A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome IV
INEQUALITIES

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The booklets in the series, **A Taste Of Mathematics** (ATOM), are published by the Canadian Mathematical Society (CMS). They are designed as enrichment materials for high school students with an interest in and aptitude for mathematics. Some booklets in the series will also cover the materials useful for mathematical competitions at national and international levels.

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Foreward

This volume contains most of the inequalities that are useful in solving problems. Many inequality problems admit several approaches. Some solutions are given, but other problems are left to the reader.

While we have tried to make the text as correct as possible, some mathematical and typographical errors might remain, for which we accept full responsibility. We would be grateful to any reader drawing our attention to errors as well as to alternative solutions.

It is the hope of the Canadian Mathematical Society that this volume may find its way to high school students who may have the talent, ambition and mathematical expertise to represent Canada internationally. Here are a few general resources for problem solving:

1. The International Mathematical Talent Search (problems can be obtained from the author, or from the magazine *Mathematics & Informatics Quarterly*, subscriptions for which can be obtained (in the USA) by writing to Professor Susan Schwartz Wildstrom, 10300 Parkwood Drive, Kensington, MD USA 20895 <ssw@ umd5.umd.edu>, or (in Canada) to Professor Ed Barbeau, Department of Mathematics, University of Toronto, Toronto, ON Canada M5S 3G3 <barbeau@ math.utoronto.ca>);
2. The journal *Crux Mathematicorum with Mathematical Mayhem* (subscriptions can be obtained from the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
3. The book *The Canadian Mathematical Olympiad 1969–1993 L'Olympiade mathématique du Canada*, which contains the problems and solutions of the first twenty-five Olympiads held in Canada (published by the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
4. The book *Five Hundred Mathematical Challenges*, by E.J. Barbeau, M.S. Klamkin & W.O.J. Moser (published by the Mathematical Association of America, 1529 Eighteenth Street NW, Washington, DC 20036, USA).
5. The book *The Mathematical Olympiad Handbook — an Introduction to Problem Solving*, by A. Gardiner (published by the Oxford University Press, ISBN 0-19-850105-6), which provides guided approaches to problems of the British Mathematical Olympiad.

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1 Introduction

Inequality is, perhaps, even more basic than equality. One of the basic principles of the real number system is the **Law of Trichotomy**, which states that every pair of real numbers x, y satisfies **exactly** one of the following three relations:

$$\begin{aligned} x &< y\,, \\ x &= y\,, \\ x &> y\,. \end{aligned}$$

This means that the real number system is **linearly-ordered**, and this leads to its representation as the real line.

Note that these inequalities do not make sense in the complex number system. That system does not lend itself to any natural linear ordering. (There are possible ways of linearly ordering the complex numbers, such as ordering by the real part first, and then by the imaginary part, but these are, in many respects, artificial orderings, and we shall not consider them in this booklet.)

As well as dealing with the **strict** inequalities as listed above, we shall be interested in the **non-strict** versions:

$$\begin{aligned} x &\leq y\,, \\ x &\geq y\,. \end{aligned}$$

There are no hard and fast rules for establishing inequalities. However, we shall give a number of guidelines that should prove useful.

1. Sometimes an inequality can be proved by working backwards from the conclusion, and reaching the given conditions.

   This process must be used with great caution, to ensure that the logic will be correct when statements in the argument are read in the proper order: from condition to conclusion.

   Be careful with your reasoning to ensure that all implications are in the right direction. This is particularly important when two statements are not equivalent.

   Unfortunately, this is a common source of error.

2. To prove an inequality of the form $A < B$, or $A \leq B$, it is often productive to examine the expression $B - A$, and to try to prove that it is positive or non-negative, respectively.

   Alternatively, when A and B are positive, it can be productive to examine one of the ratios $\frac{A}{B}$ or $\frac{B}{A}$, and examine the relationship with $\mathbf{1}$.

3. To prove something positive, examine it for positive valued expressions such as:

   (a) squares,

   (b) functions known to have positive values, such as:

      (i) exponentials, which are always positive,

      (ii) logarithms, when the variable is greater than $\mathbf{1}$,

      (iii) trigonometric functions on certain ranges,

   (c) sums or products of positive terms,

   (d) products of an **even** number of negative terms.

4. To prove a polynomial expression positive, first try to determine the roots, so that you can find the factors. It is easy to determine if a linear factor is positive or negative on an interval (between the roots), and so, if the polynomial is positive or negative on that interval.

   However, one must be careful about multiple roots here! Polynomials often do not change sign at such a root.

5. Many inequalities turn out to be standard inequalities in some sort of disguise. The next few sections will develop a number of the most common standard inequalities.

6. Finally, there are inequalities that demand the application of standard inequalities in some sort of cunning way. Experience in solving inequality problems, and a good sound knowledge of the standard inequalities are the best helpers. Surprisingly often, the Arithmetic Mean-Geometric Mean Inequality yields the desired result.

When writing up solutions involving inequalities, make sure that you proceed in logical steps from what is known or established to what has to be determined. It is a good idea to use logical connectives; for example

1. Since $\ldots$;

2. If $\ldots$, then $\ldots$;

3. Therefore $\ldots$;

4. $\ldots$ implies $\ldots$;

5. $\ldots$ if and only if $\ldots$;

6. $\Longrightarrow$;

7. $\Longleftrightarrow$.

You must be careful to distinguish between implications that go only one way ($\Longrightarrow$) and those that are reversible ($\Longleftrightarrow$). For example, $\boldsymbol{0 < x < y}$ implies that $\boldsymbol{x^2 < y^2}$, but the reverse implication does not hold: note that $\boldsymbol{3^2 < (-4)^2}$, but $\boldsymbol{3 \not< -4}$.

Many students writing up an inequality proof of $\boldsymbol{A \le P}$, use a format like:

$$\begin{aligned} & \boldsymbol{A \le P} \\ \iff \quad & \boldsymbol{B \le Q} \\ \iff \quad & \boldsymbol{C \le R} \\ & \quad \vdots \\ \iff \quad & \boldsymbol{K \le Z} \end{aligned}$$

where a slight manipulation takes $\boldsymbol{A}$ to $\boldsymbol{B}$ to $\boldsymbol{C}$ to $\boldsymbol{\ldots}$ to $\boldsymbol{K}$, and $\boldsymbol{P}$ to $\boldsymbol{Q}$ to $\boldsymbol{R}$ to $\boldsymbol{\ldots}$ to $\boldsymbol{Z}$, where $\boldsymbol{K \le Z}$ is a known inequality.

Unless you are making a change to the required inequality to obtain a simpler or more convenient form (such as might be obtained by clearing fractions or squaring both sides), it is generally advisable to **avoid** this format. The format

$$\boldsymbol{A = B = C = \cdots = K \le Z = \cdots = R = Q = P}$$

usually gives a better flow to the argument and is easier to follow.

When $\boldsymbol{x}$ and $\boldsymbol{y}$ are real numbers, and $\boldsymbol{c}$ is positive, the inequality $\boldsymbol{x \le y}$ is equivalent to the inequality $\boldsymbol{cx \le cy}$.

Often, in an argument with inequalities, it may be necessary to multiply or divide by some quantity. Always check the possibility that the quantity may take a zero or negative value. A zero value may indicate a special situation that must be handled separately. Remember particularly that division by zero is a forbidden operation.

Finally, remember that multiplication by a negative quantity reverses the inequality. For example, if $\boldsymbol{x \le y}$, then $\boldsymbol{-2y \le -2x}$.

2 Absolute Value

The absolute value of a real number $\boldsymbol{x}$ is defined by:

$$|\boldsymbol{x}| \;=\; \begin{cases} \;\;\boldsymbol{x} & \text{if } \boldsymbol{x \ge 0}\,, \\ \boldsymbol{-x} & \text{if } \boldsymbol{x < 0}\,. \end{cases}$$

Note that when $\boldsymbol{x}$ is negative, $|\boldsymbol{x}| = \boldsymbol{-x}$ is positive. So $|\boldsymbol{x}| \ge \boldsymbol{0}$ for all values of $\boldsymbol{x}$. Also, $\boldsymbol{x} \le |\boldsymbol{x}|$ and $\boldsymbol{-x} \le |\boldsymbol{x}|$, giving $-|\boldsymbol{x}| \le \boldsymbol{x} \le |\boldsymbol{x}|$.

We also see that $|\boldsymbol{x}| = \sqrt{\boldsymbol{x^2}}$.

Geometrically, $|\boldsymbol{x}|$ is the distance from the number $\boldsymbol{x}$ (on the real number line) to the origin $\boldsymbol{0}$. Also, $|\boldsymbol{a - b}|$ is the distance between the real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ on the real number line.

The corresponding quantity for a complex number $\boldsymbol{z = x + iy}$ is called the modulus, and is defined by

$$|\boldsymbol{z}| \;=\; \sqrt{\boldsymbol{x^2 + y^2}}\,.$$

It is clear that $|x| \leq |z|$ and $|y| \leq |z|$. (Remember that the complex numbers are not ordered by $<$.) Denoting, as usual, $x = \mathfrak{Re} z$ and $y = \mathfrak{Im} z$, we have $|\mathfrak{Re} z| \leq |z|$ and $|\mathfrak{Im} z| \leq |z|$.

Geometrically, $|z|$ is the distance (in the complex plane) from the point z to the origin $\mathbf{0}$. If the complex plane is thought of as the Euclidean plane, then, $|z|$ is the distance of the point (x, y) to the origin $(0, 0)$.

Also $|z - w|$ is the distance between the points z and w in the complex plane.

We also have $|z| = \sqrt{z\overline{z}}$, where $\overline{z} = x - iy$ is the complex conjugate of z. We find the complex conjugate of any complex number by replacing every occurrence of i with $-i$. The complex conjugate of the complex conjugate is the number with which we started. That is, we have that $\overline{\overline{z}} = z$.

Finally, we note that

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

3 The Triangle Inequality

The Triangle Inequality states:

$$|x + y| \leq |x| + |y|.$$

A neat way to prove this is to start with the basic principle that, when c is positive, the inequality $|x| \leq c$ is equivalent to the inequalities $-c \leq x \leq c$.

From $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$, it follows that

$$-\left(|x| + |y|\right) \leq x + y \leq |x| + |y|,$$

so that, by the principle mentioned above, it follows that

$$|x + y| \leq |x| + |y|.$$

Alternatively, the inequality can be proved from

$$\begin{aligned}
\left(|x| + |y|\right)^2 &= |x|^2 + |y|^2 + 2|x|\,|y| \\
&= x^2 + y^2 + 2|xy| \\
&\geq x^2 + y^2 + 2xy \\
&= \left|x + y\right|^2 .
\end{aligned}$$

The general form of the Triangle Inequality is

$$\left|\sum_{k=1}^{n} x_k\right| \ \leq \ \sum_{k=1}^{n} |x_k| \ .$$

This can be proved in a similar manner.[1]

A version of the Triangle Inequality is true in other settings, such as for complex numbers and for vectors. The proof given above is not valid in these settings, for it depends on order properties of the real numbers.

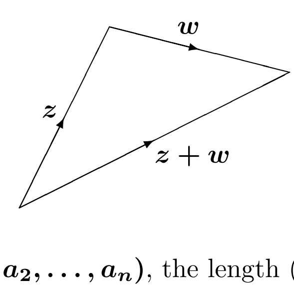
For complex numbers with $z = x + iy$ and $w = u + iv$, we have

$$|z + w| \ \leq \ |z| + |w| \, .$$

This follows from

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) \\ &= (z+w)(\overline{z}+\overline{w}) \\ &= (z\overline{z}) + (w\overline{w}) + (z\overline{w} + \overline{z}w) \\ &= |z|^2 + |w|^2 + (z\overline{w} + \overline{z\overline{w}}) \\ &= |z|^2 + |w|^2 + 2\mathfrak{Re}(z\overline{w}) \\ &\leq |z|^2 + |w|^2 + 2|z\overline{w}| \\ &= |z|^2 + |w|^2 + 2|zw| \\ &= |z|^2 + |w|^2 + 2|z|\,|w| \\ &= \big(|z| + |w|\big)^2 \ . \end{aligned}$$

The geometry lurking behind this is that the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides.



For vectors $\vec{a} = (a_1, a_2, \ldots, a_n)$, the length (or norm) is defined by

$$|\vec{a}| \ = \ \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \, .$$

[1] Those unfamiliar with "sigma" notation should see page 62.

So, for two vectors $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$, the Triangle Inequality states

$$|\vec{a} + \vec{b}| \;\leq\; |\vec{a}| + |\vec{b}|\,.$$

In two dimensions, geometrically, this states that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.

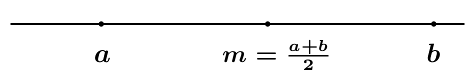
The proof of this depends on the Cauchy-Schwarz Inequality which is proved later in Section 8.

4 Means

First, some definitions of means.

4.1 Arithmetic Mean (AM)

The Arithmetic Mean (AM), m, of two numbers a, b, is the average of the two numbers, $m = \dfrac{a+b}{2}$. This has the geometric interpretation of the mid-point of the line segment joining two points on the number line.



Similarly, the AM of n numbers a_1, a_2, $\ldots$, a_n, is the average of the numbers,

$$\frac{a_1 + a_2 + \cdots + a_n}{n}\,.$$

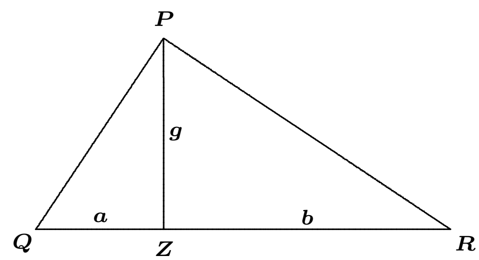
This can also be written as $\mathrm{AM}(a_1, a_2, \ldots, a_n) \;=\; \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k\,.$

Note that the sum of the differences between the numbers and the AM is zero:

$$\sum_{k=1}^{n} (a_k \;-\; \mathrm{AM}(a_1, a_2, \ldots, a_n)) \;=\; 0\,.$$

4.2 Geometric Mean (GM)

The Geometric Mean (GM), g, of two numbers a, b, is $g = \sqrt{ab}$. This has a geometric interpretation from right triangles. The length of the altitude from the vertex at the right angle is the geometric mean of the lengths of the segments into which it divides the hypotenuse.

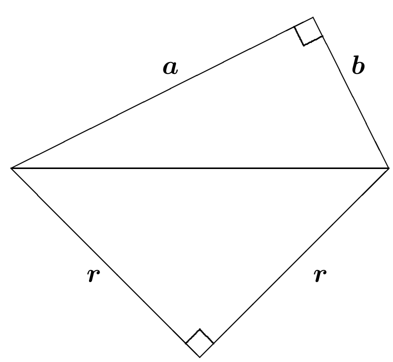


Here is a geometric representation. Draw the perpendicular, $\boldsymbol{PZ}$, from $\boldsymbol{P}$ to the hypotenuse $\boldsymbol{QR}$ of right triangle $\boldsymbol{PQR}$. Let $\boldsymbol{PZ = g}$, $\boldsymbol{QZ = a}$ and $\boldsymbol{ZR = b}$. From the similar triangles $\boldsymbol{PQZ}$ and $\boldsymbol{RPZ}$, we see that $\boldsymbol{g^2 = ab}$.

Similarly, the GM of $\boldsymbol{n}$ numbers $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{\ldots}$, $\boldsymbol{a_n}$, is $\left(\boldsymbol{a_1 a_2 \ldots a_n}\right)^{\frac{1}{n}}$. This is also written as $\left(\prod_{k=1}^{n} \boldsymbol{a_k}\right)^{\frac{1}{n}}$.

4.3 Root-Mean-Square (RMS)

The Root-Mean-Square (RMS), $\boldsymbol{r}$, of two numbers $\boldsymbol{a}$, $\boldsymbol{b}$ is $\boldsymbol{r} = \sqrt{\dfrac{\boldsymbol{a^2 + b^2}}{\boldsymbol{2}}}$. This has a geometric interpretation from right triangles. If $\boldsymbol{a}$, $\boldsymbol{b}$ are the two sides of a right triangle adjacent to the right angle (the legs of the right triangle) and $\boldsymbol{r}$ is the RMS of $\boldsymbol{a}$ and $\boldsymbol{b}$, then the right triangle with legs $\boldsymbol{r}$ and $\boldsymbol{r}$ has the same hypotenuse length as the original right triangle.



It is interesting to note that

$$\boldsymbol{2m^2 = g^2 + r^2},$$

so that the arithmetic mean is the root-mean-square of the geometric mean and the root-mean-square of the two numbers.

Similarly, the RMS of $\boldsymbol{n}$ numbers $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{\ldots}$, $\boldsymbol{a_n}$, is the quantity

$\left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}\right)^{1/2}$. This is also written as $\sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}$.

We show that

$$\min(a_1, a_2, \ldots, a_n) \ \leq \ \text{RMS} \ \leq \ \max(a_1, a_2, \ldots, a_n)\,. \tag{1}$$

Suppose, without loss of generality, that

$$0 < a_1 \leq a_2 \leq \ldots \leq a_n\,.$$

Then $\max(a_1, a_2, \ldots, a_n) = a_n$, and $\min(a_1, a_2, \ldots, a_n) = a_1$; and (1) becomes equivalent to

$$n a_1^2 \ \leq \ a_1^2 + a_2^2 + \cdots + a_n^2 \ \leq \ n\, a_n^2\,.$$

But this is easily seen from the monotonicity [2] of $\{a_k\}$.

