equiv 1 \pmod{4}$, and $26 \equiv 0 \pmod{13}$.

Note that $a \equiv b \pmod{m}$ if and only if $(a - b)$ is divisible by m.

Modular arithmetic may also be applied to polynomials with integer coefficients. For example, $2x^3 + 4x^2 + 30x - 7 \equiv 2x^3 + x^2 + 2 \pmod{3}$.

Parity: This term describes a relation between a pair of integers. If the integers are both odd or both even, they are said to have the same parity. If one integer is odd and the other is even, they have different parity. For example, 6 and 20 have the same parity, whereas 6 and 21 do not.

Pigeonhole Principle: If $n+1$ pigeons fly into n pigeonholes, then at least two pigeons must fly into the same pigeonhole. This can be generalized: if $nk+1$ pigeons fly into n pigeonholes, then at least $k+1$ pigeons must fly into the same pigeonhole.

This idea has many useful applications. For example, if there are 37 people in a room, you can immediately conclude that at least 4 people were born in the same month. (This is the Pigeonhole Principle with $n = 12$ and $k = 3$).

Sigma Notation: For any function f defined on the positive integers, we define

$$\sum_{i=1}^{n} f(i) \; = \; f(1) + f(2) + f(3) + \cdots + f(n).$$

For example, if $f(i) = i^3$ for all positive integers i, we have

$$\sum_{i=1}^{10} i^3 \; = \; 1^3 + 2^3 + \cdots + 10^3.$$

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