A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome I

MATHEMATICAL OLYMPIADS' CORRESPONDENCE PROGRAM (1995-96)

Edward J. Barbeau

University of Toronto

Published by the Canadian Mathematical Society, Ottawa, Ontario
and produced by the CMS CRUX with MAYHEM Office, St. John's,
Newfoundland

Publié par la Société mathématique du Canada, Ottawa (Ontario)
et produit par le Bureau CRUX with MAYHEM de la SMC, St. John's,
Terre-Neuve

This is a PDF version of the original printed volume. There are changes in fonts and format that could result in different line and page breaks.

Ceci est une version PDF de l'ouvrage imprim. Il est possible que des changements de polices ou de format modifient les coupures de ligne ou de page.

ISBN 0-919558-10-0

A Taste Of Mathematics

Aime–T–On les Mathématiques

Volume / Tome I

MATHEMATICAL OLYMPIADS' CORRESPONDENCE PROGRAM (1995-96)

Edward J. Barbeau

University of Toronto

The ATOM series

The booklets in the series, **A Taste of Mathematics**, are published by the Canadian Mathematical Society (CMS). They are designed as enrichment materials for high school students with an interest in and aptitude for mathematics. Some booklets in the series will also cover the materials useful for mathematical competitions at national and international levels.

La collection ATOM

Publiés par la Société mathématique du Canada (SMC), les livrets de la collection Aime-t-on les mathématiques (ATOM) sont destinés au perfectionnement des étudiants du cycle secondaire qui manifestent un intérêt et des aptitudes pour les mathématiques. Certains livrets de la collection ATOM servent également de matériel de préparation aux concours de mathématiques sur l'échiquier national et international.

Net proceeds from the sale of the booklets in the ATOM series will be donated to the CMS's Mathematical Olympiads Fund.

Le produit net de la vente des livrets de la collection ATOM sera versé au Fonds de la SMC pour les Olympiades mathématiques

Contents

Foreword

This volume contains the problems and solutions from the 1995–1996 Mathematical Olympiads' Correspondence Program. This program has several purposes. It provides students with practice at solving and writing up solutions to Olympiad-level problems, it helps to prepare student for the Canadian Mathematical Olympiad and it is a partial criterion for the selection of the Canadian IMO team.

Many of the problems admit several approaches. Accordingly, I have often indicated a number of alternative solutions to a problem in order to show how different ideas can be consummated.

While I have tried to make the text as correct as possible, some mathematical and typographical errors might remain, for which I accept full responsibility. I would be grateful to any reader drawing my attention to errors as well as to alternative solutions.

Thanks are due to Bruce Shawyer of Memorial University of Newfoundland for suggesting the publication of this book and for overseeing its publication, as well as to Cindy Hiscock, 1996 WISE student at Memorial University of Newfoundland, for producing the initial LaTeX document.

It is the hope of the Canadian Mathematical Society that this collection may find its way to high school students who may have the talent, ambition and mathematical expertise to represent Canada internationally. Those who find the problems too challenging at present can work their way up through other collections. For example:

1. The International Mathematical Talent Search (problems can be obtained from the author, or from the magazine *Mathematics & Informatics Quarterly*, subscriptions for which can be obtained by writing to Professor George Berzsenyi, Department of Mathematics, Rose-Hulman Institute of Technology, 5500 Wabash Avenue, Terre Haute, IN 47803–3999, USA);
2. The journal *Crux Mathematicorum with Mathematical Mayhem* (subscriptions can be obtained from the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
3. The book *The Canadian Mathematical Olympiad 1969–1993 L'Olympiade mathématique du Canada*, which contains the problems and solutions of the first twenty five Olympiads held in Canada (published by the Canadian Mathematical Society, 577 King Edward, PO Box 450, Station A, Ottawa, ON, Canada K1N 6N5);
4. The book *Five Hundred Mathematical Challenges*, by E.J. Barbeau, M.S. Klamkin & W.O.J. Moser (published by the Mathematical Association of America, 1529 Eighteenth Street NW, Washington, DC 20036, USA).

E.J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON
Canada M5S 3G3

PROBLEM SET 1 — Numbers

Definitions:

(1) $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x;

(2) $\tau(n)$ denotes the number of integers between 1 and n inclusive that divide evenly into the positive integer n;

(3) $\sigma(n)$ denotes the sum of all the integers between 1 and n inclusive that divide evenly into the positive integer n;

(4) $\phi(n)$ denotes the number of integers between 1 and n inclusive which have only the divisor 1 in common with n.

Exercises

These were not to be handed in and are for the student to test him/herself.

(1) Verify that, for f equal to each of the functions τ, σ and ϕ, we have that $f(mn) = f(m)f(n)$ whenever the greatest common divisor of m and n is equal to 1.

(2) Suppose that $n = \prod p^a$ is the representation of n as a product of powers of distinct primes. Verify that for f as in (1), $f(n) = \prod f(p^a)$,and so deduce a formula for $f(n)$ in each case.

Problems

1 Let n be a positive integer.

(a) By three separate arguments, determine a formula for the sum of the first n odd squares.

(b) By three separate arguments, determine a formula for the sum of the first n odd cubes.

2 Suppose that n is a positive integer. Prove that there is a positive integer k for which

$$\left(\sqrt{2} - 1\right)^n = \sqrt{k} - \sqrt{k-1}.$$

3 Show that every positive rational strictly between 0 and 1 can be written as a finite series of the form

$$\frac{1}{q_1} - \frac{1}{q_1q_2} + \frac{1}{q_1q_2q_3} - \frac{1}{q_1q_2q_3q_4} + \cdots,$$

where the q_i are positive integers.

4 Let p be an odd prime, and let

$$(1+x)^{p-2} = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{p-2}x^{p-2}.$$

Show that $a_1 + 2, a_2 - 3, a_3 + 4, \cdots, a_{p-3} - (p-2), a_{p-2} + (p-1)$ are all multiples of p.

5 Prove that

$$\sum_{k=1}^{2n} \tau(k) - \sum_{k=1}^{n} \left\lfloor \frac{2n}{k} \right\rfloor = n.$$

6 Let the sequence $\{u_n\}$ be defined recursively by

$$u_0 = 0, \quad u_1 = 1, \quad u_n = 1995u_{n-1} - u_{n-2} \quad (n \geq 2).$$

Find all the values of n exceeding 1 for which u_n is prime.