The RMS is a particular case of the Power Mean, which is discussed in Section 4.5. It is the Power Mean of order 2.

4.4 Harmonic Mean (HM)

The Harmonic Mean (HM), h, of two positive numbers a, b, is $h = \frac{2ab}{a+b}$. This may not seem a natural quantity, but it comes from

$$\frac{1}{\text{HM}} \ = \ \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \ = \ \text{AM}\left(\frac{1}{a}, \frac{1}{b}\right)\,.$$

Note that $g^2 = mh$, so that the geometric mean of two positive numbers is also the geometric mean of their arithmetic and harmonic means.

Similarly, the HM of n positive numbers a_1, a_2, $\ldots$, a_n, is given by

$$\frac{1}{\text{HM}} \ = \ \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\,.$$

This gives HM $= \dfrac{n(a_1 + a_2 + \cdots + a_n)}{S_{n-1}}$, where

$$S_{n-1} = \sum a_1 a_2 \ldots \widehat{a_k} \ldots a_n\,,$$

the sum of all products of $n-1$ of the numbers a_1, a_2, $\ldots$, a_n (the "hat" $\widehat{\ }$ denotes the deletion of the symbol underneath it).

We note that

$$\min(a_1, a_2, \ldots, a_n) \ \leq \ \text{HM} \ \leq \ \max(a_1, a_2, \ldots, a_n)\,.$$

[2] A sequence $\{a_k\}$ is said to be monotone non-decreasing if $a_k \leq a_{k+1}$ for all k, or monotone non-increasing if $a_k \geq a_{k+1}$ for all k.

4.5 Power Mean of Order p (PM(p))

The Power Mean of order p, $\big(\text{PM}(p)\big)$, for a set of n positive numbers, is defined by:

$$\big(\text{PM}(p)\big) \begin{cases} = & \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p\right)^{1/p} & p \neq 0, \quad |p| < \infty\,, \\ = & \left(\prod_{k=1}^{n} a_k\right)^{1/n} & p = 0\,, \\ = & \min\,\{a_k\} & p = -\infty\,, \\ = & \max\,\{a_k\} & p = \infty\,. \end{cases}$$

It turns out that PM(p) is a non-decreasing function of p. This will be established in Section 14.

4.6 Weighted Means (WAM), (WGM)

Positive real numbers $\{w_1, w_2, \ldots, w_n\}$ such that $w_1 + w_2 + \cdots + w_n = 1$, are called weights. Clearly, any set of positive numbers can be converted into a set of weights, simply by dividing by their sum. Students will be familiar with this idea from the sort of formulae used for calculating marks.

The weighted arithmetic mean (WAM) of n numbers $a_1, a_2, \ldots, a_n$, is given by

$$\text{WAM} \;=\; \sum_{k=1}^{n} w_k a_k \;=\; w_1 a_1 + w_2 a_2 + \cdots + w_n a_n\,.$$

Similarly, the weighted geometric mean (WGM) of n numbers $a_1, a_2, \ldots, a_n$, is given by

$$\begin{aligned} \text{WGM} &= \prod_{k=1}^{n} (a_k)^{w_k} \\ &= (a_1)^{w_1}\,(a_2)^{w_2}\,\ldots\,(a_n)^{w_n}\,. \end{aligned}$$

4.7 AM–GM Inequality

The AM–GM Inequality, for two positive numbers is

$$\frac{a+b}{2} \;\geq\; \sqrt{ab} \qquad \text{or} \qquad \text{AM} \;\geq\; \text{GM}\,,$$

with equality if and only if $a = b$.

Write $a = \alpha^2$ and $b = \beta^2$. Then

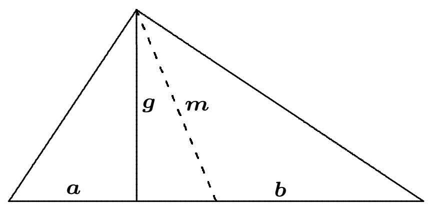
$$(\alpha - \beta)^2 \;=\; \alpha^2 + \beta^2 - 2\alpha\beta \;\geq\; 0\,,$$

so that

$$\frac{\alpha^2 + \beta^2}{2} \;\geq\; \alpha\beta\,,$$

with equality if and only if $\alpha = \beta$.

Here is a geometric way of looking at the inequality. The hypotenuse of a right triangle is divided into lengths a and b by the perpendicular from the right angle. The geometric mean, g, is the length of the perpendicular. The arithmetic mean, m, is the radius of the circumcircle.



The AM–GM Inequality, for three positive numbers is

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \qquad \text{or} \qquad \text{AM} \geq \text{GM},$$

with equality if and only if $a = b = c$.

This is not so easy to see. But if we set $a = u^3$, $b = v^3$, $c = w^3$, we are then asked to show that

$$u^3 + v^3 + w^3 - 3uvw \geq 0 .$$

It turns out that we can factor the left side of this inequality!

$$u^3 + v^3 + w^3 - 3uvw = (u+v+w)\left(u^2 + v^2 + w^2 - uv - vw - wu\right) .$$

Since u, v, w are all positive, $(u + v + w)$ is positive, so we need to show that the second factor is also positive. But we can write the second factor as

$$\frac{1}{2}\left((u-v)^2 + (v-w)^2 + (w-u)^2\right) ,$$

which is clearly positive.

The general form of the AM–GM Inequality is:

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \left(\prod_{k=1}^{n} a_k\right)^{1/n} .$$

However, the proof of the general AM–GM Inequality is best obtained by a cunning trick. First we see that it is straight-forward to obtain the AM–GM for **four** numbers, by applying the AM–GM for two numbers twice. This is the basis of a proof by induction [3] for the AM–GM Inequality for 2^N numbers. The reader

[3] See, for example, ATOM II.

should complete this proof! So we now know that

$$\frac{1}{2^N}\sum_{k=1}^{2^N} a_k \;\geq\; \left(\prod_{k=1}^{2^N} a_k\right)^{1/2^N} .$$

To complete the proof for any positive integer n, we first observe that if n is a power of 2, then we have a proof. Otherwise, n must lie strictly between two powers of two; that is, there exists a positive integer N such that $2^{N-1} < n < 2^N$.

Let $g \;=\; \left(\prod_{k=1}^{n} a_k\right)^{1/n}$, and define $a_{n+1} = a_{n+2} = \ldots = a_{2^N} = g$.

So we now have a set of 2^N numbers, to which we can apply the AM–GM Inequality! This gives:

$$\frac{a_1 + a_2 + \cdots + a_n \;+\; g + g + \cdots + g}{2^N} \;\geq\; \left(a_1.a_2 \ldots a_n \,.\, g.g \ldots g\right)^{1/2^N} ,$$

so that

$$\frac{a_1 + a_2 + \cdots + a_n \;+\; g(2^N - n)}{2^N} \;\geq\; \left(a_1.a_2 \ldots a_n \,.\, (g)^{2^N - n}\right)^{1/2^N} ,$$

or

$$\begin{aligned} \frac{a_1 + a_2 + \cdots + a_n}{2^N} + g\left(1 - \frac{n}{2^N}\right) &\geq \left(a_1.a_2 \ldots a_n\right)^{1/2^N} \times (g)^{1-\left(n/2^N\right)} \\ &= (g)^{n/2^N} \times (g)^{1-\left(n/2^N\right)} \;=\; g\,. \end{aligned}$$

Thus,

$$\frac{a_1 + a_2 + \cdots + a_n}{2^N} \;\geq\; \frac{ng}{2^N}$$

and the general form of the AM–GM Inequality follows.

There are similar inequalities for weighted means; we shall obtain these in a very general setting in Section 14.

4.8 HM–GM Inequality

It is easy to see that the harmonic mean is less than or equal to the geometric mean. This follows from the AM–GM Inequality, since $2\sqrt{ab} \leq a + b$ implies that $\dfrac{2ab}{a+b} \leq \sqrt{ab}\,$.

The proof that HM $\leq$ GM for n positive numbers can readily be obtained by applying the AM–GM Inequality to the set of numbers $1/a_1$, $1/a_2$, $\ldots$, $1/a_n$.

4.9 AM–RMS Inequality

Since $(a-b)^2 \geq 0$ implies that $2ab \leq a^2 + b^2$, we see that

$$(a+b)^2 = a^2 + b^2 + 2ab \leq 2(a^2 + b^2)\,.$$

Similarly

$$(a_1 + a_2 + \cdots + a_n)^2 \;\leq\; n(a_1^2 + a_2^2 + \cdots + a_n^2)\,,$$

and the inequality: AM $\leq$ RMS follows at once.

4.10 Summary

$$\textbf{min} \;\leq\; \text{HM} \;\leq\; \text{GM} \;\leq\; \text{AM} \;\leq\; \text{RMS} \;\leq\; \textbf{max}\;.$$

5 Tchebychev's Inequality

For a sequence $a = \{a_k\}_{k=1}^n$, we have already defined the arithmetic mean by

$$\text{AM}(a) \;=\; \frac{1}{n}\sum_{k=1}^{n} a_k\,.$$

Here, we shall deal with two sequences, $a = \{a_n\}_{k=1}^n$ and $b = \{b_n\}_{k=1}^n$, and their term-wise product, which is the sequence $\{a_n b_n\}_{k=1}^n$.

Tchebychev's Inequality states that if the sequences a and b satisfy $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$, then

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \;\leq\; \frac{1}{n}\sum_{k=1}^{n} a_k b_k\,,$$

or

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \;\leq\; n\sum_{k=1}^{n} a_k b_k\,.$$

In words, this is: **the product of the arithmetic means of two monotonic non-decreasing sequences is less than or equal to the arithmetic mean of their term-wise product**.

We give one of several ways to prove this. Note that

$$\begin{aligned}\sum_{\mu=1}^{n}\sum_{\kappa=1}^{n}(a_\mu b_\mu - a_\mu b_\kappa) &= \sum_{\mu=1}^{n}\left(n\, a_\mu\, b_\mu - a_\mu \sum_{\kappa=1}^{n} b_\kappa\right)\\ &= n\sum_{\kappa=1}^{n} a_\kappa\, b_\kappa - \sum_{\kappa=1}^{n} a_\kappa \sum_{\kappa=1}^{n} b_\kappa\,.\end{aligned}$$

Note also that

$$\begin{aligned}\sum_{\mu=1}^{n}\sum_{\kappa=1}^{n}(a_\kappa b_\kappa - a_\kappa b_\mu) &= \sum_{\kappa=1}^{n}\sum_{\mu=1}^{n}(a_\kappa b_\kappa - a_\kappa b_\mu)\\ &= \sum_{\mu=1}^{n}\sum_{\kappa=1}^{n}(a_\mu b_\mu - a_\mu b_\kappa)\ .\end{aligned}$$

For the first equality, we have interchanged the order of summation; for the second equality, we have relabelled the indices.
Hence

$$\begin{aligned}& n\sum_{\kappa=1}^{n} a_\kappa\, b_\kappa - \sum_{\kappa=1}^{n} a_\kappa \sum_{\kappa=1}^{n} b_\kappa\\ &= \frac{1}{2}\sum_{\mu=1}^{n}\sum_{\kappa=1}^{n}(a_\mu b_\mu - a_\mu b_\kappa + a_\kappa b_\kappa - a_\kappa b_\mu)\\ &= \frac{1}{2}\sum_{\mu=1}^{n}\sum_{\kappa=1}^{n}(a_\mu - a_\kappa)(b_\mu - b_\kappa) \ \geq\ 0\,,\end{aligned}$$

since $(a_\mu - a_\kappa)(b_\mu - b_\kappa) \geq 0$ for $\mu,\ \kappa = 1,\ 2,\ \ldots,\ n$.

The general form for a set of p sequences $a^{(k)}$ with $a_1^{(k)} \leq a_2^{(k)} \leq \ldots \leq a_n^{(k)}$, is

$$\mathrm{AM}(a^{(1)})\,\mathrm{AM}(a^{(2)})\ \ldots\ \mathrm{AM}(a^{(p)}) \ \leq\ \frac{1}{n}\sum_{k=1}^{n} a_k^{(1)}\, a_k^{(2)} \ldots a_k^{(p)}\,,$$

which can be written as

$$\prod_{j=1}^{p}\left(\frac{1}{n}\sum_{\mu=1}^{n} a_\mu^{(j)}\right) \ \leq\ \frac{1}{n}\sum_{\mu=1}^{n}\prod_{j=1}^{p} a_\mu^{(j)}\,,$$

or

$$\frac{1}{n^p}\prod_{j=1}^{p}\sum_{\mu=1}^{n} a_\mu^{(j)} \ \leq\ \frac{1}{n}\sum_{\mu=1}^{n}\prod_{j=1}^{p} a_\mu^{(j)}\,.$$

In words, this is: **the product of the arithmetic means of a finite set of monotonic non-decreasing sequences is less than or equal to the arithmetic mean of their termwise product**. This can be established by induction on the number of sequences.

6 Bernoulli's Inequality

Suppose that $x > -1$ and that n is a natural number. Then Bernoulli's Inequality states

$$(1+x)^n \;\geq\; 1+nx\,.$$

We shall prove this by induction on n.

TEST. The inequality is clearly true for $n = 1$.

STEP. Assume the inequality true for $n = k$. This gives

$$(1+x)^k \;\geq\; 1+kx\,.$$

Multiply this inequality by $(1+x)$ (being positive, the inequality is preserved: here we need the condition $x > -1$). This gives

$$(1+x)^{k+1} \;\geq\; (1+x)(1+kx) \;=\; 1+(k+1)x+k\,x^2 \;\geq\; 1+(k+1)x\,.$$

Hence, by induction, the inequality is proved.

This can be extended to a set of distinct values as follows:

Suppose that $n \geq 2$ and that x_1, x_2, $\ldots$, x_n are non-zero real numbers which all have the same sign and which satisfy $x_k \geq -2$. Then

$$(1+x_1)\,(1+x_2)\ldots(1+x_n) \;>\; 1+x_1+x_2+\cdots+x_n\,.$$

We shall prove this by induction on n.

TEST. If $n = 2$, since x_1 and x_2 have the same sign, we have

$$(1+x_1)\,(1+x_2) \;=\; 1+x_1+x_2+x_1x_2 \;>\; 1+x_1+x_2\,.$$

STEP. Assume the inequality true for $n = k$. Then

$$\begin{aligned}
&(1+x_1)\,(1+x_2)\ldots(1+x_k)\,(1+x_{k+1}) \\
&\quad - \quad (1+x_1+x_2+\cdots+x_k+x_{k+1}) \\
&\qquad = \big(\,(1+x_1)\,(1+x_2)\ldots(1+x_k) - (1+x_1+x_2+\cdots+x_k)\big) \\
&\qquad\quad + \;\; x_{k+1}\,\big(\,(1+x_1)\,(1+x_2)\ldots(1+x_k) - 1\big)\;.
\end{aligned}$$

The first term on the right is positive by hypothesis.

If each x_k is positive, then the second term is positive.

If each x_k satisfies $-2 \leq x_k < 0$, then the product

$$(1+x_1)\,(1+x_2)\ldots(1+x_k) \leq 1\,,$$

and the second term, being the product of two non-positive quantities, is non-negative.

Hence, by induction, the inequality is proved.

Here is an alternative argument, also by induction.

The case $n = 2$ is proved as above. Suppose that $n \geq 3$.

Suppose that all x_j are negative and at least two of them, say x_1 and x_2 lie in the interval $[-2, -1]$. Then

$$(1 + x_1)\,(1 + x_2) \ldots (1 + x_n) \geq -1 = 1 - 1 - 1 > 1 + x_1 + x_2 + \cdots + x_n$$

since $-2 \leq x_j < 0$ and $|1 + x_j| \leq 1$ for $1 \leq j \leq n$.

Henceforth, we assume that either

(i) all x_j are positive $(1 \leq j \leq n)$, or

(ii) all x_j are negative with $-2 \leq x_1 < 0$ and $-1 < x_j < 0$ $(2 \leq j \leq n)$.

As an induction hypothesis, assume that the result holds for $n = k \geq 2$.

Then $1 + x_{k+1} > 0$, so that

$$\begin{aligned}
&(1 + x_1)\,(1 + x_2) \ldots (1 + x_k)\,(1 + x_{k+1}) \\
&\quad > (1 + x_1 + x_2 + \cdots + x_k)\,(1 + x_{k+1}) \quad \text{by the induction hypothesis,} \\
&\quad > 1 + x_1 + x_2 + \cdots + x_k + x_{k+1} \qquad\quad \text{by the case } n = 2.
\end{aligned}$$

The result follows by induction.

The general form of Bernoulli's Inequality states that, for $-1 < x \neq 0$,

$$\begin{aligned}
(1+x)^\alpha &< 1 + \alpha\, x \qquad \text{for} \quad 0 < \alpha < 1\,; \\
(1+x)^\alpha &> 1 + \alpha\, x \qquad \text{for} \quad \alpha > 1 \ \text{ or } \ \alpha < 0\,.
\end{aligned}$$

7 Abel's Inequality

Suppose that a_1, a_2, $\ldots$, a_n and $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$ are real numbers. Let

$$m = \min_{1 \leq k \leq n} \left\{ \sum_{\mu=1}^{k} a_\mu \right\}, \qquad M = \max_{1 \leq k \leq n} \left\{ \sum_{\mu=1}^{k} a_\mu \right\}.$$

Then

$$m\, b_1 \;\leq\; \sum_{\mu=1}^{n} a_\mu b_\mu \;\leq\; M\, b_1\,.$$

Proof. Let $s_k = \sum_{\mu=1}^{k} a_\mu$, so that $m \le s_k \le M$. We use "partial summation":

$$\begin{aligned}\sum_{\mu=1}^{n} a_\mu b_\mu &= s_1 b_1 + \sum_{\mu=2}^{n} b_\mu (s_\mu - s_{\mu-1}) \\ &= \sum_{\mu=1}^{n-1} s_\mu (b_\mu - b_{\mu+1}) + s_n\, b_n \\ &\le \sum_{\mu=1}^{n-1} M (b_\mu - b_{\mu+1}) + M\, b_n = M\, b_1 .\end{aligned}$$

The other side is proved similarly.

8 Cauchy-Schwarz Inequality

For real sequences,

$$\left| \sum_{k=1}^{n} (a_k b_k) \right| \le \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} \left(\sum_{k=1}^{n} b_k^2 \right)^{1/2} .$$

To obtain this inequality, form the polynomial $\sum_{k=1}^{n} (a_k x + b_k)^2$, which, when expanded, gives

$$x^2 \left(\sum_{k=1}^{n} a_k^2 \right) + 2\, x \left(\sum_{k=1}^{n} a_k\, b_k \right) + \left(\sum_{k=1}^{n} b_k^2 \right) .$$

This is a quadratic polynomial in x which, being non-negative, either has no real roots or coincident roots (in the case that b_k/a_k has the same value for each k). Hence, its discriminant is non-positive, giving

$$\left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \left(\sum_{k=1}^{n} a_k b_k \right)^2 \ge 0 ,$$

which proves the result.

A more direct approach is to note that the left side of this inequality can be expressed as a sum of squares:

$$\sum_{1 \le j,k \le n} (a_j b_k - a_k b_j)^2 .$$

9 Newton's Inequalities

Let x_1, x_2, $\ldots$, x_n be positive real numbers. For each $k = 1, 2, \ldots, n$, let

$$u_k \;=\; \frac{1}{\binom{n}{k}} \sum x_1 x_2 \ldots x_k \qquad \text{and} \qquad v_k \;=\; u_k^{1/k}\,,$$

where $\sum x_1 x_2 \ldots x_k$ denotes the sum of all k–fold products of distinct x_j. This sum has $\binom{n}{k}$ terms, so that u_k is the average of the k–fold products. In particular, u_1 is the arithmetic mean of the x_k, and v_n is the geometric mean of the x_k. Then (Newton's Inequalities)

$$v_n \leq v_{n-1} \leq \ldots \leq v_2 \leq v_1\,.$$

Outline of the proof

Before embarking on the proof properly, we need a little calculus:

1. The derivative of the polynomial

$$p(t) \;=\; a_n t^n + a_{n-1} t^{n-1} + \cdots + a_k t^k + \cdots + a_1 t + a_0\,,$$

   is equal to

$$p'(t) \;=\; n a_n t^{n-1} + (n-1) a_{n-1} t^{n-2} + \cdots + k a_k t^{k-1} + \cdots + a_1\,.$$

2. **Rolle's Theorem**
   If $p(a) = p(b) = 0$, then there exists a number, c, between a and b for which $p'(c) = 0$.

NOTE: Rolle's Theorem is more generally true; it applies to any function, continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

In words, Rolle's Theorem says that the graphs of such functions must have a point with level tangent in the open interval (a, b).

A degenerate case of Rolle's theorem occurs when $a = b$, so that $p(t)$ has a double root at a. Then $p'(a) = 0$.

Corollary to Rolle's Theorem

If a polynomial of degree n has n real roots (counting multiplicity of roots), then $p'(t)$ has $n-1$ real roots, and, for each k $(1 \leq k \leq n-1)$, the k^{th} derivative of $p(t)$ has $n-k$ real roots.

We now proceed with the proof:

(a) Let $p(t) = \displaystyle\prod_{k=1}^{n} (t - x_k)$. Then

$$p(t) \;=\; t^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} u_k t^{n-k}\,.$$

(b) Applying Rolle's Theorem, we see that the $(n-2)^{\text{nd}}$ derivative

$$p^{(n-2)}(t) \;=\; \tfrac{1}{2}(n!)\left(t^2 - 2u_1 t + u_2\right)$$

has real roots, and so, from the discriminant condition on this quadratic, we see that

$$u_2 \le u_1^2 .$$

(c) Note that

$$\begin{aligned} \binom{n}{2}\frac{u_{n-2}}{u_n} &= \frac{1}{x_1x_2} + \frac{1}{x_1x_3} + \cdots + \frac{1}{x_{n-1}x_n} , \\ \binom{n}{1}\frac{u_{n-1}}{u_n} &= \frac{1}{x_1} + \cdots + \frac{1}{x_n} . \end{aligned}$$

Applying (b) to $\dfrac{1}{x_1}, \ldots, \dfrac{1}{x_n}$ yields

$$u_{n-2}u_n \le u_{n-1}^2 .$$

(d) The result of (b) and (c) can be generalized to obtain

$$u_{k-1}u_{k+1} \le u_k^2$$

for $k = 2, 3, \ldots, n-1$.

This is done using induction on n, the number of the x_k.

TEST The result holds for $n = 2$ and $n = 3$.

STEP Suppose that the result holds when the number of x_k does not exceed $n-1$.

By differentiating the second expression for $p(t)$ in (a), we find that

$$p'(t) \;=\; n\left(t^{n-1} + \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}u_k t^{n-k-1}\right) .$$

Suppose that the roots of $p'(t)$ are $y_1, \ldots, y_{n-1}$. For $1 \le k \le n-1$, define z_r by

$$\binom{n-1}{k} z_k \;=\; \sum y_1 y_2 \ldots y_k ,$$

so that the z_k are to the y_j what the u_k are to the x_j.

The argument to establish the second expression for $p(t)$ in (a) can be used on $p'(t)$ to obtain

$$p'(t) \;=\; n\left(t^{n-1} + \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}z_k t^{n-1-k}\right) ,$$

whereupon we have that $u_k = z_k$ for $1 \le k \le n-1$. Now we use the induction hypothesis on $y_1, y_2, \ldots, y_{n-1}$ to obtain that, for $2 \le k \le n-2$,

$$u_{k-1}u_{k+1} \le u_k^2 .$$

(e) From (b), (c) and (d), we find that, for $1 \leq k \leq n-1$,

$$u_2\,(u_1u_3)^2\,(u_2u_4)^3\ldots(u_{k-1}u_{k+1})^k \;\leq\; u_1^2u_2^4u_3^6\ldots u_k^{2k}\,,$$

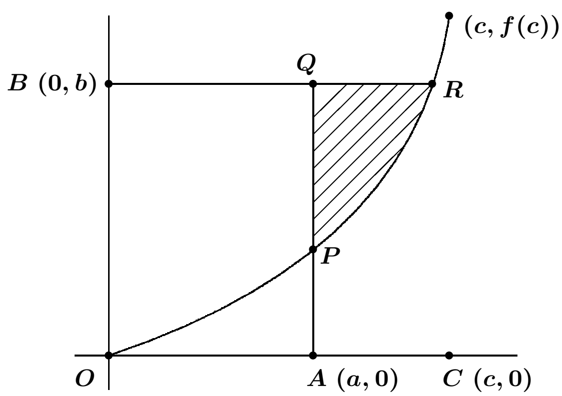
from which we obtain

$$u_{k+1}^k \leq u_k^{k+1} \qquad \text{or, recalling that } v_k = u_k^{1/k}, \qquad v_{k+1} \leq v_k\,.$$

10 Young's Inequality

Young's Inequality requires some knowledge of calculus, but since it leads to some nice results, we give it here.

We need a function, $f(x)$, positive and strictly increasing on an interval $[0,c]$. We may assume, without loss of generality, that $f(0)=0$. Let Q have coordinates (a,b) with $0<a<c$, $0<b<f(c)$.



The area of the region OAP is given by $\int_0^a f(x)\,dx$, and the area of the region OBR is given by $\int_0^b f^{-1}(x)\,dx$. The sum of these areas exceeds the area of the rectangle $OAQB$ — the resulting excess is shaded. This gives Young's Inequality

$$\int_0^a f(x)\,dx \;+\; \int_0^b f^{-1}(x)\,dx \;\geq\; ab\,.$$

Equality holds if and only if $b = f(a)$.

As an application of Young's Inequality, take $f(x) = x^{p-1}$ with $p > 1$. This gives

$$\int_0^a x^{p-1}\,dx \;+\; \int_0^b x^{1/(p-1)}\,dx \;\geq\; ab\,,$$

so that

$$\frac{1}{p}\,a^p \;+\; \frac{p-1}{p}\,b^{p/(p-1)} \;\geq\; ab\,,$$

which can be re-written as

$$\frac{1}{p}\,a^p \;+\; \frac{1}{q}\,b^q \;\geq\; ab\,, \tag{2}$$

where $a, b \geq 0,\; p > 1,\; \dfrac{1}{p} + \dfrac{1}{q} \;=\; 1$.

11 Hölder's Inequality

Hölder's Inequality is a generalisation of the Cauchy-Schwarz Inequality.

For positive sequences, and with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $(p > 1)$, we have

$$\sum_{k=1}^{n} (a_k b_k) \;\leq\; \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}.$$

This can be obtained from (2) in section 10, by replacing a and b with

$$\frac{a_\mu}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}} \qquad \text{and} \qquad \frac{b_\mu}{\left(\sum_{k=1}^{n} b_k^q\right)^{1/q}}\,,$$

respectively. This gives

$$\frac{a_\mu^p}{p\sum_{k=1}^{n} a_k^p} + \frac{b_\mu^q}{q\sum_{k=1}^{n} b_k^q} \;\geq\; \frac{a_\mu\, b_\mu}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}}.$$

We sum this over μ from 1 to n to get

$$1 \;=\; \frac{1}{p} + \frac{1}{q} \;\geq\; \frac{\sum_{k=1}^{n} a_k\, b_k}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}}\,,$$

from which Hölder's Inequality follows.

12 Minkowski's Inequality

Minkowski's Inequality is a generalisation of the Triangle Inequality.

For positive sequences, and with $p > 1$

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n}a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n}b_k^p\right)^{1/p} . \tag{3}$$

We note that

$$(a_k+b_k)^p = a_k(a_k+b_k)^{p-1} + b_k(a_k+b_k)^{p-1}\,,\,, \tag{4}$$

so that

$$\sum_{k=1}^{n}(a_k+b_k)^p = \sum_{k=1}^{n}a_k(a_k+b_k)^{p-1} + \sum_{k=1}^{n}b_k(a_k+b_k)^{p-1} .$$

We apply Hölder's Inequality to each sum on the right side of (4) and get

$$\sum_{k=1}^{n}a_k(a_k+b_k)^{p-1} \le \left(\sum_{k=1}^{n}a_k^p\right)^{1/p}\left(\sum_{k=1}^{n}(a_k+b_k)^{q(p-1)}\right)^{1/q} ,$$

$$\sum_{k=1}^{n}b_k(a_k+b_k)^{p-1} \le \left(\sum_{k=1}^{n}b_k^p\right)^{1/p}\left(\sum_{k=1}^{n}(a_k+b_k)^{q(p-1)}\right)^{1/q} .$$

Putting these inequalities into (4), and noting that $q(p-1) = p$, leads to the required inequality.

Note that Minkowski's Inequality (3) is an equality if we allow $p = 1$. For $0 < p < 1$, the inequality is reversed.

13 A Comparison Technique

Suppose that we have to prove an inequality of the form

$$f(x_1, x_2, \ldots, x_n) \le g(x_1, x_2, \ldots, x_n) ,$$

where f and g are symmetric functions and the x_i are non-negative real variables. We can look at the problem in the following light: over all vectors $(x_1, x_2, \ldots, x_n)$ for which $g(x_1, x_2, \ldots, x_n)$ is a specified constant k, maximize $f(x_1, x_2, \ldots, x_n)$ and show that this maximum does not exceed k. (A similar minimization problem can be formulated if "≤" is replaced by "≥".)

We begin by showing where the maximum of f, under the constraints, cannot occur, by replacing a test vector $(x_1, x_2, \ldots, x_n)$ by a vector $(y_1, y_2, \ldots, y_n)$, derived from $(x_1, x_2, \ldots, x_n)$ in some way so that

$$g(x_1, x_2, \ldots, x_n) = g(y_1, y_2, \ldots, y_n) = k ,$$

$$\text{while} \quad f(x_1, x_2, \ldots, x_n) < f(y_1, y_2, \ldots, y_n) .$$

For example, if it turns out that $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$ when $x_1 = x_2 = \cdots = x_n$, then we may wish to show that the maximum of f cannot occur if, say, $x_1 \neq x_2$. Given such a test vector $(x_1, x_2, \ldots, x_n)$, we may try a new test vector $(y_1, y_2, \ldots, y_n)$ where $y_i = x_i$ $(3 \leq i \leq n)$ and y_1 and y_2 are each equal to some mean of x_1 and x_2 (for example, the AM of the GM).

Then the structure of the reasoning is as follows:

1. the maximum of $f(x_1, x_2, \ldots, x_n)$, subject to $g(x_1, x_2, \ldots, x_n) = k$, must occur somewhere. Often this requires a result in analysis to the effect that the continuous function $f(x_1, x_2, \ldots, x_n)$ assumes its maximum value on a closed and bounded set of vectors for which $g(x_1, x_2, \ldots, x_n) = k$;
2. the maximum cannot occur if the x_i are not all equal;
3. hence the maximum must occur when the x_i are all equal and $f(x_1, x_2, \ldots, x_n) = k$.
4. Thus, $f(x_1, x_2, \ldots, x_n) \leq g(x_1, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n)$, since we can apply the reasoning to each value of k assumed by $g(x_1, x_2, \ldots, x_n)$.

We illustrate this technique with two examples.

Example 1. Suppose that $n \geq 2$ and let u_1, u_2, $\ldots$, u_n be real numbers, all not less than 1. Prove that

$$\frac{1}{1+u_1} + \frac{1}{1+u_2} + \cdots + \frac{1}{1+u_n} \geq \frac{n}{1 + \sqrt[n]{u_1 u_2 \ldots u_n}}\,.$$

Solution. We see that equality occurs when $u_1 = u_2 = \cdots = u_n$. Let k be an arbitrary real number, not less than 1, and let

$$S = \{(u_1, u_2, \ldots, u_n) \ : \ u_1 u_2 \cdots u_n = k,\, u_1 \geq 1,\, u_2 \geq 1,\ \ldots\ , u_n \geq 1\}\,.$$

Now, S is a closed and bounded subset of real n–space, and the continuous function $f(u_1, u_2, \ldots, u_n) = \sum_{1 \leq i \leq n} \frac{1}{1+u_i}$ assumes its minimum value somewhere in S.

Suppose, with no loss of generality, that $u_1 \neq u_2$. Suppose that $u_1 = a^2$, and $u_2 = b^2$. Keeping the geometric mean on the right side in mind, we define v_1 and v_2 so that $v_1 = v_2$ and $v_1 v_2 = u_1 u_2$. Thus $v_1 = v_2 = ab$. Let $v_i = u_i$ $(3 \leq i \leq n)$. Now,

$$\begin{aligned} f(u_1, u_2, \ldots, u_n) - f(v_1, v_2, \ldots, v_n) &= \frac{1}{1+a^2} + \frac{1}{1+b^2} - \frac{2}{1+ab} \\ &= \frac{(ab-1)(a-b)^2}{(1+a^2)(1+b^2)(1+ab)} \geq 0\,. \end{aligned}$$

The desired result follows.

In this problem, we can actually complete the argument without recourse to the properties of continuous functions. We first establish the result when n is a power of 2. For the case $n = 2$, we have already shown that $\frac{1}{1+a^2} + \frac{1}{1+b^2} \geq \frac{2}{1+ab}$. Suppose that the inequality is established for $n = 2^{m-1}$. Let u_1, u_2, $\dots$, u_{2^m} be given and define $v_1 = \sqrt{u_1u_2}$, $v_2 = \sqrt{u_3u_4}$, $\dots$, $v_k = \sqrt{u_{2k-1}u_{2k}}$, $\dots$, $v_{2^{m-1}} = \sqrt{u_{2^m-1}u_{2^m}}$.

Then $\sum_{i=1}^{2^m} \frac{1}{1+u_i} \geq \sum_{i=1}^{2^{m-1}} \frac{2}{1+v_i} \geq 2\left(\frac{2^{m-1}}{1+\sqrt[2^{m-1}]{v_1v_2\cdots v_{2^{m-1}}}}\right) = \frac{2^m}{1+\sqrt[2^m]{u_1u_2\cdots u_{2^m}}}$.

By induction, we have that the inequality holds for $n = 2^m$, where m is a positive integer.