7 Let p be a prime number and let $(a_1, a_2, \cdots a_p)$ and $(b_1, b_2, \cdots, b_p)$ each be arbitrary arrangements of the finite p-tuple $(0, 1, 2, \cdots, p-1)$. For each i, let c_i be the non-negative remainder when the product a_ib_i is divided by p. Show that $(c_1, c_2, \cdots, c_p)$ cannot be a rearrangement of $(0, 1, 2, \cdots, p-1)$.

8 Show that, for $k \geq 2$, the equation

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$$

does not have any solution, where $x_1, x_2, \cdots, x_{k+1}$ are distinct nonzero integers.

9 Prove that, for any non-negative integer n, the equation

$$\left\lfloor n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n+2)^{\frac{1}{2}} \right\rfloor = \left\lfloor (9n+8)^{\frac{1}{2}} \right\rfloor$$

holds.

10 The integers $a_1, a_2, a_3, \cdots$ are determined by the recursion

$$a_1 = 1, \qquad a_2 = 2,$$

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n, & \text{if } a_n \cdot a_{n+1} \text{ is even,} \\ a_{n+1} - a_n, & \text{if } a_n \cdot a_{n+1} \text{ is odd.} \end{cases}$$

Prove that:

(a) The sequence $\{a_n\}$ contains infinitely many positive and infinitely many negative terms;

(b) $\{a_n\}$ is never equal to zero;

(c) if $n = 2^k - 1$ ($k = 2, 3, 4, \cdots$), then a_n is divisible by 7.

11 Denote by $s(n)$ the sum of the base–10 digits of the natural number n. The function $f(n)$ is defined on the natural numbers by

$$f(0) = 0 \qquad f(n) = f(n - s(n)) + 1 \quad (n = 1, 2, \cdots).$$

Prove, or disprove, that $f(m) \leq f(n)$ whenever $1 \leq m \leq n$.

12 Prove that the positive integer n is the product of exactly two primes that differ by 2 if and only if

$$\phi(n)\sigma(n) = (n-3)(n+1).$$

Solutions

1 (a) *Solution 1*

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n \\ &= \frac{n(4n^2-1)}{3}.\end{aligned}$$

1 (a) *Solution 2*

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= \left[4\sum_{i=1}^{n} i(i-1)\right] - n \\ &= \left[\frac{4}{3}\sum_{i=1}^{n}\{(i+1)i(i-1) - i(i-1)(i-2)\}\right] - n \\ &= \frac{4}{3}(n+1)n(n-1) - n \\ &= \frac{n}{3}\left[4(n^2-1) - 3\right] \\ &= \frac{n(4n^2-1)}{3}.\end{aligned}$$

1 (a) *Solution 3*

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= \sum_{j=1}^{2n} j^2 - \sum_{i=1}^{n}(2i)^2 \\ &= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6} \\ &= \frac{2n(2n+1)}{6}\left[(4n+1) - 2(n+1)\right] \\ &= \frac{2n(2n+1)(2n-1)}{6}.\end{aligned}$$

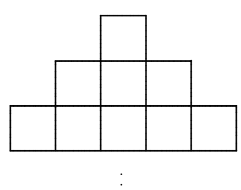
1 (a) *Solution 4* Since
$(2i-1)^2 = \frac{1}{6}\left[(2i+1)2i(2i-1) - (2i-1)(2i-2)(2i-3)\right],$
we have

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 & \\ &= \frac{1}{6}\left[\sum_{i=1}^{n}(2i+1)2i(2i-1) - \sum_{j=0}^{n-1}(2j+1)2j(2j-1)\right] \\ &= \frac{1}{6}(2n+1)2n(2n-1).\end{aligned}$$

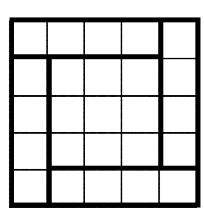
1 (a) *Solution 5* (by induction) The result $\sum_{i=1}^{n}(2i-1)^2 = \frac{n(4n^2-1)}{3}$ holds for $n = 1$.
Assume that it holds for $n = k$. Then,

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1)^2 &= \frac{k(4k^2-1)}{3} + (2k+1)^2 \\ &= \frac{(k+1)[4(k+1)^2-1]}{3}.\end{aligned}$$

1 (a) *Solution 6* The sum is equal to the number of blocks in a pyramid n blocks high with $(2i-1)^2$ blocks in the i^{th} layer from the top.



When $i \geq 2$, in the i^{th} layer, there are $4(2i-2) = 8(i-1)$ blocks not covered by higher blocks. The blocks are the top of a column of $n-(i-1) = n-i+1$ blocks.



3rd layer
from top
($i = 3$)

Hence,

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= n+\sum_{i=2}^{n}\Big(n-(i-1)\Big)\cdot 8(i-1)\\ &= n+8n\sum_{i=2}^{n}(i-1)-8\sum_{i=2}^{n}(i-1)^2\\ &= n+4n^2(n-1)-\frac{4}{3}(n-1)n(2n-1)\\ &= \frac{4n^3-n}{3}.\end{aligned}$$

1 (a) *Solution 7* Observe that $6r^2=(r+2)^3-r^3-12r-8$.
Hence,

$$\begin{aligned}&6\sum_{i=1}^{n}(2i-1)^2\\ &= \sum_{i=1}^{n}(2i+1)^3-\sum_{i=1}^{n}(2i-1)^3-12\sum_{i=1}^{n}(2i-1)-\sum_{i=1}^{n}8\\ &= (2n+1)^3-1-12n^2-8n\\ &= 8n^3-2n=2n(4n^2-1)\end{aligned}$$

so that

$$\sum_{i=1}^{n}(2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

1 (a) *Solution 8* [A. Martin] Let k be a positive integer and let $n\geq 2$. Then

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^k &= 1\cdot 1^{k-1}+3\cdot 3^{k-1}+\cdots+(2n-1)\cdot(2n-1)^{k-1}\\ &= (2n-1)\left[1^{k-1}+\cdots+(2n-1)^{k-1}\right]\\ &\quad -2\left[(n-1)1^k+\cdots+(2n-3)^{k-1}\right]\\ &= (2n-1)\sum_{i=1}^{n}(2i-1)^{k-1}\\ &\quad -2\sum_{k=1}^{n-1}\left[1^{k-1}+\cdots+(2i-1)^{k-1}\right].\end{aligned}$$

Taking $k=1$ yields

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) &= (2n-1)n-2\sum_{i=1}^{n-1} i=(2n^2-n)-(n^2-n)\\ &= n^2.\end{aligned}$$

Taking $k=2$ yields

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= (2n-1)n^2-2\sum_{i=1}^{n-1} i^2\\ &= 2n^3-n^2-\frac{(n-1)(n)(2n-1)}{3}\\ &= \frac{4n^3-n}{3}.\end{aligned}$$

1 (a) *Solution 9* [B.Chun] Note that

$$\begin{aligned}&(2n)^2-(2n-1)^2+(2n-2)^2-(2n-3)^2+\cdots\\ &= 2n+\overline{2n-1}+\overline{2n-2}+\cdots+2+1,\end{aligned}$$

since $(a+1)^2-a^2=(a+1)+a$.