Now, let n be arbitrary with $2^{m-1} < n < 2^m$. Given u_1, u_2, $\dots$, u_n, define $v_i = u_i$ for $1 \leq i \leq n$, and $v_i = (u_1u_2\cdots u_n)^{1/n}$ for $n+1 \leq i \leq 2^m$. Then

$$\sum_{i=1}^{n} \frac{1}{1+u_i} + \frac{2^m - n}{1+\sqrt[n]{u_1u_2\cdots u_n}} = \sum_{i=1}^{2^m} \frac{1}{1+v_i} \geq \frac{2^m}{1+(v_1v_2\cdots v_{2^m})^{1/2^m}}.$$

Since

$$v_1v_2\cdots v_{2^m} = (u_1u_2\cdots u_n)(u_1u_2\cdots u_n)^{(2^m-n)/n} = (u_1u_2\cdots u_n)^{2^m/n},$$

we obtain

$$\sum_{i=1}^{n} \frac{1}{1+u_i} + \frac{2^m - n}{1+\sqrt[n]{u_1u_2\cdots u_n}} \geq \frac{2^m}{1+\sqrt[n]{u_1u_2\cdots u_n}},$$

and the result follows.

Example 2. (IMO, 1999) Determine the smallest real number C for which

$$\sum_{1\leq i\leq j\leq n} x_ix_j\left(x_i^2 + x_j^2\right) \leq C(x_1 + x_2 + \cdots + x_n)^4$$

for each positive integer $n \geq 2$ and non-negative real x. Determine when equality occurs.

Solution. The inequality is homogeneous of degree 4, so it is enough to deal with the case $x_1 + x_2 + \cdots + x_n = 1$. The minimum value of C is the maximum value of $h(x_1, x_2, \dots, x_n) = \sum_{1\leq i\leq j\leq n} x_ix_j\left(x_i^2 + x_j^2\right)$ subject to the constraint $x_1 + x_2 + \cdots + x_n = 1$.

Let C_n be this maximum over n–tuples $(x_1, x_2, \ldots, x_n)$. Since these n–tuples contain some with their last $n - m$ entries equal to 0, it is true that $C_m \le C_n$ when $m < n$.

To show the reverse inequality, we need to show that each value of h does not exceed its value when some of the x_i are equal to 0.

Suppose that $n \ge 3$ and that $x_1 \ge x_2 \ge \cdots \ge x_{n-1} \ge x_n \ge 0$. When $n = 3$, $x_1 \ge \frac{1}{3}$, so that $x_2 + x_3 \le \frac{2}{3}$; when $n \ge 4$, then $x_n \le \frac{1}{n}$ and $x_{n-1} \le \frac{1}{n-1}$, so that $x_n + x_{n-1} \le \frac{1}{4} + \frac{1}{3} < \frac{3}{4}$.

Since
$$\begin{aligned} h(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i \le j \le n} x_i^3 x_j + x_j^3 x_i \;=\; \sum_{i=1}^{n} x_i^3 \left(\sum_{j \ne i} x_j \right) \\ &= \sum_{i=1}^{n} x_i^3 (1 - x_i) \;=\; \sum_{i=1}^{n} (x_i^3 - x_i^4)\,, \end{aligned}$$
it follows that
$$\begin{aligned} &h(x_1, x_2, \ldots, x_{n-2}, x_{n-1} + x_n, 0) - h(x_1, x_2, \ldots, x_{n-2}, x_{n-1}, x_n) \\ &\quad= \big(x_{n-1} + x_n\big)^3 - \big(x_{n-1} + x_n\big)^4 - x_{n-1}^3 + x_{n-1}^4 - x_n^3 + x_n^4 \\ &\quad= x_{n-1} x_n \big(3x_{n-1} + 3x_n - 4x_{n-1}^2 - 6x_{n-1}x_n - 4x_n^2\big) \\ &\quad= x_{n-1} x_n \Big(\big(x_{n-1} + x_n\big) \big(3 - 4\big(x_{n-1} + x_n\big)\big) + 2x_{n-1}x_n \Big) \;\ge\; 0\,. \end{aligned}$$
Since $h(x_1, x_2, \ldots, x_{n-2}, x_{n-1} + x_n, 0) \le C_{n-1}$ for each h, it follows that $C_n \le C_{n-1}$ for each $n \ge 3$.

We now have that $C_n = C_2$. Suppose that $x_1 + x_2 = 1$. Then $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = 1 - 2x_1x_2$ and
$$x_1 x_2 \big(x_1^2 + x_2^2\big) \;=\; \tfrac{1}{2} \big(2x_1x_2\big(1 - 2x_1x_2\big)\big) \;\le\; \tfrac{1}{8}\,,$$
with equality if and only if $2x_1x_2 = \frac{1}{2}$; that is, $x_1 = x_2 = \frac{1}{2}$. Summing up, we can say that the value of C desired in the problem is $\frac{1}{8}$ and equality occurs if and only if $x_1 = x_2$ and $x_3 = x_4 = \cdots = x_n = 0$.

At the IMO, a much simpler solution to this problem came to light.

Let $S = x_1^2 + x_2^2 + \cdots + x_n^2$. Then
$$\begin{aligned} \sum_{1 \le i \le j \le n} x_i x_j \big(x_i^2 + x_j^2\big) &\le \sum_{1 \le i \le j \le n} x_i x_j S \;=\; \tfrac{1}{2} \Big(S \times 2 \sum x_i x_j \Big) \\ &\le \frac{1}{2} \left(\frac{S + 2\sum x_i x_j}{2} \right)^2 \\ &= \frac{1}{2} \frac{(x_1 + x_2 + \cdots + x_n)^4}{4} \end{aligned}$$
where equality occurs in both places if and only if at most two of the x_i are non-zero, and these two x_i are equal in value.

14 Averages and Jensen's Inequality

Many of the inequalities established so far can be put into a very general setting with arguments that are brief and elegant. The setting for these results is the space of real-valued functions on an arbitrary set S. On the class of functions u from S to $\mathbb{R}$, we define an **average** $A(u)$ which satisfies these five axioms:

(1) $A(u)$ is a real number;

(2) $A(cu) = cA(u)$ where c is any real constant;

(3) $A(u + v) = A(u) + A(v)$ for any two functions u and v;

(4) $A(u) \geq 0$ whenever the function u assumes non-negative values;

(5) $A(1) = 1$, where $\mathbf{1}$ denotes the constant function that assumes the real value $\mathbf{1}$ at each point of S.

The fourth axiom has an important consequence. Suppose that u and v are two functions on S for which $u(s) \leq v(s)$ for all s belonging to S. Then $v - u$ assumes only non-negative values, so that $A(v) - A(u) = A(v - u) \geq 0$ and $A(v) \geq A(u)$. Thus, the operator A is monotone in the sense that, the larger the function, the larger the value assigned to it by A.

Here are some examples of an average:

Example 1: Let S be the set $\{1, 2, 3, \cdots, n\}$ and let u be that function that maps the integer k to u_k. In this case, we can conveniently represent the function u by a $n-$tuple that displays its values: $(u_1, u_2, \cdots, u_n)$. One example of an average is the ordinary mean of these values defined by

$$A(u) = \frac{1}{n}(u_1 + u_2 + \cdots + u_n)\,.$$

More generally, we can assign a system of weights w_1, w_2, $\cdots$, w_n (see page 9). This allows us to define a weighted average

$$A_w(u) = w_1u_1 + w_2u_2 + \cdots + w_nu_n\,.$$

Example 2. Let S be the closed unit interval $[0, 1] \equiv \{t : 0 \leq t \leq 1\}$ and let $u(t)$ be any continuous real-valued function defined on S. Then $A(u) = \int_0^1 u(t)dt$ defines an average.

We can formulate a very general version of the Arithmetic-Geometric Mean Inequality. For each positive real-valued function u on S, we define its logarithm [4] by

$$(\log u)(s) = \log(u(s))\,.$$

[4] Here, as in higher mathematics, we use "**log**" for the natural logarithm, whereas, in elementary mathematics, "**ln**" is more commonly used.

Of course, the logarithm is taken to the natural base $e = 2.1828\ldots$. The geometric mean $G(u)$ of a positive function u is given by

$$G(u) = \exp A(\log u)$$

where $\exp t$ is equal to e^t.

Example 3. Let $S = \{1, 2\}$ and let $A(u) = \frac{1}{2}(u_1 + u_2)$.
Then $A(\log u) = \frac{1}{2}(\log u_1 + \log_2) = \frac{1}{2}\log(u_1 u_2) = \log(u_1 u_2)^{1/2}$ so that $G(u) = (u_1 u_2)^{1/2}$.

Example 4. Let $S = \{1, 2, \cdots, n\}$, $\{w_1, w_2, \cdots, w_n\}$ be a set of non-negative weights summing to 1 and $A_w(u) = w_1 u_1 + w_2 u_2 + \cdots + w_n u_n$. Then the corresponding geometric mean with the same weights is $G_w(u) = u_1^{w_1} u_2^{w_2} \cdots u_n^{w_n}$.

The Arithmetic-Geometric Mean Inequality. If u is a positive real-valued function on S, then

$$G(u) \leq A(u)\,.$$

Proof. In the most general situation, there are complications in the case when the function u assumes the value 0 or when $A(u) = 0$. We will not handle them here, but will restrict ourselves to the case that the geometric mean is well-defined and $A(u)$ is non-zero. This will cover the situations that students will encounter in practice and will allow us to focus on the main ideas.

By sketching the graph of $\log t$ and its tangent when $t = 1$, we can see that $\log t \leq t - 1$ whenever $t > 0$. Let s be any point of S and let u be any positive function on S with a positive average $A(u)$. Applying the Logarithm Inequality for $t = u(s)/A(u)$, we have that

$$\log\left(\frac{u}{A(u)}\right) \leq \frac{u}{A(u)} - 1$$

for both sides evaluated at any point s in S. This implies that

$$\log u - \log A(u) \leq \frac{u}{A(u)} - 1\,.$$

We can regard this as an inequality between two functions, so that

$$\log u - (\log A(u))\,1 \leq \frac{u}{A(u)} - 1\,.$$

When we take the average of the left side, we find that

$$\begin{aligned} A(\log u - \log A(u)\,1) &= A(\log u) - (\log A(u))A(1) \\ &= A(\log u) - (\log A(u))\,. \end{aligned}$$

Taking the average of the right side yields

$$A\left(\frac{u}{A(u)} - 1\right) = A\left(\frac{u}{A(u)}\right) - A(1) = \left(\frac{1}{A(u)}\right)A(u) - 1 = 1 - 1 = 0\,.$$

Since the operator A is monotone, the average of the left side is less than the average of the right side, yielding

$$A(\log u) - \log A(u) \le 0\,,$$

or

$$A(\log u) \le \log A(u)\,.$$

Taking exponentials yields the desired result.

Corollary. Let p and q be two positive real numbers for which

$$\frac{1}{p} + \frac{1}{q} = 1\,,$$

and suppose that $x \ge 0$ and $y \ge 0$. Then

$$xy \le \frac{1}{p}x^p + \frac{1}{q}y^q\,.$$

Proof. This is an application of the Arithmetic-Geometric Mean Inequality to $S = \{1, 2\}$ and the function (x^p, y^q), where A is the average with weights $1/p$ and $1/q$.

The next major result involving the concept of average is Jensen's Inequality. To formulate this, we first need to define the concept of concave up and concave down functions. Suppose that $a \le b$ and that $f(x)$ is a real-valued function defined on the real interval $[a, b] \equiv \{x : a \le x \le b\}$.

The function f is **concave up** on the interval $[a, b]$ if and only if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

whenever $0 \le t \le 1$ and $a \le x, y \le b$. The chord joining any two points on the graph of a concave up function lies above the graph. The tangent to any point of the graph of a concave up function lies beneath the graph. If f is a twice-differentiable function, then f is concave up if and only if its second derivative f'' is everywhere non-negative.

The function f is **concave down** on the interval $[a, b]$ if and only if

$$f(tx + (1-t)y) \ge tf(x) + (1-t)f(y)$$

whenever $0 \le t \le 1$ and $a \le x, y \le b$. The chord joining any two points on the graph of a concave down function lies beneath the graph. The tangent to any point of the graph of a concave down function lies above the graph. If f is a twice-differentiable function, then f is concave down if and only if its second derivative f'' is everywhere non-positive.

For example, $f(x) = \sin x$ is concave down on $[0, \pi]$ since its second derivative is non-positive there. We can show "mid-point" concavity directly. For

$$\begin{aligned} \frac{f(\alpha) + f(\beta)}{2} &= \frac{\sin\alpha + \sin\beta}{2} = \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\ &\le \sin\left(\frac{\alpha+\beta}{2}\right) = f\left(\frac{\alpha+\beta}{2}\right)\,, \end{aligned}$$

since $0 \leq \cos\left(\frac{\alpha-\beta}{2}\right) \leq 1$ for $\alpha, \beta \in [0, \pi]$.

Jensen's Inequality: Let u denote a function defined on a set S taking real values, and A be an average defined on the set of such functions.

(i) Suppose that f is a concave up real-valued function of a real variable. Then

$$f(A(u)) \leq A(f \circ u)$$

where $f \circ u(s) = f(u(s))$ denotes the composition of the functions u and f.

(ii) Suppose that f is a concave down real-valued function of a real variable. Then

$$f(A(u)) \geq A(f \circ u)\,.$$

Example 5. Let $S = \{1, 2, \cdots, n\}$, $u(i) = u_i$ for $1 \leq i \leq n$ and let $f(t)$ be concave up. Consider the average defined by

$$A(u) = \frac{1}{n}(u_1 + u_2 + \cdots + u_n)\,.$$

Then Jensen's Inequality leads to

$$f\left(\frac{1}{n}\sum_{k=1}^{n} u_k\right) \;\leq\; \frac{1}{n}\sum_{k=1}^{n} f(u_k)\,.$$

In particular, when $f(t) = t^2$, we are led to

$$(u_1 + u_2 + \cdots + u_n)^2 \;\leq\; n\left(u_1^2 + u_2^2 + \cdots + u_n^2\right)\,.$$

When $f(t)$ is concave down, we obtain

$$f\left(\frac{1}{n}\sum_{k=1}^{n} u_k\right) \;\geq\; \frac{1}{n}\sum_{k=1}^{n} f(u_k)\,.$$

Example 6. Let $S = \{1, 2\}$, $u(1) = a$, $u(2) = b$, $f(t) = \log t$ for $t > 0$ and $A(u) = (1/2)(a + b)$. Since $\log t$ is concave down, when a and b are positive, Jensen's Inequality provides that

$$\log(\frac{1}{2}(a+b)) \geq \frac{1}{2}\log a + \frac{1}{2}\log b = \log\sqrt{ab}\,;$$

taking exponentials yields the basic Arithmetic-Geometric Mean Inequality. In a similar way, we can establish the Arithmetic-Geometric Mean Inequality for a weighted average of n positive real numbers.

Proof of (i). Let $c = A(u)$ and let $r < c < s$. Then

$$f(c) \;\leq\; \frac{c-r}{s-r}f(s) + \frac{s-c}{s-r}f(r)$$

$$\Longrightarrow [(s-c)+(c-r)]f(c) \;=\; (s-r)f(c) \;\le\; (c-r)f(s)+(s-c)f(r)$$
$$\Longrightarrow (s-c)(f(c)-f(r)) \;\le\; (c-r)(f(s)-f(c))$$
$$\Longrightarrow \frac{f(c)-f(r)}{c-r} \;\le\; \frac{f(s)-f(c)}{s-c}\,.$$

Since this is true for r and s independently with $r < c < s$, there exists a real number k for which

$$\frac{f(c)-f(r)}{c-r} \;\le\; k \;\le\; \frac{f(s)-f(c)}{s-c}\,.$$

Therefore $f(c)-f(r) \le k(c-r)$ or $f(c)-kc \le f(r)-kr$ for $r < c$. Also $f(s)-f(c) \ge k(s-c)$ or $f(c)-kc \le f(s)-ks$ for $s > c$.

Thus, for **all** real values t, we have that

$$f(c)-kc \;\le\; f(t)-kt\,.$$

Setting $t = u(s)$ leads to

$$f(c)-kc \;\le\; f(u(s))-ku(s) \;=\; (f\circ u)(s)-ku(s)\,.$$

Because this holds for all s in S, we get the inequality

$$(f(c)-kc)\,\mathbf{1} \;\le\; (f\circ u)-ku\,.$$

Applying the average A and using its monotonicity, we obtain

$$f(A(u))-kA(u) = f(c)-kc \le A(f\circ u \;-\; ku) \;=\; A(f\circ u)-kA(u)\,,$$

from which $f(A(u)) \le A(f\circ u)$ as required.

The proof of (ii) is similar and left to the reader.

The final general result we shall obtain concerns power means. Let A be an average and suppose that u is a **positive** function defined on S. For a real number r, we define the function $u^r(s) \equiv [u(s)]^r$.

When $r \neq 0$, let the power mean $M_r \equiv M_r(A,u)$ of u be defined by

$$M_r = [A(u^r)]^{1/r}\,.$$

In addition, we define

$$\begin{aligned} M_0 &= \exp A(\log u) \;=\; G(u)\,,\\ M_{+\infty} &= \sup u\,,\\ M_{-\infty} &= \inf u\,, \end{aligned}$$

where $\sup u$ is the smallest number greater than or equal to every value assumed by u and $\inf u$ is the largest number less than or equal to every value assumed by u. If u actually assumes a maximum value, then this maximum value is equal to

$\sup u$; if u assumes a minimum value, then this minimum value is equal to $\inf u$. Thus

$$\inf u \;\leq\; u(s) \;\leq\; \sup u$$

for every element s in S, and these bounds are as tight as possible.

Example 7. Let $S = \{1, 2, \cdots, n\}$ and let $A(u) = w_1u_1 + w_2u_2 + \cdots + w_nu_n$ be the weighted average considered earlier. Then, for $r \neq 0$,

$$M_r = \big(\sum_{i=1}^{n} w_i u_i^r\big)^{1/r} .$$

In particular,

$$M_1 = \sum w_i u_i \qquad \text{the usual weighted mean}\,,$$

$$M_2 = \sqrt{\sum w_i u_i^2} \qquad \text{the root-mean-square}\,,$$

$$M_{-1} = \frac{1}{\sum w_i/u_i} \qquad \text{the harmonic mean}\,.$$

Also

$$M_0 = \prod_{i=0}^{n} u_i^{w_i} \qquad \text{the geometric mean}\,,$$

$$M_{+\infty} = \max u_i \qquad \text{and} \qquad M_{-\infty} \;=\; \min u_i\,.$$

The Power Mean Inequalities. Let $r < s$. Then

$$M_{-\infty} \leq M_r \leq M_s \leq M_{+\infty}\,.$$

Proof. We break the proof into several parts.

(1) If $r > 0$, then $M_0 \leq M_r$.

$$M_0(A,u)^r \;=\; M_0(A,u^r) \;\leq\; M_1(A,u^r) \;=\; M_r(A,u)^r$$

by the definitions of M_0, M_1, M_r and the Arithmetic-Geometric Mean Inequality.

(2) If $0 < r < 1$, then $M_0 \leq M_r \leq M_1$.

Since t^r is a concave down function of t, Jensen's Inequality yields

$$A(u)^r \geq A(u^r)\,,$$

which gives the inequality on the right.

(3) If $0 < r < s$, then $M_r \leq M_s$.

Putting $r = ms$ with $0 < m < 1$ and applying (2), we obtain

$$A(u^r) = A(u^{ms}) \leq A(u^s)^m \,.$$

Now raise to the power $1/ms$.

(4) If $r < s < 0$, then $M_r \leq M_s \leq M_0$.

To get this, use the fact that, for $r \neq 0$,

$$M_r(A, u) \;=\; M_{-r}(A, u^{-1})^{-1} \,.$$

(5) For each real r, $M_{-\infty} \leq M_r \leq M_{+\infty}$.

This is evident.

15 Problems Involving Basic Ideas

15.1 The Problems

Problem 15/1

Suppose that $a > b > c > d > 0$ *and that* $a + d = b + c$. *Show that* $ad < bc$.

Problem 15/2

Suppose that a, b, p, q, r, s *are positive integers for which* $\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$ *and* $qr - ps = 1$. *Prove that* $b \geq q + s$.

Problem 15/3

Suppose that a *and* c *are fixed real numbers with* $a \leq 1 \leq c$. *Determine the largest value of* b *which is compatible with*

$$a + bc \;\leq\; b + ac \;\leq\; c + ab\,.$$

Problem 15/4

Suppose that a_k $(k = 1, 2, \ldots)$ *are real numbers for which* $a_1 = 0$ *and, for* $k > 1$, $|a_k| = |a_{k-1} + 1|$. *Prove that, for* $n = 1, 2, \ldots$,

$$a_1 + a_2 + \cdots + a_n \;\geq\; -\frac{n}{2}\,.$$

Problem 15/5

Suppose that $a \geq 1$ *and that* x *is real. Prove that*

$$\frac{x^2 + a}{\sqrt{x^2 + a - 1}} \;\geq\; 2\,.$$

Problem 15/6

Let $f(a, b, c, d) = (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2$. *For* $a < c < b < d$, *prove that*

$$f(a, c, b, d) \;>\; f(a, b, c, d) \;>\; f(a, b, d, c)\,.$$

Problem 15/7

For real x, y, z, *prove that*

$$x^2 + y^2 + z^2 \;\geq\; |xy + yz + zx|\,.$$

Problem 15/8 (*Lithuanian Team Contest 1987*)

For real x, y, z, *prove that*

$$\left|\sqrt{x^2 + y^2} - \sqrt{x^2 + z^2}\right| \;\leq\; |y - z|\,.$$

Problem 15/9

For $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ *such that* $\boldsymbol{a^2+b^2=\alpha^2+\beta^2=1}$, *prove* (*without applying the Cauchy-Schwarz Inequality*) *that*

$$|a\alpha+b\beta| \;\le\; 1\,.$$

Problem 15/10

(a) *For* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ *such that* $\boldsymbol{a\beta-b\alpha=1}$, *prove that*

$$a^2+\alpha^2+b^2+\beta^2+a\alpha+b\beta \;>\; 1\,.$$

(b) *Under the same hypotheses, strengthen the inequality to*

$$a^2+\alpha^2+b^2+\beta^2+a\alpha+b\beta \;\ge\; \sqrt{3}\,.$$

Problem 15/11

Prove that, for all integers $\boldsymbol{n\ge 2}$,

$$\sum_{k=1}^{n}\frac{1}{k^2} \;>\; \frac{3n}{2n+1}\,.$$

Problem 15/12

For real numbers $\boldsymbol{0<a<b}$, *prove that*

$$\sqrt{b^2-a^2}+\sqrt{2ab-a^2} \;>\; b\,.$$

Problem 15/13

For positive $\boldsymbol{x,y,z}$, *prove Schur's Inequality:*

$$x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y) \;\ge\; 0\,.$$

Problem 15/14

(a) *For natural numbers* $\boldsymbol{k<n}$, *prove that*

$$n! \;>\; k!\,(n-k)!\,.$$

(b) *For natural numbers* $\boldsymbol{n>4}$ *and* $\boldsymbol{k<n}$, *prove that*

$$n! \;<\; (k!\,(n-k)!)^2\,.$$

Find all exceptional cases when $\boldsymbol{n\le 4}$. [$\boldsymbol{k!}$ *is the running product of the integers from* $\boldsymbol{1}$ *to* $\boldsymbol{k}$.]

Problem 15/15

For natural numbers $\boldsymbol{n}$ *and for* $\boldsymbol{x>y>0}$, *prove that*

$$x^{1/n}-y^{1/n} \;<\; (x-y)^{1/n}\,.$$

Problem 15/16

For $p \geq 1$*, prove that*

$$|x+y|^p \;\leq\; 2^p(|x|^p + |y|^p)\,.$$

Problem 15/17

For $0 < x < \dfrac{1}{n}$ *and integer* n*, prove that*

$$(1+x)^n \;<\; \frac{1}{1-nx}\,.$$

Problem 15/18

For $n = 1, 2, \ldots$*, let* $s_n = \left(1+\dfrac{1}{n}\right)^n$ *and* $t_n = \left(1+\dfrac{1}{n}\right)^{n+1}$*. Prove that, for all positive integers* j *and* k*,*

$$s_j \;<\; s_{j+1} \;<\; t_{k+1} \;<\; t_k\,,$$

and show that $\lim\limits_{n\to\infty}(t_n - s_n) \;=\; 0$.

Problem 15/19

Prove that $\lim\limits_{n\to\infty} n^{1/n} \;=\; 1$.

Problem 15/20 (a) **(CRUX** [1975: 8]**)** *Prove that*

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{999999}{1000000} < \frac{1}{1000}\,.$$

(b) *For natural numbers* n*, prove that* $\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{5}{6}\cdots\dfrac{2n-1}{2n} \;\leq\; \dfrac{1}{\sqrt{3n+1}}$,

Problem 15/21

Which of the following inequalities are true, which are false? (a *and* b *are real numbers, while* n *is a positive integer*). [*Note:* $\lfloor x\rfloor$ *means the greatest integer less than or equal to* x.]

(a) $\lfloor a\rfloor + \lfloor b\rfloor \;\leq\; \lfloor a+b\rfloor$,

(b) $\lfloor a+b\rfloor \;\leq\; \lfloor a\rfloor + \lfloor b\rfloor + 1$,

(c) $\lfloor a\rfloor\,\lfloor b\rfloor \;\leq \lfloor ab\rfloor$,

(d) $\lfloor ab\rfloor \;\leq\; \lfloor a\rfloor\,\lfloor b\rfloor + \lfloor a\rfloor + \lfloor b\rfloor$,

(e) $\lfloor\sqrt{a^2}\rfloor \;=\; \lfloor\sqrt{\lfloor a^2\rfloor}\rfloor$,

(f) $\lfloor\sqrt{n}\rfloor^2 \;\leq\; n$,

(g) $n \;\leq\; \lfloor\sqrt{n}\rfloor^2 + 2\lfloor\sqrt{n}\rfloor$,

(h) $\lfloor\sqrt[3]{n}\rfloor^3 \;\leq\; n$,

(i) $n \;\leq\; \lfloor\sqrt[3]{n}\rfloor^3 + 3\lfloor\sqrt[3]{n}\rfloor^2 + 3\lfloor\sqrt[3]{n}\rfloor$.

Problem 15/22 (*Lithuanian Team Contest 1987*)

For positive reals x, y, z, prove that

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2} \;\geq\; \frac{x+y+z}{3}\,.$$

Problem 15/23

For positive a, b, α, β, prove that

$$\frac{ab}{a+b}+\frac{\alpha\beta}{\alpha+\beta} \;\leq\; \frac{(a+\alpha)(b+\beta)}{a+b+\alpha+\beta}\,.$$

Problem 15/24

For natural numbers $n \geq 2$, prove that

(a) $n^{n/2} \;<\; n! \;\leq\; 2^{n(n-1)/2}$,

(b) $n! \;<\; \left(\dfrac{n+1}{n}\right)^n$.

Problem 15/25

For $n \geq 2$, prove that $2!\,4!\,\ldots\,(2n)! \;>\; \big\{(n+1)!\big\}^n$.

Problem 15/26

For $n \geq 2$, prove that $(n+1)^{n-1}(n+2)^n \;>\; 3^n(n!)^2$.

Problem 15/27

For $n \geq 3$, prove that $n^{(n+1)} \;>\; (n+1)^n$.

Problem 15/28

Let n be a positive integer.

(a) *By considering the coefficient of x^n in the identity*

$$(1+x)^{2n} \;=\; (1+x)^n(1+x)^n\,,$$

or otherwise, verify that

$$\binom{2n}{n} \;=\; \sum_{k=0}^{n}\binom{n}{k}^2.$$

(b) *Prove that*

$$\frac{4^n}{n+1} \;<\; \frac{(2n)!}{(n!)^2}\,.$$

Problem 15/29 (*Australian Interstate Finals 1989*)

Let u, v, c be real numbers which satisfy

$$u^2 < c^2 \qquad \text{and} \qquad v^2 < c^2\,.$$

Prove that

$$\left[\frac{u+v}{1+(uv)/c^2}\right]^2 \;<\; c^2\,.$$

15.2 The Solutions

Solution 15/1 Here are three possible ways to get the solution. The strategy is to examine the difference between the two sides of the inequality.

1. Since $c = a + d - b$, we have that
   $$bc - ad = b(a+d-b) - ad = (a-b)b - (a-b)d = (a-b)(b-d) > 0\,.$$
2. Let $u = a + d = b + c$. Then
   $$bc - ad = b(u-b) - (u-d)d = u(b-d) - (b^2 - d^2) = (b-d)(u-b-d)\,.$$
   Now, $u = b + c > b + d$, so that $b - d > 0$ and $u - b - d > 0$. Thus $bc - ad > 0$.
3. Let $x = a - b > 0$. Since $a - b = c - d$, we have that $a = b + x$ and $d = c - x$. Hence
   $$\begin{aligned} bc - ad &= bc - (b + x)(c - x) \\ &= bc - bc + bx - cx + x^2 \;=\; x^2 + x(b - c) > 0\,. \end{aligned}$$

Solution 15/2 We have $aq - bp > 0$ and $br - as > 0$. Since all the variables represent integers, $aq - bp \geq 1$ and $br - as \geq 1$. Thus
$$b = b(qr - ps) = q(br - as) + s(aq - bp) \geq q + s\,.$$

Solution 15/3 Observe that
$$(b + ac) - (a + bc) = \;(c - 1)(a - b)$$
and
$$(c + ab) - (b + ac) = \;(1 - a)(c - b)\,.$$
Therefore, the inequalities are equivalent to
$$(c - 1)(a - b) \geq \;0$$
and
$$(1 - a)(c - b) \geq \;0\,.$$

(i) If $a = c = 1$, then the inequalities hold for all values of b, and there is no largest value of b.

(ii) If $a = 1 < c$, then the inequalities hold if and only if $b \leq a = 1$, and so the largest value of b is 1.

(iii) If $a < 1 = c$, then the inequalities hold if and only if $b \leq c = 1$, and so the largest value of b is 1.

(iv) If $a < 1 < c$, then the inequalities hold if and only if $b \leq a$, and so the largest value of b is a.

Solution 15/4 1. In this solution, we square in order to avoid the awkwardness of dealing with absolute values. We have that

$$a_k^2 = (a_{k-1}+1)^2 \;=\; a_{k-1}^2 + 2a_{k-1} + 1 \qquad (k = 2, \ldots, n+1)$$

Adding these equations yields

$$\begin{aligned} &a_2^2 + a_3^2 + \cdots + a_{n+1}^2 \\ &\quad = \;\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) + 2\,(a_1 + a_2 + \cdots + a_n) + n\,, \end{aligned}$$

so that

$$2\,(a_1 + a_2 + \cdots + a_n) \;=\; -n + a_{n+1}^2 \;\geq\; -n\,.$$

2. We establish the result by induction on the length of all possible sequences with the stated property.

   Let $P(n)$ be the statement: *suppose that* $\{x_1,\ x_2,\ \ldots,\ x_n\}$ *is a finite sequence of real numbers for which* $x_1 = 0$ *and* $|x_j| = |x_{j-1}+1|$ *for* $j = 2,\ 3,\ \ldots,\ n$*; then* $x_1 + x_2 + \cdots + x_n \geq -\frac{n}{2}$.

   We see that $P(1)$ and $P(2)$ are clearly true.

   Suppose that $P(k)$ holds for $k = 1,\ 2,\ \ldots,\ n-1$. Let $\{x_1,\ x_2,\ \ldots,\ x_n\}$ be a sequence as specified above.

   If $x_j \geq 0$ $(j = 1, 2, \ldots, n)$, then $P(n)$ is true. On the other hand, suppose that at least one of the x_j is negative. Let x_r be the first negative entry, so that $x_1 \geq 0$, $x_2 \geq 0$, $\ldots$, $x_{r-1} \geq 0$, $x_r < 0$.

   (i) $r > 2$. We must have

$$\begin{aligned} x_{r-1} &= x_{r-2} + 1\,, \\ x_r &= -\,(x_{r-1}+1) \\ &= -\,(x_{r-2}+2)\,, \\ |x_{r+1}| &= |-\,(x_{r-2}+2)+1| \\ &= |-\,(x_{r-2}+1)| \\ &= |x_{r-2}+1|\,. \end{aligned}$$

   Define the finite sequence $\{y_1,\ y_2,\ \ldots,\ y_{n-2}\}$ by

$$\begin{aligned} y_j &= x_j \qquad (1 \leq j \leq r-2)\,, \\ y_j &= x_{j+2} \qquad (r-1 \leq j \leq n-2)\,. \end{aligned}$$

   Thus, the y–sequence is obtained from the x–sequence by removing x_{r-1} and x_r.

   Since $P(n-2)$ is true, and since $|y_j| = |y_{j-1}+1|$, $(2 \leq j \leq n-2)$, we have

$$\begin{aligned} &x_1 + x_2 + \cdots + x_{r-1} + x_r + \cdots + x_n \\ &\quad = \;(x_{r-1} + x_r) + (y_1 + y_2 + \cdots + y_{n-2}) \\ &\quad = \;-1 + (y_1 + y_2 + \cdots + y_{n-2}) \\ &\quad \geq \;-1 - \frac{n-2}{2} \;=\; -\frac{n}{2}\,. \end{aligned}$$

(ii) $r = 2$. In this case we have $x_1 = 0$, $x_2 = -1$, $x_3 = 0$, $\ldots$. Define $y_j = x_{j+2}$ $(j = 1, 2, \ldots, n-2)$. Then

$$(x_1 + x_2) + (x_3 + \cdots + x_n) \ \geq\ -1 - \frac{n-2}{2} \ = \ -\frac{n}{2}\,.$$

Now, consider the infinite sequence $\{a_n\}$. Then $\{a_1, a_2, \ldots, a_n\}$ satisfies

$$a_1 = 0\,, \quad |a_j| = |a_{j-1} + 1| \quad (2 \leq j \leq n)\,,$$

so, by $P(n)$, the required result follows for all values of n.

Solution 15/5 The inequality is equivalent to $\left(x^2 + a\right)^2 \ \geq\ 4\left(x^2 + a - 1\right)$. Taking the right side from the left, we get

$$x^4 + (2a - 4)\,x^2 + \left(a^2 - 4a + 4\right) \ = \ \left(x^2 + a - 2\right)^2 \ \geq\ 0\,.$$

Equality holds if and only if $1 \leq a \leq 2$ and $x = \pm\sqrt{2 - a}$.