Hence

$$\begin{aligned}&1^2+3^2+\cdots+(2n-1)^2\\ &= 2^2+4^2+\cdots+(2n)^2-(1+2+\cdots+2n)\\ &= 4\sum_{k=1}^{n} k^2-\frac{2n(2n+1)}{2}\\ &= \frac{2n(n+1)(2n+1)}{3}-n(2n+1)\\ &= \frac{n(2n+1)(2n+2-3)}{3}\\ &= \frac{n(2n+1)(2n-1)}{3}.\end{aligned}$$

1 (a) *Solution 10* [D. Cheung] If $a_n=1^2+3^2+\cdots+(2n-1)^2$, then it can be verified that $a_{n+4}-4a_{n+3}+6a_{n+2}-4a_{n+1}+a_n=0$, a recursion whose characteristic polynomial is $(t-1)^4$.

The general solution is $a_n=(\alpha n^3+\beta n^2+\gamma n+\delta)1^n$, and checking a_1, a_2, a_3, a_4 will yield the coefficients $\alpha=\frac{4}{3}$, $\gamma=-\frac{1}{3}$, $\beta=\delta=0$.

1 (a) *Solution 11* [D. Cheung]

$$
\begin{aligned}
&1^2+3^2+\cdots+(2n-1)^2\\
&= (1^2-1^2)+(3^2-1^2)+\cdots+\left[(2n-1)^2-1^2\right]+n\\
&= 0+2\times 4+4\times 6+\cdots+(2n-2)\times 2n+n\\
&= 4(1\times 2+2\times 3+\cdots+(n-1)\times n)+n\\
&= 4\sum_{r=1}^{n} r^{(2)}+n\\
&= 4\frac{(n+1)^{(3)}}{3}+n\\
&= n\left[\frac{4(n^2-1)}{3}+1\right]\\
&= \frac{n(4n^2-1)}{3}.
\end{aligned}
$$

[Here we have used the definition:

$x^{(k)} = x(x-1)\cdots(x-k+1) = k!\binom{x}{k}$].

1 (b) *Solution 1*

$$
\begin{aligned}
\sum_{i=1}^{n}(2i-1)^3 &= \sum_{j=1}^{2n} j^3 - \sum_{i=1}^{n}(2i)^3\\
&= \left[\frac{2n(2n+1)}{2}\right]^2 - 8\left[\frac{n(n+1)}{2}\right]^2\\
&= n^2\left[(2n+1)^2-2(n+1)^2\right]\\
&= n^2(2n^2-1).
\end{aligned}
$$

1 (b) *Solution 2* Observe that

$$
\begin{aligned}
a^3+(b-a)^3 &= b\left[a^2-a(b-a)+(b-a)^2\right]\\
&= b\left[3a^2-3ab+b^2\right]
\end{aligned}
$$

Therefore

$$\begin{aligned} 2\sum_{i=1}^{n}(2i-1)^3 &= \sum_{i=1}^{n}(2i-1)^3 + \sum_{i=1}^{n}(2n-2i+1)^3 \\ &= \sum_{i=1}^{n}\left[(2i-1)^3 + (2n-\overline{2i-1})^3\right] \\ &= \sum_{i=1}^{n} 2n\left[3(2i-1)^2 - 6n(2i-1) + 4n^2\right] \\ &= 6n\left[\frac{n(4n^2-1)}{3}\right] - 12n^2\cdot n^2 + 8n^4 \\ &= n^2\left[2(4n^2-1) - 4n^2\right] \\ &= 2n^2(2n^2-1), \end{aligned}$$

from which the result follows.

1 (b) *Solution 3*

$$\begin{aligned} n^2 + \sum_{i=1}^{n}(2i-1)^3 &= \sum_{i=1}^{n}\left[(2i-1) + (2i-1)^3\right] \\ &= \sum_{i=1}^{n}\left(8i^3 - 12i^2 + 8i - 2\right) \\ &= \sum_{i=1}^{n} 2\left[i^4 - (i-1)^4\right] \\ &= 2\sum_{i=1}^{n} i^4 - 2\sum_{i=1}^{n-1} i^4 \\ &= 2n^4, \end{aligned}$$

from which the result follows.

1 (b) *Solution 4* (by induction) The result $\sum_{i=1}^{n}(2i-1)^3 = n^2(2n^2-1)$ holds for $n=1$. Assume it holds for $n=k$. Then

$$\begin{aligned} \sum_{i=1}^{k+1}(2i-1)^2 &= k^2(2k^2-1) + (2k+1)^3 \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\ &= 2(k+1)^4 - (k+1)^2 \end{aligned}$$

as required for $n=k+1$.

1 (b) *Solution 5* Take $k = 3$ in Solution 8 of 1 (a).

1 (b) *Solution 6*

$$\begin{aligned}\sum_{k=1}^{n}\left[(2k)^3-(2k-1)^3\right] &= 12\sum_{k=1}^{n}k^2-6\sum_{k=1}^{n}k+n\\ &= 2n(n+1)(2n+1)-3n(n+1)+n\\ &= n^2[4n+3].\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n}(2k-1)^3 &= 8\sum_{k=1}^{n}k^3-n^2[4n+3]\\ &= n^2(2n^2-1).\end{aligned}$$

2 *Solution 1* Observe that

$$\left(1+\sqrt{2}\right)^n=\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i}2^i+\sqrt{2}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i+1}2^i,$$

$$\left(1-\sqrt{2}\right)^n=\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i}2^i-\sqrt{2}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i+1}2^i,$$

so that, if $\left(1+\sqrt{2}\right)^n=a_n+\sqrt{2}b_n$ then $\left(1-\sqrt{2}\right)^n=a_n-\sqrt{2}b_n$.
Also

$$\begin{aligned}(-1)^n &= \left[\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)\right]^n\\ &= \left(a_n+\sqrt{2}b_n\right)\left(a_n-\sqrt{2}b_n\right)=a_n^2-2b_n^2.\end{aligned}\tag{1}$$

Hence $\left(\sqrt{2}-1\right)^n=(-1)^n\left(a_n-\sqrt{2}b_n\right)=(-1)^n\left(\sqrt{a_n^2}-\sqrt{2b_n^2}\right)$.

When n is even, we have

$$\left(\sqrt{2}-1\right)^n=\sqrt{a_n^2}-\sqrt{2b_n^2}=\sqrt{2b_n^2+1}-\sqrt{2b_n^2}$$

by (1),

and when n is odd, we have

$$\left(\sqrt{2}-1\right)^n=\sqrt{2b_n^2}-\sqrt{a_n^2}=\sqrt{a_n^2+1}-\sqrt{a_n^2}$$

by (1).

In either case, $\left(\sqrt{2}-1\right)^n$ has the required form.

2 *Solution 2* For each $n = 1, 2, 3, \cdots$, we see from the binomial expansion that there are integers x_n and y_n such that

$$\left(\sqrt{2}-1\right)^n = x_n + y_n\sqrt{2}, \quad (-1)^n x_n > 0, (-1)^n y_n < 0.$$

Since $\left(\sqrt{2}-1\right)^{n+1} = \left(\sqrt{2}-1\right)\left(\sqrt{2}-1\right)^n$, we have that

$$-x_{n+1} = x_n - 2y_n, \quad y_{n+1} = x_n - y_n.$$

The desired result holds for $n = 1$ and 2 since $\sqrt{2}-1 = \sqrt{2}-\sqrt{1}$ and $\left(\sqrt{2}-1\right)^2 = \sqrt{9}-\sqrt{8}$.

Suppose it holds for $n = 2m$, and indeed,

$$\left(\sqrt{2}-1\right)^{2m} = \sqrt{x_{2m}^2} - \sqrt{2y_{2m}^2}$$

where $x_{2m}^2 - 2y_{2m}^2 = 1$. Then

$$\left(\sqrt{2}-1\right)^{2m+1} = \sqrt{2y_{m+1}^2} - \sqrt{x_{2m+1}^2}$$

and

$$\begin{aligned} 2y_{2m+1}^2 - x_{2m+1}^2 &= 2\left(x_{2m} - y_{2m}\right)^2 - \left(x_{2m} - 2y_{2m}\right)^2 \\ &= x_{2m}^2 - 2y_{2m}^2 = 1, \\ \left(\sqrt{2}-1\right)^{2m+2} &= \sqrt{x_{2m+2}^2} - \sqrt{2y_{2m+2}^2} \end{aligned}$$

and

$$\begin{aligned} x_{2m+2}^2 - 2y_{2m+2}^2 &= (x_{2m+1} - 2y_{2m+1})^2 - 2(x_{2m+1} - y_{2m+1})^2 \\ &= 2y_{2m+1}^2 - x_{2m+1}^2 = 1. \end{aligned}$$

The result follows by induction.

2 *Solution 3* Let $u = \left[\frac{\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n}{2}\right]^2$.

Using the Binomial Expansion, we see that u is an integer. Now

$$u = \frac{\left(\sqrt{2}+1\right)^{2n} + 2 + \left(\sqrt{2}-1\right)^{2n}}{4},$$

so that

$$u - 1 = \left[\frac{\left(\sqrt{2}+1\right)^n - \left(\sqrt{2}-1\right)^n}{2}\right]^2,$$

is also an integer.

Thus, $\left(\sqrt{2}-1\right)^n = \sqrt{u} - \sqrt{u-1}$ is the desired representation.

2 *Solution 4* By induction, using

$$\left(a\sqrt{2}-b\right)\left(3-2\sqrt{2}\right)=(3a+2b)\sqrt{2}-(4a+3b),$$

it can be shown that

$$\left(\sqrt{2}-1\right)^n=a\sqrt{2}-b$$

where $2a^2-b^2=1$, when, n is odd.

Similarly, when n is even,

$$\left(\sqrt{2}-1\right)^n=c-d\sqrt{2}$$

where $c^2-2d^2=1$.

3 *Solution 1* If the rational has numerator 1, then the result is obvious (the right side has one term).

Suppose, as an induction hypothesis, we have established the result for fractions of numerator n for $n=1,2,\cdots,k-1$.

Let $0<\frac{k}{m}<1$. Then $m>k$, so we can write $m=kq+r$, where q, r are integers for which $q\geq 1$ and $0\leq r\leq k-1$.

If $r=0$, then $\frac{k}{m}=\frac{1}{q}$ and the required representation is obtained.

Let $r>0$. Then

$$\frac{1}{q}-\frac{k}{m}=\frac{1}{q}-\frac{k}{kq+r}=\frac{r}{q(kq+r)}$$

so that

$$\frac{k}{m}=\frac{1}{q}\left(1-\frac{r}{kq+r}\right).$$

By the induction hypothesis, we can write

$$\frac{r}{kq+r}=\frac{1}{q_2}-\frac{1}{q_2q_3}+\cdots$$

as a finite sum. Taking $q=q_1$, we now obtain the required representation for $\frac{k}{m}$.

3 *Solution 2* Let $0<\frac{a}{b}<1$. By repeated use of the division algorithm, we find that, for integers q_i, r_i with $b>a>r_1>r_2>\cdots\geq 0$.

$$b=aq_1+r_1=r_1q_2+r_2=r_2q_3+r_3=\cdots=r_{s+1}q.$$

The process must terminate. Hence

$$\begin{aligned} \frac{a}{b} &= \frac{1}{q_1} - \frac{r_1}{bq_1} \\ &= \frac{1}{q_1}\left(1 - \frac{r_1}{b}\right) \\ &= \frac{1}{q_1}\left(1 - \left[\frac{1}{q_2} - \frac{r_2}{bq_2}\right]\right) \\ &= \frac{1}{q_1}\left(1 - \left[\frac{1}{q_2}\left(1 - \frac{r_2}{b}\right)\right]\right) \\ &= \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \cdots . \end{aligned}$$

4 *Solution 1* Observe that $(1+x)^p = \sum_{i=0}^{p} \binom{p}{i} x^i$. When $1 \le i \le p-1$,

$$\binom{p}{i} = \frac{p(p-1)\cdots(p-i+1)}{1\cdot 2\cdots(i-1)i}$$

is a fraction (as well as being an integer), whose numerator is divisible by p, but whose denominator is not.

Hence the coefficients of x^i for $1 \le i \le p-1$ are all multiples of p.