Solution 15/6

$$\begin{aligned}
f(a,c,b,d) - f(a,b,c,d) &= (a-c)^2 - (a-b)^2 + (b-d)^2 - (c-d)^2 \\
&= (b-c)(2a-b-c) + (b-c)(b+c-2d) \\
&= 2(c-b)(d-a) \ > \ 0\,; \\
f(a,b,c,d) - f(a,b,d,c) &= (b-c)^2 - (b-d)^2 + (d-a)^2 - (c-a)^2 \\
&= (d-c)(2b-c-d) + (d-c)(c+d-2a) \\
&= 2(d-c)(b-a) \ > \ 0\,.
\end{aligned}$$

Solution 15/7

1. Observe that

$$\begin{aligned}
&x^2 + y^2 + z^2 - |x||y| - |y||z| - |z||x| \\
&= \ \tfrac{1}{2}\left(|x| - |y|\right)^2 + \tfrac{1}{2}\left(|y| - |z|\right)^2 + \tfrac{1}{2}\left(|z| - |x|\right)^2 \\
&\geq \ 0\,.
\end{aligned}$$

Hence

$$|xy + yz + zx| \ \leq\ |x||y| + |y||z| + |z||x| \ \leq\ x^2 + y^2 + z^2\,.$$

2. Apply the Cauchy-Schwarz Inequality to (x, y, z) and (y, z, x).

Solution 15/8

$$\begin{aligned}
\left|\sqrt{x^2+y^2} - \sqrt{x^2+z^2}\right| &= \frac{|y^2 - z^2|}{\sqrt{x^2+y^2} + \sqrt{x^2+z^2}} \\
&= \frac{|y-z||y+z|}{\sqrt{x^2+y^2} + \sqrt{x^2+z^2}}\,.
\end{aligned}$$

The result follows since

$$|y + z| \ \leq\ |y| + |z| \ = \ \sqrt{y^2} + \sqrt{z^2} \ \leq\ \sqrt{x^2+y^2} + \sqrt{x^2+z^2}\,.$$

Solution 15/9 Squaring to dispose of the absolute value gives

$$(a\alpha + b\beta)^2 \;=\; (a^2+b^2)(\alpha^2+\beta^2) - (a\beta - b\alpha)^2 \;\leq\; 1\,,$$

with equality if and only if $a\beta = b\alpha$.

Solution 15/10

(a) Taking the difference of the two sides yields

$$\begin{aligned} &a^2+\alpha^2+b^2+\beta^2+a\alpha+b\beta-a\beta+b\alpha \\ &\quad= \;\tfrac{1}{2}\left((a+\alpha)^2+(a-\beta)^2+(b+\alpha)^2+(b+\beta)^2\right) \;\geq\; 0\,. \end{aligned}$$

Equality cannot occur, since this would require $a = -\alpha = \beta = -b = 0$.

(b) [For those who know some calculus.] We can select positive reals u and v, and real θ and ϕ for which $a = u\cos\theta$, $b = u\sin\theta$, $\alpha = v\cos\phi$, $\beta = v\sin\phi$. The condition $a\beta - b\alpha = 1$ says that $uv\sin(\phi-\theta) = 1$, so that $uv \geq 1$, and $\cos^2(\phi-\theta) = (u^2v^2-1)(uv)^{-2}$. The left side of the proposed inequality becomes

$$u^2+v^2 \pm \sqrt{u^2v^2-1}\,,$$

and this is not less than $2uv - \sqrt{u^2v^2-1}\,$.

Let $f(t) = 2t - \left(t^2-1\right)^{1/2}$ for $t \geq 1$. Then $f'(t) = 2 - t\left(t^2-1\right)^{-1/2}$. Now $f'(t) \geq 0$ if and only if $4(t^2-1) \geq t^2$. or equivalently, $t \geq \frac{2}{\sqrt{3}}$. Thus, $f(t)$ attains its minimum value of $\sqrt{3}$ when $t = \frac{2}{\sqrt{3}}$. The result follows from this.

Solution 15/11 The inequality holds for $n = 2$. It is natural to try a proof by induction, which will succeed if we can establish that

$$\frac{3n}{2n+1} + \frac{1}{(n+1)^2} \;>\; \frac{3(n+1)}{2(n+1)+1} \;=\; \frac{3(n+1)}{2n+3}$$

for $n \geq 2\,$. We find that

$$\frac{1}{(n+1)^2} + \frac{3n}{2n+1} - \frac{3(n+1)}{2n+3} \;=\; \frac{n(n+2)}{(n+1)^2(2n+1)(2n+3)} \;>\; 0\,,$$

and so the induction proof can be constructed.

Solution 15/12 The inequality is equivalent to

$$\sqrt{2ab-a^2} > b - \sqrt{b^2-a^2}\,.$$

Squaring and dividing by $2b$ gives the equivalent inequality

$$\sqrt{b^2-a^2} > b - a\,.$$

Dividing by $\sqrt{b-a}$ gives another equivalent inequality,

$$\sqrt{b+a} > \sqrt{b-a}\,,$$

which clearly holds.

Solution 15/13 Since the expression is completely symmetrical in x, y and z, we may suppose that $x \geq y \geq z > 0$. The third term is non-negative. The sum of the first two terms is

$$(x-y)\left(x^2 - xz - y^2 + yz\right) \;=\; (x-y)(x+y-z)\,,$$

which, again, is positive. The result follows.

Solution 15/14

(a) Note that $n! = n(n-1)\cdots(n-k+1)\cdot(n-k)!$

(b) $n! \;<\; (k!\,(n-k)!)^2$ for $n = 2$, 3, and for $(n,k) = (4,2)$.

Suppose now that $n \geq 5$. It is straightforward to verify that $k!(n-k)!$ assumes its maximum value when $k = \left\lfloor \frac{n}{2} \right\rfloor$. Thus it is sufficient to show that

$$(2m-1)! \;<\; \left((m-1)!m!\right)^2 \qquad \text{and} \qquad (2m)! \;<\; (m!)^4 \qquad \text{for } m \geq 3.$$

This holds when $m = 3$. Use the fact that $(2m+1) < (m+1)^2$ and $(2m+2) < (m+1)^2$ to construct an induction argument for the general case.

Solution 15/15 Observe that

$$\left((x-y)^{\frac{1}{n}} + y^{\frac{1}{n}}\right)^n \;=\; (x-y) + \left(\sum_{k=1}^{n-1} \binom{n}{k} (x-y)^{\frac{n-k}{n}} y^{\frac{k}{n}}\right) + y\,.$$

Solution 15/16 Let $|x| \geq |y|$. Then

$$2\left(|x|^p + |y|^p\right)^{\frac{1}{p}} \;\geq\; 2|x| \;\geq\; |x| + |y| \;\geq\; |x+y|\,.$$

Solution 15/17 For each positive real x, we have

$$\begin{aligned}
1-(1+x)^n(1-nx) \;&=\; \sum_{k=0}^{n} n\binom{n}{k}x^{k+1} - \sum_{k=1}^{n}\binom{n}{k}x^k \\
&=\; nx^{n+1} + \sum_{k=1}^{n}\left(n\binom{n}{k-1} - \binom{n}{k}\right)x^k \\
&=\; nx^{n+1} + \sum_{k=1}^{n}\frac{n!}{k!(n+1-k)!}(n+1)(k-1)x^k \\
&>\; 0\,.
\end{aligned}$$

The result follows for $0 < x < \dfrac{1}{n}$ by dividing through by the positive quantity $1 - nx$.

Solution 15/18

1. Note that

$$\begin{aligned}\frac{s_{n+1}}{s_n} &= \left(1+\frac{1}{n}\right)\left(\frac{\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n}\right)}\right)^{n+1} \\ &= \left(\frac{n+1}{n}\right)\left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} \\ &= \left(\frac{n+1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1} \\ &> \left(\frac{n+1}{n}\right)\left(1-\frac{1}{n+1}\right) = 1\,,\end{aligned}$$

since $(1-x)^{n+1} > 1-(n+1)x$ for $0 < x < 1$.

Hence $s_{n+1} > s_n$ for every positive integer n.

$$\begin{aligned}\frac{t_n}{t_{n+1}} &= \left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n+1}\right)}\right)^{n+2} \\ &= \left(\frac{n}{n+1}\right)\left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} \\ &= \left(\frac{n}{n+1}\right)\left(1+\frac{1}{n(n+2)}\right)^{n+2} \\ &> \left(\frac{n}{n+1}\right)\left(1+\frac{1}{n}\right) = 1\,,\end{aligned}$$

since $(1+x)^{n+2} > 1+(n+2)x$ for $0 < x$.

Hence $t_n > t_{n+1}$ for every positive integer n.

For any positive integers, k, j, let $m = \max(k,j)$. Then $s_{m+1} < t_{m+1}$, so that

$$s_k < s_{k+1} \le s_{m+1} < t_{m+1} \le t_{j+1} < t_j\,,$$

as desired. Also,

$$\begin{aligned}0 \;<\; t_n - s_n &= \left(1+\frac{1}{n}\right)^n \left(\left(1+\frac{1}{n}\right)-1\right) \\ &= s_n\left(\frac{1}{n}\right) \le t_1\left(\frac{1}{n}\right) \\ &= \frac{4}{n}\,.\end{aligned}$$

Hence $t_n - s_n \to 0$ as $n \to \infty$.

2. [Solution by Richard Hoshino]

We start with a lemma:

Lemma 1 *Suppose that a, $b > 0$. Then, by the AM–GM Inequality, for each $k = 1, 2, \ldots$,*

$$\frac{a+kb}{k+1} \ge \left(ab^k\right)^{1/k+1}\,.$$

Hence,

$$\left(\frac{a+kb}{k+1}\right)^{k+1} \;\geq\; ab^k \,,$$

with equality if and only if $a = b$.

Taking $a = 1$, $b = 1 + \frac{1}{k}$, we obtain

$$\left(\frac{k+2}{k+1}\right)^{k+1} \;>\; \left(1+\tfrac{1}{k}\right)^{k} \,,$$

so that $s_{k+1} > s_k$.

Taking $a = \frac{j+2}{j}$, $b = 1 + \frac{1}{j+1}$, $k = j+1$, we obtain

$$\left(\frac{\left(\frac{j+2}{j}\right) + (j+2)}{j+2}\right)^{j+2} \;>\; \left(\frac{j+2}{j}\right)\left(1+\frac{1}{j+1}\right)^{j+1} \,,$$

so that $\left(\text{since } \frac{1}{j} + 1 = \frac{j+1}{j}\right)$,

$$\left(\frac{j+1}{j}\right)^{j+2} \;>\; \left(\frac{j+2}{j}\right)\left(1+\frac{1}{j+1}\right)^{j+1} \,.$$

Since $1 + \frac{1}{j+1} = \frac{j+2}{j+1}$, we obtain that

$$\left(\frac{j+1}{j}\right)\left(1+\frac{1}{j}\right)^{j+1} \;>\; \left(\frac{j+1}{j}\right)\left(1+\frac{1}{j+1}\right)^{j+2} \,.$$

This means that $t_j > t_{j+1}$, and we can now complete this solution as in the previous solution.

Solution 15/19

1. Suppose that $n \in \{2\ 4, 3, 4, \ldots\}$. Let $u_n = n^{1/n} - 1$. Then $u_n > 0$ and

$$\begin{aligned} n \;&=\; (1+u_n)^n \;=\; 1 + nu_n + \binom{n}{2}u_n^2 + \binom{n}{3}u_n^3 + \cdots \\ &>\; \binom{n}{2}u_n^2 \;=\; \frac{n(n-1)}{2}\,u_n^2 \,. \end{aligned}$$

Thus, $u_n^2 < \frac{2}{n-1}$, so that $u_n < \sqrt{\frac{2}{n-1}}$.

Since $\lim\limits_{n\to\infty} \frac{2}{n-1} = 0$, we have that $\lim\limits_{n\to\infty} u_n = 0$; therefore $\lim\limits_{n\to\infty} n^{1/n} = 1$.

Comment: we can sharpen the inequality to $u_n < \sqrt{\frac{2}{n}}$.

Note that $\left(1+\sqrt{\frac{2}{n}}\right)^n = 1+\sqrt{2n}+\frac{n(n-1)}{2}\cdot\frac{2}{n}+\cdots > 1+n-1 = n\,.$

2. For $n \in \mathbb{N}$, by truncating the binomial expansion, we have that

$$\left(1+\frac{1}{\sqrt{n}}\right)^n \;\geq\; 1+n\left(\frac{1}{\sqrt{n}}\right) \;=\; 1+\sqrt{n} \;>\; \sqrt{n}$$

Thus

$$\left(1+\frac{1}{\sqrt{n}}\right)^2 \;>\; n^{1/n}\,.$$

Since $n \geq 1$, we have

$$1 \;\leq\; n^{1/n} \;<\; \left(1+\frac{1}{\sqrt{n}}\right)^2\,.$$

Let $n \to \infty$ and the result follows.

3. By taking a single term of the binomial expansion, we find that, for $n \geq 2$,

$$\begin{aligned}(\sqrt{n}+1)^{2n} \;&>\; \binom{2n}{2}(\sqrt{n})^{(2n-2)} \\ &=\; \frac{2n(2n-1)}{2}\,n^{n-1} \;=\; (2n-1)\,n^n \;>\; n^{n+1}\end{aligned}$$

so that

$$\left(1+\frac{1}{\sqrt{n}}\right)^{2n} \;=\; \frac{(\sqrt{n}+1)^{2n}}{n^n} \;>\; n\,.$$

Therefore,

$$\left(1+\frac{1}{\sqrt{n}}\right)^2 \;>\; n^{1/n}\,.$$

The result follows as before.

4. [Solution by Reza Shahidi]

By the AM–GM Inequality,

$$\begin{aligned}1 \;\leq\; n^{1/n} \;&=\; \Big(\underbrace{1\cdot 1\cdot 1\cdots 1}_{n-2}\sqrt{n}\sqrt{n}\Big)^{1/n} \\ &\leq\; \frac{n-2+2\sqrt{n}}{n} \;=\; 1-\frac{2}{n}+\frac{2}{\sqrt{n}} \\ &=\; 1+2\left(\frac{1}{\sqrt{n}}-\frac{1}{n}\right) \;=\; 1+\frac{2\left(\sqrt{n}-1\right)}{n} \;<\; 1+\frac{2}{\sqrt{n}}\,.\end{aligned}$$

Since $\lim\limits_{n\to\infty}\left(1+\frac{2}{\sqrt{n}}\right) = 1$, it follows that $\lim\limits_{n\to\infty} n^{1/n} = 1$.

Solution 15/20

(b) The result holds when $n = 1$. Assuming the result up to $n-1$ (≥ 1), we have

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \;\leq\; \frac{1}{\sqrt{3n-2}}\cdot\frac{2n-1}{2n}\,.$$

So we need to show that

$$\left(\frac{2n}{2n-1}\right)^2 \geq \frac{3n+1}{3n-2}\,.$$

This is equivalent to $n \geq 1$, and so the result follows by induction.

(a) From (b), we find that the left side does not exceed

$$\frac{1}{\sqrt{1500001}} < \frac{1}{\sqrt{1000000}} = \frac{1}{1000}\,.$$

Solution 15/21

(a) True. Note that $\lfloor a\rfloor + \lfloor b\rfloor \leq a+b$.

(b) True. Add the inequalities $a < \lfloor a\rfloor + 1$ and $b < \lfloor b\rfloor + 1$.

(c) False. Take $a = b = -\frac{1}{2}$.

(d) False. Take $a = b = -\frac{1}{2}$.

However, if a and b are non-negative, then

$$\lfloor ab\rfloor \leq ab < (\lfloor a\rfloor + 1)(\lfloor b\rfloor + 1) = \lfloor a\rfloor\lfloor b\rfloor + \lfloor a\rfloor + \lfloor b\rfloor + 1\,,$$

and the inequality holds.

(e) True. Since $\lfloor a^2\rfloor \leq a^2$, it follows that $\sqrt{\lfloor a^2\rfloor} \leq \sqrt{a^2}$, and that $\left\lfloor\sqrt{\lfloor a^2\rfloor}\right\rfloor \leq \left\lfloor\sqrt{a^2}\right\rfloor$.

For the reverse inequality, we first note that, for any non-negative integer n, we have $\sqrt{n+1} \leq \sqrt{n} + 1$, so that $\left\lfloor\sqrt{n+1}\right\rfloor \leq \lfloor n\rfloor + 1$. Since $a^2 < \lfloor a^2\rfloor + 1$, it follows that $\sqrt{a^2} < \sqrt{\lfloor a^2\rfloor + 1}$, so that $\left\lfloor\sqrt{a^2}\right\rfloor \leq \left\lfloor\sqrt{\lfloor a^2\rfloor}\right\rfloor$, as desired.

(f) True. Note that $\lfloor\sqrt{n}\rfloor \leq \sqrt{n}$.

(g) True. Square the identity $\sqrt{n} < (\lfloor\sqrt{n}\rfloor + 1)$, and note that n is an integer.

(h) True. Note that $\lfloor\sqrt[3]{n}\rfloor \leq \sqrt[3]{n}$.

(i) True. Cube the inequality $\sqrt[3]{n} < (\lfloor\sqrt[3]{n}\rfloor + 1)$, and note that n is an integer.

Solution 15/22 Since $\dfrac{x^3-y^3}{x^2+xy+y^2} = x-y$, $\dfrac{y^3-z^3}{y^2+yz+z^2} = y-z$, and $\dfrac{z^3-x^3}{z^2+zx+x^2} = z-x$, it follows that the left side is equal to

$$\frac{y^3}{x^2+xy+y^2} + \frac{z^3}{y^2+yz+z^2} + \frac{x^3}{z^2+zx+x^2}$$
$$= \frac{1}{2}\left(\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2}\right)\,.$$

Now, $\dfrac{x^3+y^3}{x^2+xy+y^2} - \dfrac{x+y}{3} \;=\; \dfrac{2(x-y)^2(x+y)}{3(x^2+xy+y^2)} \;\geq\; 0$, with similar inequalities for the other terms. The result follows.