Now

$$\begin{aligned} (1+x)^p &= (1+x)^{p-2}(1+x)^2 \\ &= \left(1 + a_1 x + \cdots + a_{p-2}x^{p-2}\right)\left(1 + 2x + x^2\right) \\ &= 1 + (a_1+2)x + (a_2 + 2a_1 + 1)x^2 \\ &\quad + \cdots + (a_{p-2} + 2a_{p-3} + a_{p-4})x^{p-2} \\ &\quad +(2a_{p-2} + a_{p-3})x^{p-1} + a_{p-2}x^p \end{aligned}$$

so that

$$a_1 + 2 \equiv 0 \pmod{p}.$$

Thus

$$a_1 \equiv -2 \pmod{p}.$$

Also

$$a_2 + 2a_1 + 1 \equiv 0 \pmod{p}$$

implies that

$$a_2 \equiv -2a_1 - 1 \equiv 4 - 1 \equiv 3 \pmod{p}.$$

Suppose, as an induction hypothesis, we have established that

$$a_1 \equiv (-1)^i (i+1) \quad \text{for some } i \le p-3.$$

Then

$$a_{i+1} + 2a_i + a_{i-1} \equiv 0 \pmod{p} \quad (a_0 \equiv 1)$$

implies that

$$\begin{aligned} a_{i+1} &\equiv -2a_i - a_{i-1} \equiv (-1)^{i+1}2(i+1) + (-1)^i i \\ &= (-1)^{i+1}(2i+2-i) = (-1)^{i+1}(i+2) \pmod{p}. \end{aligned}$$

Hence

$$a_1 + 2, a_2 - 3, a_3 + 4, \cdots, a_{p-2} + (p-1)$$

are all multiples of p.

4 *Solution 2* For $1 \le k \le p-2$,

$$a_k = \binom{p-2}{k} = \frac{(p-2)(p-3)\cdots(p-k-1)}{k!}$$

is an integer. Now

$$(p-2)(p-3)\cdots(p-k-1) = b_k p + (-1)^k (k+1)!$$

where

$$b_k = p^{k-1} - \left(2 + 3 + \cdots + \overline{k+1}\right) p^{k-2} + \cdots .$$

Since both $(p-2)(p-3)\cdots(p-k-1)$ and $(k+1)!$ are divisible by $k!$ and since $\gcd(p, k!) = 1$, we must have $b_k = c_k k!$ for some integer c_k. Hence

$$a_k = c_k p + (-1)^k (k+1).$$

This implies that

$$a_k + (-1)^{k+1}(k+1) = c_k p,$$

and the result holds.

4 *Solution 3* For $1 \le k \le p-2$,

$$\begin{aligned} & a_k + (-1)^{k+1}(k+1) \\ &= \binom{p-2}{k} + (-1)^{k+1}(k+1) \\ &= \frac{(p-2)\cdots\left(p-\overline{k+1}\right) + (-1)^{k+1}(k+1)!}{k!} \\ &= \frac{p^k + u_1 p^{k-1} + \cdots + u_{k-1}p + (-1)^k (k+1)! + (-1)^{k+1}(k+1)!}{k!}. \end{aligned}$$

This is an integer whose numerator is divisible by p and whose denominator is not. The result follows.

4 *Solution 4* When $1 \le r \le p-1$,

$$\binom{p}{r} = \frac{p(p-1)\cdots(p-r+1)}{r!} \equiv 0 \pmod{p}.$$

Now

$$\begin{aligned}\binom{p}{r} &= \binom{p-1}{r} + \binom{p-1}{r-1} \\ &= \binom{p-2}{r} + 2\binom{p-2}{r-1} + \binom{p-2}{r-2} \\ &= a_r + 2a_{r-1} + a_{r-2} \text{ for } 2 \le r \le p-2,\ a_0 = 1.\end{aligned}$$

Hence

$$a_r + 2a_{r-1} + a_{r-2} \equiv 0 \pmod{p}$$

We conclude as in Solution 1.

5 *Solution 1* A positive integer r gets counted in the function $\tau(k)$ if and only if k is a multiple of r.

Hence the number of times that r gets counted in the sum $\sum_{k=1}^{2n} \tau(k)$ is the number of multiples of r not exceeding $2n$, namely $\lfloor \frac{2n}{r} \rfloor$ times.

If $n+1 \le r \le 2n$, then r gets counted once.

If $r > 2n$, then r does not get counted at all.

Hence

$$\begin{aligned}\sum_{k=1}^{2n} \tau(k) &= \sum_{r=1}^{2n} \left\lfloor \frac{2n}{r} \right\rfloor \\ &= \sum_{r=1}^{n} \left\lfloor \frac{2n}{r} \right\rfloor + \sum_{r=n+1}^{2n} \left\lfloor \frac{2n}{r} \right\rfloor \\ &= \sum_{k=1}^{n} \left\lfloor \frac{2n}{k} \right\rfloor + n,\end{aligned}$$

as desired.

5 *Solution 2* (by induction) When $n = 1$, we have

$$\tau(1) + \tau(2) - \left\lfloor \frac{2}{1} \right\rfloor = 1 + 2 - 2 = 1.$$

Suppose the result holds for $n = r$. Then

$$\begin{aligned}
&\sum_{k=1}^{2(r+1)} \tau(k) - \sum_{k=1}^{r+1} \left\lfloor \frac{2(r+1)}{k} \right\rfloor \\
&= \tau(2r+1) + \tau(2r+2) + \sum_{k=1}^{2r} \tau(k) - \sum_{k=1}^{r+1} \left\lfloor \frac{2(r+1)}{k} \right\rfloor \\
&= \tau(2r+1) + \tau(2r+2) + r \\
&\quad - \sum_{k=1}^{r} \left(\left\lfloor \frac{2(r+1)}{k} \right\rfloor - \left\lfloor \frac{2r}{k} \right\rfloor \right) - \left\lfloor \frac{2(r+1)}{r+1} \right\rfloor .
\end{aligned}$$

Now,

$$\left\lfloor \frac{2r+2}{k} \right\rfloor = \left\lfloor \frac{2r}{k} \right\rfloor + 1$$

is equivalent to either $2r+1$ or $2r+2$ being a multiple of k and $k \neq 1$.

Also,

$$\left\lfloor \frac{2r+2}{k} \right\rfloor = \left\lfloor \frac{2r}{k} \right\rfloor + 2$$

is equivalent to $k = 1$.

Otherwise, $\left\lfloor \frac{2r+2}{k} \right\rfloor = \left\lfloor \frac{2r}{k} \right\rfloor$.