Solution 15/23 Multiply both sides by $(a+b)+(\alpha+\beta)$, and then subtract $ab+\alpha\beta$ from both sides. We find that the inequality is equivalent to

$$\frac{ab}{a+b}(\alpha+\beta) + \frac{\alpha\beta}{\alpha+\beta}(a+b) \;\leq\; \alpha b + \beta a\,,$$

and hence to

$$ab(\alpha+\beta)^2 + \alpha\beta(a+b)^2 \;\leq\; (\alpha b+\beta a)(a+b)(\alpha+\beta)\,.$$

Taking the left side from the right side yields

$$(\alpha b)^2 - 2\alpha b\beta a + (\beta a)^2 \;=\; (\alpha b - \beta a)^2\,,$$

which is non-negative and vanishes if and only if $\alpha : \beta = a : b$. The result follows.

Solution 15/24

(a) For the left inequality, use the fact that

$$(n+1-k)k - n \;=\; (n-k)(k-1) \;\geq\; 0$$

with equality if and only if $k=1$. For the right inequality, use the fact that $k \leq 2^{k-1}$ with equality if and only if $k=1$, 2.

(b) Note that, for $1 \leq k \leq n$,

$$\begin{aligned}\left(\frac{n+1}{2}\right)^2 - k(n+1-k) \;&=\; \frac{1}{4}\left(n^2+2n+1-4kn-4k+4k^2\right)\\ &=\; \frac{1}{4}(n-2k+1)^2 \;\geq\; 0\,.\end{aligned}$$

Solution 15/25 For specificity, let n be odd. (A similar argument pertains when n is even.) The inequality is equivalent to

$$\frac{2!}{(n+1)!}\,\frac{4!}{(n+1)!}\cdots\frac{(n-1)!}{(n+1)!} > \frac{(n+1)!}{(n+3)!}\,\frac{(n+1)!}{(n+5)!}\cdots\frac{(n+1)!}{(2n)!}\,.$$

This can be obtained from a pairwise comparison of terms using

$$\begin{aligned}\frac{k!}{(n+1)!} \;&=\; \frac{1}{(k+1)\cdots(n+1)}\\ &>\; \frac{1}{(n+2)\cdots(2n+2-k)} \;=\; \frac{(n+1)!}{(2n+2-k)!}\end{aligned}$$

for $k=2,\,4,\,\ldots,\,n-1$, where the denominators of the middle terms each have $n+1-k$ terms.

Solution 15/26 The proof is by induction. When $n = 1$, we in fact get equality. Suppose, for $n \geq 2$, that $n^{n-2}(n+1)^{n-1} \geq 3^{n-1}((n-1)!)^2$. Then

$$\begin{aligned}(n+1)^{n-1}(n+2)^n &= n^{n-2}(n+1)^{n-1}\left(\frac{n+2}{n}\right)^{n-2}(n+2)^2 \\ &\geq 3^{n-1}((n-1)!)^2\left(\frac{n+2}{n}\right)^{n-2}(n+2)^2 \\ &= 3^{n-1}(n!)^2\left(\frac{n+2}{n}\right)^n = 3^{n-1}(n!)^2\left(1+\frac{2}{n}\right)^n \\ &> 3^{n-1}(n!)^2\left(1+\frac{2n}{n}\right) = 3^n(n!)^2 ,\end{aligned}$$

as desired.

Solution 15/27 We first show that $\left(1+\frac{1}{n}\right)^n < 3$ for each positive integer n exceeding 2. This relies on knowing that $2^{k-1} \leq k!$ for $k = 1, 2, \ldots$. So, we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &= 1+1+\sum_{k=2}^{n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{k!} \\ &< 1+1+\sum_{k=2}^{n}\frac{1}{k!} = 1+1+\sum_{k=2}^{n}\frac{1}{2^{k-2}} < 3 .\end{aligned}$$

Thus, for $n \geq 3$, we obtain $\left(1+\frac{1}{n}\right)^n < 3 \leq n$, so that $(n+1)^n < n^{n+1}$ as desired.

Solution 15/28 (b) By Tchebychev's Inequality, we have that

$$\frac{(2n)!}{(n!)^2} = \binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2 \geq \frac{1}{n+1}\left(\sum_{k=0}^{n}\binom{n}{k}\right)^2 = \frac{(2^n)^2}{n+1} = \frac{4^n}{n+1} .$$

In fact, it is straightforward to check from the proof of the inequality that it is strict here.

On the other hand,

$$\binom{2n}{n} < \sum_{k=0}^{2n}\binom{2n}{k} = (1+1)^{2n} = 4^n .$$

Solution 15/29 We may take $c > 0$. Let $x = \frac{u}{c}$, $y = \frac{v}{c}$. Then we must establish that $\left(\frac{x+y}{1+xy}\right)^2 < 1$, subject to $-1 < x < 1$, $-1 < y < 1$.

But,

$$\begin{aligned} 1 - \left(\frac{x+y}{1+xy}\right)^2 &= \frac{(1+xy)^2 - (x+y)^2}{(1+xy)^2} \\ &= \frac{((1+xy) - (x+y))\,((1+xy) + (x+y))}{(1+xy)^2} \\ &= \frac{(1-x)(1-y)(1+x)(1+y)}{(1+xy)^2} \;>\; 0\,. \end{aligned}$$

16 Problems Involving Standard Results

16.1 The Problems

Problem 16/1

(a) *Suppose that* $x \geq 0$. *Prove that*

$$x + \frac{1}{x} \;\geq\; 2\,.$$

(b) *For positive reals* x, y, z, *prove that*

$$xz + \frac{y}{z} \;\geq\; 2\sqrt{xy}\,.$$

Problem 16/2

Suppose that a_1, a_2, $\ldots$, a_n *is a set of positive numbers. Prove that*

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \;\geq\; n^2\,.$$

More generally, prove that for non-negative b_1, b_2, $\ldots$, b_n,

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{b_k}{a_k}\right) \;\geq\; \left(\sum_{k=1}^{n} b_k\right)^2 .$$

Problem 16/3 (CRUX [1976: 297])

If $0 < b \leq a$, *prove that*

$$a + b - 2\sqrt{ab} \geq \frac{1}{2}\frac{(a-b)^2}{a+b}\,.$$

Problem 16/4

For real numbers x, $y \geq 0$, *prove that*

$$\sqrt{x^2 + y^2} \;\geq\; x + y - (2 - \sqrt{2})\sqrt{xy}\,.$$

Problem 16/5

For non-negative reals a, b, c, *prove that*

$$a^2b^2 + b^2c^2 + c^2a^2 \;\geq\; abc(a + b + c)\,.$$

Problem 16/6

For positive reals a, b, c *with* $a+b+c=1$, *prove that*

$$ab+bc+ca \ \le\ \frac{1}{3}\,.$$

Problem 16/7

For positive real numbers, a, b, c, *prove that*

(a) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ \le\ \frac{a^2+b^2+c^2}{abc}\,.$

(b) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ \le\ \frac{\left(a^{3n-1}+b^{3n-1}+c^{3n-1}\right)}{(abc)^n}$ *for integer* $n \ge 1$.

Problem 16/8

For $0 < x < 1$ *and integer* n, *prove that*

$$(1-x)^n \ <\ \frac{1}{1+nx}\,.$$

Problem 16/9

For positive reals a, b, c, *prove that*

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \ \ge\ \frac{a}{c}+\frac{c}{b}+\frac{b}{a}\,.$$

Problem 16/10

For positive a, b *such that* $a+b=1$, *prove that*

$$\frac{2}{a/x+b/y} \ \le\ ax+by\,.$$

Problem 16/11 (*Eötvös 1910*)

If a, b, c *are real numbers such that* $a^2+b^2+c^2=1$, *prove that*

$$-\frac{1}{2} \ \le\ ab+bc+ca \ \le\ 1\,.$$

Can either equality occur?

Problem 16/12

For positive reals a, b, c, d, *prove that*

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \ \ge\ 2\,.$$

Problem 16/13

Suppose that $x_1, x_2, \ldots, x_n > 0$. *Prove that*

$$\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\cdots+\frac{x_n^2}{x_n+x_1} \ \ge\ \frac{1}{2}\left(x_1+x_2+\cdots+x_n\right)\,.$$

Problem 16/14

Suppose that $a_1 \ge a_2 \ge \ldots \ge a_n > 0$ *and that*

$$b_1 \ge a_1, \quad b_1 b_2 > a_1 a_2, \quad b_1 b_2 b_3 > a_1 a_2 a_3\,,$$
$$\ldots, \quad b_1 b_2 \ldots b_n > a_1 a_2 \ldots a_n\,.$$

Prove that

$$b_1 + b_2 + \cdots + b_n \;\ge\; a_1 + a_2 + \cdots + a_n\,.$$

Problem 16/15 (*Sharp Calculator Competition 1995* [*South Africa*]).

For positive a, b, α, β, *prove that*

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{\alpha} + \frac{16}{\beta} \;\ge\; \frac{64}{a + b + \alpha + \beta}\,.$$

Problem 16/16

For positive integers n, *define* $n!! = n(n-2)(n-4)\cdots$, *terminating with* 1 *if* n *is odd, or with* 2 *if* n *is even.*

Prove that

(a) $n^n \;>\; (2n-1)!!\,,$

(b) $(n+1)^n \;>\; (2n)!!\,,$

(c) $\dfrac{(2n-1)!!}{(2n)!!} \;<\; \dfrac{1}{\sqrt{n}}\,.$

Problem 16/17

For two positive numbers, x *and* y, *let* a *be their arithmetic mean,* g, *their geometric mean, and* h, *their harmonic mean. Prove that* $a + h \ge 2g$.

Does this result extend to the general case of more than two positive numbers, or to weighted means?

Problem 16/18

For each positive integer n, *prove that*

$$(n+1)^n (2n+1)^n \;\ge\; 6^n \,(n!)^2\,.$$

Hint: where have you seen $(n+1)(2n+1)$ *in a formula?*

16.2 The Solutions

Solution 16/1
(a) Use the AM–GM Inequality on x and $\frac{1}{x}$.
(b) Use the AM–GM Inequality on xz and $\frac{y}{z}$.

Solution 16/2 Use the HM–AM Inequality on $\{a_1, a_2, \ldots, a_n\}$.

Alternatively, apply the Cauchy-Schwarz Inequality to $\{\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\}$, and to $\{\sqrt{\frac{1}{a_1}}, \sqrt{\frac{1}{a_2}}, \ldots, \sqrt{\frac{1}{a_n}}\}$.

Solution 16/3 By the AM–GM Inequality, we have

$$a+b-\frac{1}{2}\frac{(a-b)^2}{a+b} \;=\; \frac{a^2+6ab+b^2}{2(a+b)} \;=\; \frac{1}{2}\left(a+b+\frac{4ab}{a+b}\right) \;\geq\; 2\sqrt{ab}\,.$$

Solution 16/4 Let r, a, g, be respectively the root-mean-square, the arithmetic mean, and the geometric mean of x and y. The proposed inequality is equivalent to $r-g \geq \sqrt{2}(a-g)$. Since both sides are non-negative, squaring gives the equivalent inequality

$$r^2-2rg+g^2 \;\geq\; 2a^2-4ag+2g^2\,.$$

Since $r^2+g^2 = 2a^2$, this in turn is equivalent to $a \geq \frac{r+g}{2}$. But, by the RMS-AM Inequality, we have

$$a \;=\; \sqrt{\frac{r^2+g^2}{2}} \;\geq\; \frac{r+g}{2}\,,$$

and the result follows.

Solution 16/5 If $abc = 0$, the result is clear. If $abc > 0$, then we have

$$\begin{aligned}\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \;&=\; \frac{1}{2}\left(a\left(\frac{b}{c}+\frac{c}{b}\right)+b\left(\frac{c}{a}+\frac{a}{c}\right)+c\left(\frac{a}{b}+\frac{b}{a}\right)\right)\\ &\geq\; \frac{1}{2}\left(2a+2b+2c\right)\,,\end{aligned}$$

and the result follows.

Solution 16/6 Note that $1 = \left(a^2+b^2+c^2\right)+2(ab+bc+ca)$, and use the inequality between the root-mean-square and arithmetic means of a, b, c.

Solution 16/7
(a) Since $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$, we have

$$a^2+b^2+c^2 \;\geq\; bc+ca+ab \;=\; abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\,.$$

(b) The result holds for $n = 1$. We prove the result by induction.

Let integer $r > 0$, $u_r = \left(\dfrac{a^r + b^r + c^r}{3}\right)^{1/r}$ and $g = (abc)^{\frac{1}{3}}$.

Then $u_{r+3} \geq u_r \geq g$, so that

$$u_{r+3}^{r+3} \;=\; u_{r+3}^{r}\, u_{r+3}^{3} \;\geq\; u_r^r\, g^3 \,.$$

Suppose that the desired result holds for $n = k$. Then

$$\begin{aligned} \frac{a^{3k+2} + b^{3k+2} + c^{3k+2}}{(abc)^{k+1}} \;&=\; \frac{3u_{3k+2}^{3k+2}}{(g^3)^{k+1}} \;\geq\; \frac{3u_{3k-1}^{3k-1}\, g^3}{(g^3)^{k+1}} \\ &=\; \frac{a^{3k-1} + b^{3k-1} + c^{3k-1}}{(abc)^k} \;\geq\; \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,, \end{aligned}$$

so that the result holds for $n = k + 1$.

The induction step is now complete.

Solution 16/8 Let $y = 1 - x$. The inequality is equivalent to

$$(1 + n)y^n \;<\; 1 + ny^{n+1}\,,$$

which is a straight-forward consequence of the Weighted AM–GM Inequality.

Solution 16/9 By the Cauchy-Schwarz Inequality, we have

$$\frac{a}{c} + \frac{c}{b} + \frac{b}{a} \;=\; \frac{a}{b}\cdot\frac{b}{c} + \frac{c}{a}\cdot\frac{a}{b} + \frac{b}{c}\cdot\frac{c}{a} \;\leq\; \left(\frac{a^2}{b^2} + \frac{c^2}{a^2} + \frac{b^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{b^2}{c^2} + \frac{a^2}{b^2} + \frac{c^2}{a^2}\right)^{\frac{1}{2}}\,,$$

which gives the desired result.

Solution 16/10

$$(ax + by)\left(\frac{a}{x} + \frac{b}{y}\right) \;=\; a^2 + b^2 + ab\left(\frac{x}{y} + \frac{y}{x}\right) \;\geq\; a^2 + b^2 + 2ab \;=\; 1\,.$$

Solution 16/11 The upper inequality can be established by using the Cauchy-Schwarz Inequality (compare 16/9); equality occurs when $a = b = c = \dfrac{1}{\sqrt{3}}$.

The lower inequality results from using $(a + b + c)^2 \geq 0$; equality occurs when $a = -b = \dfrac{1}{\sqrt{2}}$ and $c = 0$.

Solution 16/12 Let $s = a + b + c + d$. By the AM–GM Inequality with $u = a + b$, we have $4u(s - u) \leq s^2$. Thus is also true with u being any other pair from a, b, c and d. We shall need it with $u = d + a$.

Note that

$$\begin{aligned} &2\left(a(d + a) + c(b + c) + b(a + b) + d(c + d)\right) - (a + b + c + d)^2 \\ &\quad=\; a^2 + b^2 + c^2 + d^2 - 2ac - 2bd \\ &\quad=\; (a - c)^2 + (b - d)^2 \;\geq\; 0\,. \end{aligned}$$

Hence

$$\begin{aligned}
&\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\\
&= \frac{a(d+a)+c(b+c)}{(d+a)\,(s-(d+a))}+\frac{b(a+b)+d(c+d)}{(a+b)\,(s-(a+b))}\\
&\geq \frac{4\,(a(d+a)+c(b+c))}{s^2}+\frac{4\,(b(a+b)+d(c+d))}{s^2}\\
&= \frac{2\cdot 2\,(a(d+a)+c(b+c)+b(a+b)+d(c+d))}{(a+b+c+d)^2} \;\geq\; 2\,,
\end{aligned}$$

as desired.

Solution 16/13

First Solution. By the Cauchy-Schwarz Inequality (see 16/2), we have

$$\begin{aligned}
&((x_1+x_2)+(x_2+x_3)+\cdots+(x_n+x_1))\\
&\quad\times\left(\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\cdots+\frac{x_n^2}{x_n+x_1}\right) \;\geq\; (x_1+x_2+\cdots+x_n)^2\,,
\end{aligned}$$

whence the result.

Second Solution. Note that, with $x_{n+1}=x_1$, we have, for $1\leq i\leq n$,

$$\frac{x_i^2}{x_i+x_{i+1}} \;=\; x_i-\frac{x_ix_{i+1}}{x_i+x_{i+1}} \;\geq\; x_i-\frac{x_i+x_{i+1}}{4}$$

by the AM–GM Inequality. The result follows.

Third Solution. Suppose that $1\leq i\leq n$ and $x_{n+1}=x_1$.
Since $\dfrac{x_i^2-x_{i+1}^2}{x_i+x_{i+1}} \;=\; x_i-x_{i+1}$, we have

$$\begin{aligned}
\sum_{i=1}^{n}\frac{x_i^2}{x_i+x_{i+1}} \;&=\; \sum_{i=1}^{n}\frac{x_{i+1}^2}{x_i+x_{i+1}} \;=\; \frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2+x_{i+1}^2}{x_i+x_{i+1}}\\
&\geq\; \frac{1}{4}\sum_{i=1}^{n}\frac{(x_i+x_{i+1})^2}{x_i+x_{i+1}} \;=\; \frac{1}{2}\sum_{i=1}^{n}x_i
\end{aligned}$$

using the AM–GM–RMS Inequality.