$2r+1$ is a multiple of k for $\tau(2r+1) - 2$ values of k not exceeding r and exceeding 1.

$2r+2$ is a multiple of k for $\tau(2r+2) - 3$ values of k not exceeding r and exceeding 1.

Hence,

$$\begin{aligned}
&\sum_{k=1}^{r} \left\lfloor \frac{2(r+1)}{k} \right\rfloor - \left\lfloor \frac{2r}{k} \right\rfloor \\
&= 2 + \sum_{k=2}^{r} \left\lfloor \frac{2(r+1)}{k} \right\rfloor - \left\lfloor \frac{2r}{k} \right\rfloor \\
&= 2 + [\tau(2r+1) - 2] + [\tau(2r+2) - 3] \\
&= \tau(2r+1) + \tau(2r+2) - 3.
\end{aligned}$$

Since $\left\lfloor \frac{2(r+1)}{r+1} \right\rfloor = 2$, it follows that

$$\sum_{k=1}^{2(r+1)} \tau(k) - \sum_{k=1}^{r+1} \left\lfloor \frac{2(r+1)}{k} \right\rfloor = r + 3 - 2 = r + 1$$

as desired.

6 *Solution 1* Let $1995 = k$. By trial and error on early terms in the sequence, we conjecture the following lemma.

LEMMA. For each positive integer m,

$$\begin{aligned} u_{2m} &= u_m(u_{m+1} - u_{m-1}), \\ u_{2m+1} &= u_{m+1}^2 - u_m^2 = (u_{m+1} + u_m)(u_{m+1} - u_m). \end{aligned}$$

PROOF (by induction).
When $m = 1, u_2 = k = u_1(u_2 - u_0)$ and $u_3 = k^2 - 1 = u_2^2 - u_1^2$.

Suppose that the result holds for $m = p \geq 1$. Then,

$$\begin{aligned} u_{2(p+1)} &= ku_{2p+1} - u_{2p} \\ &= k\left(u_{p+1}^2 - u_p^2\right) - (u_p u_{p+1} - u_p u_{p-1}) \\ &= u_{p+1}(ku_{p+1} - u_p) - u_p(ku_p - u_{p-1}) \\ &= u_{p+1}u_{p+2} - u_p u_{p+1} \\ &= u_{p+1}(u_{p+2} - u_p), \\ u_{2(p+1)+1} &= u_{2p+3} \ = \ ku_{2p+2} - u_{2p+1} \\ &= ku_{p+1}(u_{p+2} - u_p) - \left(u_{p+1}^2 - u_p^2\right) \\ &= (u_{p+2} + u_p)(u_{p+2} - u_p) - \left(u_{p+1}^2 - u_p^2\right) \\ &= u_{p+2}^2 - u_{p+1}^2, \end{aligned}$$

so that the result holds for $m = p + 1$.

Now $u_{n+1} - u_n = (k-1)u_n - u_{n-1} = (u_n - u_{n-1}) + (k-2)u_n$, so, by induction, it follows that $u_{n+1} - u_n > 1$ for $n \geq 2$, and $u_{n+1} - u_{n-1} > 1$ for $n \geq 1$.

Hence u_{2m} is composite for $m \geq 2$ and u_{2m+1} is composite for $m \geq 1$. This leaves only u_2 to consider.

But $u_2 = 1995 = 5 \times 399 = 5 \times 3 \times 7 \times 19$. So, for $n \geq 2$, u_n is not prime.

6 *Solution 2* For $1 \leq r \leq n$,

$$u_n = u_r u_{n+1-r} - u_{r-1}u_{n-r}.$$

The proof is by induction. The equation is trivial for $r = 1$.

Assuming its truth for $r \leq n - 1$, we have

$$\begin{aligned} & u_{r+1}u_{n-r} - u_r u_{n-r-1} \\ &= u_{n-r}(ku_r - u_{r-1}) - u_r u_{n-r-1} \\ &= u_r(ku_{n-r} - u_{n-r-1}) - u_{r-1}u_{n-r} \\ &= u_r u_{n+1-r} - u_{r-1}u_{n-r} \end{aligned}$$

as desired.

If $n = 2m$, $r = m$, we have $u_{2m} = u_m(u_{m+1} - u_{m-1})$; if $n = 2m+1$, $r = m+1$, we have $u_{2m+1} = u_{m+1}^2 - u_m^2$, and we can complete the solution as above.

7 *Solution 1* If $a_i = b_j = 0$, then $c_i = c_j = 0$. If $(c_1, c_2, \cdots, c_p)$ is a rearrangement of $(0, 1, \cdots, p-1)$, then no entry can appear more than once. Hence $i = j$.

WOLOG, we may suppose that $a_p = b_p = 0$ so that $c_p = 0$.

For any counterexample, we must have

$$\begin{aligned}(p-1)! &\equiv a_1a_2\cdots a_{p-1} \equiv b_1b_2\cdots b_{p-1}\\ &\equiv c_1c_2\cdots c_{p-1}\\ &\equiv (a_1\cdots a_{p-1})(b_1\cdots b_{p-1})\\ &\equiv ((p-1)!)^2\end{aligned}$$

which implies that

$$(p-1)! \equiv 1 \ (\mathrm{mod}\ p).$$

But Wilson's Theorem states that $(p-1)! \equiv -1 \ (\mathrm{mod}\ p)$, so we have a contradiction.

[To prove Wilson's Theorem, note that if $x \not\equiv 1, -1, 0 \ (\mathrm{mod}\ p)$, there exists y such that $xy \equiv 1 (\mathrm{mod}\ p)$. Now pair off elements in the product $1.2.\cdots, p-1$.

For example, when $p = 11$, we have

$$\begin{aligned}&1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\\ &= 1\cdot(2\times 6)\cdot(3\times 4)\cdot(5\times 9)\cdot(7\times 8)\cdot 10\\ &= 1\times 1\times 1\times 1\times 1\times(-1) = -1 \ (\mathrm{mod}\ 11).]\end{aligned}$$

Comment The result is false when $p = 2$. We need the hypothesis that p is odd.

8 *Solution 1*
LEMMA. Let r, s be positive integers. Then

(a) $2^s r^r < (r+s)^s (r+s)^r = (r+s)^{r+s}$.

(b) $1 + 2^2 + 3^3 + \cdots + r^r < (r+1)^{r+1} - 2$.

[*PROOF* by induction. (b) holds for $r = 1$.
Assume it holds for $r = m - 1 \geq 1$. Then

$$\begin{aligned}&1 + 2^2 + \cdots + (m-1)^{m-1} + m^m\\ &< (m^m - 2) + m^m = 2m^m - 2 < (m+1)^{m+1} - 2\end{aligned}$$

by (a).]

(c) $\dfrac{1}{r^r} > \dfrac{1}{(r+1)^{r+1}} + \dfrac{1}{(r+2)^{r+2}} + \cdots + \dfrac{1}{(r+s)^{r+s}}$.
[*PROOF*. By (a),

$$\begin{aligned} & \frac{1}{(r+1)^{r+1}} + \cdots + \frac{1}{(r+s)^{r+s}} \\ < \quad & \frac{1}{r^r}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^s}\right) \\ < \quad & \frac{1}{r^r}.] \end{aligned}$$

(d) If m, n are integers with $m < n$, then $|m^m| \leq |n^n|$ with equality only if $-m = n = 1$.

Let $\{u_1, u_2, \cdots, u_{k+1}\}$ be any collection of distinct nonzero integers with $u_1 < u_2 < \cdots < u_{k+1} = w$.
Case (*i*): $w \geq 1$. Choose $v < 0, v < u_1$. By (b) and (c),

$$w^w - 2 > (w-1)^{w-1} + (w-2)^{w-2} + \cdots + 1,$$

and

$$2 > 1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{|v|^{|v|}}$$

so

$$w^w > \sum_{\substack{i=v \\ i \neq 0}}^{w-1} \left|i^i\right| \geq \sum_{i=1}^{k} |u_i^{u_i}| \geq \sum_{i=1}^{k} \epsilon_i u_i^{u_i}.$$

This implies that

$$u_{k+1}^{u_{k+1}} > \sum_{i=1}^{k} \epsilon_i u_i^{u_i}$$

where $\epsilon_i = \pm 1$.
It follows from this that there are no integral solutions to any equation of the type $\sum_{i=1}^{k+1} \epsilon_i u_i^{u_i} = 0$.
Thus, if $\{x_1, \cdots, x_{k+1}\}$ is a rearrangement of $\{u_1, \cdots, u_{k+1}\}$, then $x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$ can never hold.
Case (*ii*): $w \leq -1$. Let $v = u_1 < 0$. Then, by (c),

$$|w^w| = \left|u_{k+1}^{u_{k+1}}\right| > \sum_{i=1}^{k} |u_i^{u_i}|,$$

so no equation of the type $\sum_{i=1}^{k+1} \epsilon_i u_i^{u_i} = 0$ has integral solutions and we can complete the argument as in Case (i).

Comment If each $x_i > 0$, it is clear for the equation that x_{k+1} is the largest of the x_i. If some or all x_i are negative, this is no longer clear and the above argument is designed to deal with this.

Comment Note

$$\begin{aligned} n^n - (n-1)^{n-1} &> n(n-1)^{n-1} - (n-1)^{n-1} \\ &> (n-1)^{n-1}, \end{aligned}$$

so we can, by summing over n, obtain

$$m^m > (m-1)^{m-1} + (m-2)^{m-2} + \cdots + 1$$

for each $m \in \mathbb{N}$.

9 *Solution 1*
LEMMA. For each nonnegative integer n,

$$2\sqrt{n+1} > \sqrt{n} + \sqrt{n+2}.$$

PROOF.

$$\begin{aligned} \sqrt{n+1} - \sqrt{n} &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &> \frac{1}{\sqrt{n+2} + \sqrt{n+1}} \\ &= \sqrt{n+2} - \sqrt{n+1}, \end{aligned}$$

from which the result follows.

We now prove the required result. It holds by inspection for $n = 0$ and $n = 1$. Let $n \geq 2$. It will be shown that
$(9n+8)^{\frac{1}{2}} < n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n+2)^{\frac{1}{2}} < (9n+9)^{\frac{1}{2}}$. $(*)$
For the right inequality, the lemma gives us

$$n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n+2)^{\frac{1}{2}} < 3(n+1)^{\frac{1}{2}} = (9n+9)^{\frac{1}{2}}.$$

For the left inequality, note first that

$$n(n+2) - \left(n + \frac{7}{9}\right)^2 = \frac{4}{9}n - \frac{49}{81} \geq \frac{8}{9} - \frac{49}{81} > 0$$

which implies that

$$\sqrt{n(n+2)} > n + \frac{7}{9}.$$

Hence, using the lemma again,

$$\begin{aligned}\left[n^{\frac{1}{2}}+(n+1)^{\frac{1}{2}}+(n+2)^{\frac{1}{2}}\right]^2 &> \left[\frac{3}{2}\left(n^{\frac{1}{2}}+(n+2)^{\frac{1}{2}}\right)\right]^2\\ &= \frac{9}{4}\left[n+(n+2)+2\sqrt{n(n+2)}\right]\\ &> \frac{9}{4}\left[2n+2+2\left(n+\frac{7}{9}\right)\right]\\ &= \frac{9}{4}\left[4n+\frac{32}{9}\right]\\ &= 9n+8.\end{aligned}$$

Hence, $(*)$ is shown. The integers $9n+8$ and $9n+9$ are consecutive, so no perfect square lies between them.

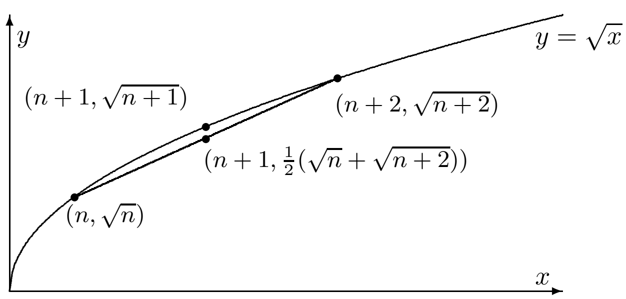
Hence there is no integer between $(9n+8)^{\frac{1}{2}}$ and $(9n+9)^{\frac{1}{2}}$ and the result follows from $(*)$, since $(9n+9)^{\frac{1}{2}} < (9n+8)^{\frac{1}{2}}+1$.

Comments

(1) Arriving at the proof involved a lot of working backwards. In particular, we had the question – what do we need to know about n to conclude

$$\frac{3}{2}\left[n^{\frac{1}{2}}+(n+2)^{\frac{1}{2}}\right] > (9n+8)^{\frac{1}{2}}?$$

(2) The lemma is actually a consequence of the concavity of $\sqrt{x}$.