Fourth Solution. Note that, with $x_{n+1}=x_1$, we have for $1\leq i\leq n$, that

$$\frac{x_i^2}{x_i+x_{i+1}} \;=\; x_i-\frac{1}{2}\left(\frac{\frac{1}{x_i}+\frac{1}{x_{i+1}}}{2}\right)^{-1} \;\geq\; x_i-\frac{1}{4}(x_i+x_{i+1})$$

by the HM–AM Inequality. The result follows.

Fifth Solution. Note that the Weighted HM–AM Inequality states that

$$\left(\sum_{i=1}^{n}\alpha_i a_i^{-1}\right)^{-1} \;\leq\; \sum_{i=1}^{n}\alpha_i a_i \qquad\text{for}\qquad \alpha_i\geq 0\,,\quad \sum_{i=1}^{n}\alpha_i=1\,.$$

Apply this to $\alpha_i=\dfrac{x_i}{x_1+x_2+\cdots+x_n}$, $a_i=\dfrac{x_i}{x_i+x_{i+1}}$.

Solution 16/14 Observe that, since $b_1 b_2 \ldots b_r > 0$ $(1 \le r \le n)$, we have that each b_i must be positive. Let $c_0 = 1$, and

$$c_1 = \frac{b_1}{a_1}\,, \quad c_2 = \frac{b_1 b_2}{a_1 a_2}\,, \ldots\,, c_n = \frac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_n}\,.$$

For $1 \le i \le n$, we have $c_i \ge 1$ and $b_i = a_i \left(\frac{c_i}{c_{i-1}}\right)$. Hence

$$\begin{aligned}
&(b_1 + b_2 + \cdots + b_n) - (a_1 + a_2 + \ldots + a_n) \\
&= \left(\frac{c_1}{c_0} - 1\right) a_1 + \left(\frac{c_2}{c_1} - 1\right) a_2 + \left(\frac{c_3}{c_2} - 1\right) a_3 + \cdots + \left(\frac{c_n}{c_{n-1}} - 1\right) a_n \\
&= (c_1 - 1)(a_1 - a_2) + \left(c_1 + \frac{c_2}{c_1} - 2\right)(a_2 - a_3) \\
&\quad + \left(c_1 + \frac{c_2}{c_1} + \frac{c_3}{c_2} - 3\right)(a_3 - a_4) \\
&\quad + \cdots + \left(c_1 + \frac{c_2}{c_1} + \frac{c_3}{c_2} + \frac{c_i}{c_{i-1}} - i\right)(a_i - a_{i+1}) \\
&\quad + \left(c_1 + \frac{c_2}{c_1} + \frac{c_3}{c_2} + \cdots + \frac{c_n}{c_{n-1}} - n\right) a_n\,.
\end{aligned}$$

By the AM–GM Inequality for each i, we obtain

$$\frac{1}{i}\left(c_1 + \frac{c_2}{c_1} + \frac{c_3}{c_2} + \cdots + \frac{c_i}{c_{i-1}}\right) \;\ge\; \left(c_1 \left(\frac{c_2}{c_1}\right) \cdots \left(\frac{c_i}{c_{i-1}}\right)\right)^{\frac{1}{i}} \;=\; c_i^{\frac{1}{i}} \;\ge\; 1\,,$$

and the result follows.

Solution 16/15 The Cauchy-Schwarz Inequality applied to the vectors $\left(\sqrt{a}, \sqrt{b}, \sqrt{\alpha}, \sqrt{\beta}\right)$, $\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{2}{\sqrt{\alpha}}, \frac{4}{\sqrt{\beta}}\right)$ yields the result.

Solution 16/16 For (a) and (b), we use the Arithmetic–Geometric Mean Inequality to show that $(a-b)b \le \left(\frac{a}{2}\right)^2$ when $0 \le b \le a$, and apply this to pairs of terms in the products $(2n-1)!!$ and $(2n)!!$.

As for (c), it is straight forward to verify that

$$\frac{n-1}{n} - \frac{(2n-1)^2}{4n^2} \;=\; \frac{1}{4n^2} \;>\; 0\,,$$

and to formulate the induction argument.

Solution 16/17 Let a, h, g, be the respective AM, HM and GM of the positive reals x and y. Then $\dfrac{a+h}{2} \ge \sqrt{ah} = g$, with equality if and only if $a = g = h = x = y$.

The result fails to hold in general. Let t be a positive real and let a, h, g, be the respective AM, HM and GM of 1, 1 and t^3 (or, equivalently, the weighted mean of 1 and t^3 with weights $\frac{2}{3}$ and $\frac{1}{3}$). Then

$$a \;=\; \frac{2+t^3}{3}\,, \qquad h \;=\; \frac{3t^3}{1+2t^3}\,, \qquad g \;=\; t\,,$$

so that

$$\begin{aligned} a+h-2g &= \frac{2+t^3}{3}+\frac{3t^3}{1+2t^3}-2t \\ &= \frac{2\left(t^6-6t^4+7t^3-3t+1\right)}{3\left(1+2t^3\right)} = \frac{2(t-1)^3\left(t^3+3t^2-1\right)}{3\left(1+2t^3\right)}. \end{aligned}$$

This is negative when $t \in (2\cos(2\pi/9)-1, 1)$; approx. $(.5321, 1)$. For example, when $t = \frac{2}{3}$, we have $t^3+3t^2-1 > 3t^2-1 = \frac{1}{3} > 0$, while $(t-1)^3 < 0$, so that $a-h < 2g$.

Solution 16/18 By the AM–GM Inequality, we have

$$\frac{(n+1)(2n+1)}{6} = \frac{1^2+2^2+\cdots+n^2}{n} \geq (n!)^{2/n},$$

from which the result follows.

17 Problems Without Solutions

Problem 17/1

For positive a, b. *Prove that*

$$\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}.$$

Problem 17/2

For positive reals a, b, c, *prove that*

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Problem 17/3

Suppose that $x \neq 1$ *is a positive real number and that* n *is a positive integer. Prove that*

$$\frac{1-x^{2n+1}}{1-x} \geq (2n+1)\,x^n.$$

Problem 17/4

For positive reals a, b, c, *prove that*

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a} \leq \frac{a+b+c}{2}.$$

Problem 17/5

For positive reals a, b, c, *prove that*

$$a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c).$$

Problem 17/6

For non-negative reals $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, *prove that*

$$8\left(x^3+y^3+z^3\right)^2 \ \geq\ 9\left(x^2+yz\right)\left(y^2+xz\right)\left(z^2+xy\right)\ .$$

Problem 17/7

For positive reals $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *such that* $(1+a)(1+b)(1+c)=8$, *prove that*

$$abc \ \leq\ 1\,.$$

Problem 17/8

For reals $0 \leq a,\ b,\ c \leq 1$, *prove that*

$$\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c) \ \leq\ 1\,.$$

Problem 17/9

For positive reals $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ *such that* $a^2+b^2=(c^2+d^2)^3$, *prove that*

$$\frac{c^3}{a}+\frac{d^3}{b} \ \geq\ 1\,.$$

Problem 17/10

For non-negative reals $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, *prove that*

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd} \ \leq\ \frac{3(a+b+c+d)}{2}\,.$$

Problem 17/11

For non-negative reals $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, *prove that*

$$(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \ \geq\ 9a^2b^2c^2\,.$$

Problem 17/12

Prove that

$$\left(\sum_{k=1}^{n} a_kb_kc_k\right)^2 \ \leq\ \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right)\,.$$

Problem 17/13

For non-negative reals $\boldsymbol{a}$, $\boldsymbol{b}$, *prove that*

(a) $(a+b)(a^2+b^2)(a^3+b^3) \ \leq\ 4(a^6+b^6)\,,$

(b) $(a+b)(a^3+b^3)(a^7+b^7) \ \leq\ 4(a^{11}+b^{11})\,,$

(c) $ab(a^2+b^2) \ \leq\ a^4+b^4\,,$

(d) $a^2b^2(a^5+b^5) \ \leq\ a^9+b^9\,.$

For positive unequal reals a, b, prove that

(e) $(a^4+b^4)(a^5+b^5) \;\le\; 2(a^9+b^9)\,,$

(f) $(a^2+b^2)(a^3+b^3) \;\le\; 2(a^5+b^5)\,.$

Problem 17/14

For positive reals a, b, prove that

$$\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}} \;\ge\; \sqrt{a}+\sqrt{b}\,.$$

Problem 17/15

For integers $n>7$, prove that $\sqrt{n}^{\sqrt{n+1}} \;>\; \sqrt{n+1}^{\sqrt{n}}\,.$

Problem 17/16 (*Australian MO 1987*)
For natural numbers $n>1$, prove that

$$\sqrt{n+1}+\sqrt{n}-\sqrt{2} \;>\; \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\,.$$

Problem 17/17 (*Eötvös 1896*)
Prove that $\log(n) \;\ge\; k\log(2)$ *where n is a natural number and k is the number of distinct primes that divide n.*

Problem 17/18 (*Eötvös 1911*)
If real numbers a, b, c, α, β, γ satisfy

$$a\gamma-2b\beta+c\alpha \;=\; 0 \qquad \text{and} \qquad ac-b^2 \;>\; 0\,,$$

prove that

$$\alpha\gamma-\beta^2 \;\le\; 0\,.$$

Problem 17/19 (*Eötvös 1913*)
If real numbers a, b, c and $-1\le x\le 1$ satisfy

$$-1 \;\le\; ax^2+bx+c \;\le\; 1\,,$$

prove that

$$-4 \;\le\; 2ax+b \;\le\; 4\,.$$

Problem 17/20

Examine the particular method given for the AM–GM Inequality for three positive numbers. Can you extend this method for four numbers, five numbers?

Problem 17/21

Prove the WAM–WGM Inequality, using the method of Mathematical Induction.

HINT: To set up the induction step, let

$$\begin{aligned} u &= \frac{w_{n-1}}{(w_{n-1}+w_m)}, \\ v &= \frac{w_n}{(w_{n-1}+w_m)}, \\ a &= u\,a_{n-1} + v\,a_n \end{aligned}$$

and verify that

$$\sum_{k=1}^{n} w_k a_k \;=\; \sum_{k=1}^{n-2} w_k a_k \;+\; (w_{n-1}+w_n)\,a\,.$$

Problem 17/22

Suppose that each of a, b, c, α, β *and* γ *are all positive real numbers and that* $a \neq b \neq c \neq a$. *Prove that*

$$a^{\alpha+\beta+\gamma} + b^{\alpha+\beta+\gamma} + c^{\alpha+\beta+\gamma} \;\geq\; a^{\alpha}b^{\beta}c^{\gamma} + a^{\beta}b^{\gamma}c^{\alpha} + a^{\gamma}b^{\alpha}c^{\beta}\,.$$

Problem 17/23

Prove that, for all of a, b, c *and* d *in* $[1,2]$,

$$\frac{1}{2} \;<\; \frac{a(c-d)+2d}{b(d-c)+2c} \;\leq\; 2\,.$$

Problem 17/24

Suppose that $x^5 - x^3 + x = p > 0$. *Prove that* $x^6 \geq 2p - 1$.

Problem 17/25

Prove that $\left(x^3+x^2+3\right)^2 \;>\; 4x^3(x-1)^2$ *for all real* x.

Problem 17/26

Prove that

$$(x+y)(y+z)(z+x) \;\geq\; 8(x+y-z)(y+z-x)(z+x-y)$$

for all real positive x, y *and* z.

Problem 17/27

Prove that $11 \times \sqrt[1993]{10} \;>\; 10 + \sqrt[1000]{10}$.

Problem 17/28

Suppose that $a_1, a_2, \ldots, a_n$ *are all positive real numbers such that* $a_1 a_2 \ldots a_n = 1$.
Prove that

$$\prod_{k=1}^{n} (k + a_k) \ \geq \ n^{n/2} \,.$$

Problem 17/29

Suppose that $\{a_k\}_{k=1}^{n}$ *is a sequence of distinct positive integers. Prove that*

$$\left(\sum_{k=1}^{n} a_k^7\right) + \left(\sum_{k=1}^{n} a_k^5\right) \ \geq \ 2\left(\sum_{k=1}^{n} a_k^3\right)^2 \ .$$

Also, find all sequences for which equality holds.

Problem 17/30

Suppose that $s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. *Prove that*

$$2\sqrt{n+1} - 2 \ < \ s_n \ < \ 2\sqrt{n} - 1 \,.$$

Problem 17/31

Suppose that $\{a_k\}$, $\{b_k\}$, $1 \leq k \leq n$ *are* $2n$ *positive real numbers.*

Prove that either

$$\prod_{k=1}^{n} \frac{a_k}{b_k} \ \geq \ n \qquad \text{or} \qquad \prod_{k=1}^{n} \frac{b_k}{a_k} \ \geq \ n \,.$$

Problem 17/32

Suppose that the polynomial $\sum_{k=0}^{n} a_{n-k} x^k$ $(a_n = 1)$ *has* n *real zeros. Prove that*

$$(n-1)\, a_1^2 \ \geq \ 2n\, a_2 \,.$$

Problem 17/33

Suppose that $a_k \geq 1$ *for all* $k \geq 1$. *For all positive integers* n, *prove that*

$$n + \prod_{k=1}^{n} a_k \ \geq \ 1 + \sum_{k=1}^{n} a_k \,,$$

with equality if and only if no more that one member of the set $\{a_k\}$ *is different from* 1.

Problem 17/34 (*Lithuanian Team Contest 1986*)

Solve the inequality:

$$\sqrt{2x-1}+\sqrt{3x-2} \;<\; \sqrt{4x-3}+\sqrt{5x-4}\,.$$

Problem 17/35 (*Lithuanian Team Contest 1990*)

Prove the inequality:

$$\sqrt{x+1}+\sqrt{2x-3}+\sqrt{50x-3x} \;<\; 12\,.$$

Problem 17/36 (*Lithuanian Team Contest 1987*)

Solve the system of inequalities:

$$\begin{aligned} x^2+y^2 &\leq ax+by\,; \\ |a-b+y-x| &\leq a+b-x-y\,; \\ |x-y| &\leq -x-y\,. \end{aligned}$$

Problem 17/37

Suppose that $1 \geq x_1 \geq x_2 \geq \ldots \geq x_n > 0$ *and that* $0 \leq t \leq 1$. *Prove that*

$$(1+x_1+x_2+\cdots+x_n)^t \;\leq\; 1+x_1^t+2^{t-1}x_2^t+\cdots+n^{t-1}x_n^t\,.$$

Problem 17/38

Let $b_n \;=\; \sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n}}}}$. *Prove that* $b_n < 3$.

Problem 17/39 (CRUX [1978: 12] **)**

Solve the following inequality:

$$\sin x \sin 3x \;>\; \frac{1}{4}\,.$$

18 Appendix

“Sigma” notation

This is a way of representing a sum of quantities without having to write it all out! The symbol used is an upper case Greek letter — sigma — written as $\sum$. We also need a symbol to represent the numbers that are being used to indicate what we must add together, and a symbol to represent the formula for each number to be added.

Here are some examples to show you how it works:

1. For $1+2+3+4+5+6+7+8+9+10$, we can write $\displaystyle\sum_{k=1}^{10} k$.

   Here, the symbol being used to indicate the numbers is k, and the formula for the numbers to be added is (also) k.

   The subscript, “$k = 1$” means that we start with the value of k equal to 1, and proceed to add each value of the formula with each successive value of k until we reach the number in the superscript, here 10.

2. For $1^2+2^2+3^2+4^2+5^2+6^2$, we can write $\displaystyle\sum_{n=1}^{6} n^2$.

   Here, the symbol being used to indicate the numbers is n, and the formula for the numbers to be added is n^2.

   The subscript, “$n = 1$” means that we start with the value of n equal to 1, and proceed to add each value of the formula with each successive value of n until we reach the number in the superscript, here 6.

   This notation is very useful for long sums!

   For $\dfrac{1}{1}+\dfrac{1}{3}+\cdots+\dfrac{1}{1001}$, we can write $\displaystyle\sum_{m=0}^{500}\frac{1}{2m+1}$.

   Here, the symbol being used to indicate the numbers is m, and the formula for the numbers to be added is $\dfrac{1}{2m+1}$.

   We start with $m = 0$, and substituting into the formula, we get $\dfrac{1}{0+1}=\dfrac{1}{1}$. We then substitute $m = 1$, and get $\dfrac{1}{2+1}=\dfrac{1}{3}$, and so on, until we substitute $m = 500$, giving $\dfrac{1}{1000+1}=\dfrac{1}{1001}$.

The general form looks like $\displaystyle\sum_{k=a}^{b} x_k$. The symbol being used to indicate the numbers is k, and the formula for the numbers to be added is given by some formula which we call x_k.

We start with the value $k = a$ and so substitute a for k in the formula x_k. We now add one to a, and so substitute $a + 1$ for k in the formula x_k. We repeat this, until we reach the value $k = b$, which is the last value to be substituted into the formula x_k. Now we add up all the numbers obtained, and the answer is what we mean by $\displaystyle\sum_{k=a}^{b} x_k$.

A similar notation is used for products — $\displaystyle\prod_{k=a}^{b} x_k$.

ATOM

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