9 *Solution 2* [D. Khosla] Observe that

$$2(\sqrt{n(n+1)}+\sqrt{n(n+2)}+\sqrt{(n+1)(n+2)}\,) < 6n+6$$

by the AM–GM Inequality. Since, for $n \geq 2$,

$$\begin{aligned} n(n+1) &> \left(n+\frac{3}{8}\right)^2,\\ (n+1)(n+2) &> \left(n+\frac{11}{8}\right)^2,\\ (n+2)n &> \left(n+\frac{3}{4}\right)^2,\end{aligned}$$

it follows that

$$\begin{aligned}
&2\left(\sqrt{n(n+1)}+\sqrt{n(n+2)}+\sqrt{(n+1)(n+2)}\right)\\
&\quad > \quad 2\left(n+\frac{3}{8}+n+\frac{11}{8}+n+\frac{3}{4}\right)\\
&\quad = \quad 6n+5.
\end{aligned}$$

Hence

$$\begin{aligned}
&9n+8\\
&\quad = \quad (3n+3)+(6n+5)\\
&\quad < \quad 3n+3+2\left(\sqrt{n(n+1)}+\sqrt{n(n+2)}+\sqrt{(n+1)(n+2)}\right)\\
&\quad < \quad (3n+3)+(6n+6) \;=\; 9n+9.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right)^2\\
&\quad = \quad 3n+3+2\left(\sqrt{n(n+1)}+\sqrt{n(n+2)}+\sqrt{(n+1)(n+2)}\right),
\end{aligned}$$

the result follows.

10 *Solution 1* The sequence is

$$\{1,2,7,29,22,23,49,26,-17,-163,-146,\cdots,\}$$

with a cyclic pattern of parity of period 3. Let

$$u_i = a_{3i-2}, \quad v_i = a_{3i-1}, \quad w_i = a_{3i} \quad (i = 1, 2, \cdots)$$

so that u_i and w_i are odd and v_i is even. We have that

$$w_i = 5v_i - 3u_i, \quad v_i = u_i - w_{i-1} \quad u_i = 5w_{i-1} - 3v_{i-1}.$$

We derive recursions for these equations.

$$w_i = 5v_i - 3u_i = u_{i+1} - v_{i+1} = \frac{1}{5}u_{i+1} + \frac{3}{5}v_i$$

which implies that

$$u_{i+1} - v_{i+1} = \frac{1}{5}u_{i+1} + \frac{3}{5}v_i,$$

and further, that

$$4u_{i+1} = 5v_{i+1} + 3v_i,$$

and

$$5v_i - 3u_i = \frac{1}{5}u_{i+1} + \frac{3}{5}v_i$$

implies that

$$22v_i = 15u_i + u_{i+1}.$$

Hence

$$4u_{i+1} = \frac{75}{22}u_{i+1} + \frac{5}{22}u_{i+2} + \frac{45}{22}u_i + \frac{3}{22}u_{i+1}$$

which implies that

$$10u_{i+1} = 5u_{i+2} + 45u_i$$

and further that

$$u_{i+2} = 2u_{i+1} - 9u_i \quad (i = 1, 2, \cdots).$$

Similarly, we have

$$\left.\begin{array}{r} v_{i+2} = 2v_{i+1} - 9v_i \\ w_{i+2} = 2w_{i+1} - 9w_i \end{array}\right\} \quad (i = 1, 2, \cdots)$$

where $u_1 = 1$, $u_2 = 29$, $v_1 = 2$, $v_2 = 22$, $w_1 = 7$, $w_2 = 23$.

(a) Consider the sequence $\{u_n\}$. If u_{i-1} and u_i have the same sign, then $u_{i+2} = 2u_{i+1} - 9u_i = 2(2u_i - 9u_{i-1}) - 9u_i = -(5u_i + 18u_{i-1})$ has the opposite sign. It follows that for each u_n, either u_{n+1} or u_{n+3} has the opposite sign to u_n. Since $\{u_n\} \subseteq \{a_n\}$, $\{a_n\}$ has infinitely many terms of each sign.

(b) If $a_n = 0$, then $n = 3m - 1$ for some m, and so $v_m = 0$. For each i, we have $v_{i+1} = 2v_i - 9v_{i-1} \equiv 2v_i (\text{mod } 9)$. Hence

$$0 = v_m \equiv 2v_{m-1} \equiv 2^2 v_{m-2} \equiv \cdots \equiv 2^{m-1}v_1 = 2^m (\text{mod } 9),$$

a contradiction.

(c) It can be shown by induction (exercise!) that

$$u_{4j+3} \equiv a_{12i+7} \equiv 0 (\text{mod } 7)$$

and

$$v_{4j+1} \equiv a_{12i+3} \equiv 0 (\text{mod } 7) \quad i = 0, 1, 2, \cdots$$

Now $\quad 2^k - 1 \equiv \begin{cases} 3(\text{mod } 12) & \text{for } k \text{ even}, \quad k \geq 2, \\ 7(\text{mod } 12) & \text{for } k \text{ odd}, \quad k \geq 3, \end{cases}$

so $\quad a_{2^k-1} \equiv 0(\text{mod } 7)$ for $k = 2, 3, 4, \cdots$.

10 *Solution 2*

(b) We show that

if n is odd, then $a_n \equiv 1(\text{mod } 3)$,

if n is even, then $a_n \equiv -1(\text{mod } 3)$.

This is true for $n = 1$, 2. Assume it holds up to $n = k$. Then, if $a_k \cdot a_{k-1}$ is even, $a_{k+1} \equiv 5a_k \equiv -a_k (\text{mod } 3)$, while if $a_k \cdot a_{k-1}$ is odd $a_{k+1} \equiv a_k - a_{k-1} \equiv a_k + a_k \equiv 2a_k \equiv -a_k (\text{mod } 3)$.

In either case, a_{k+1} has the right remainder upon division by 3. The desired result follows by induction.

It follows that $a_n \not\equiv 0 \pmod 3$ for each n so a_n never vanishes.

(a) By a similar argument to Solution 1 of this problem, we find that, for $n \geq 1$

$$a_{3n+11} = -20a_{3n+3} - 3a_{3n+2},$$

from which the desired result can be found.

(c) [W.L. Yee] Observe that

$$a_{13} \equiv 3 \equiv 3a_1 \pmod 7$$

and

$$a_{14} \equiv 6 \equiv 3a_2 \pmod 7.$$

It follows by induction that $a_{12+r} \equiv 3a_r \pmod 7$ for $r = 1, 2, 3, \cdots$ and so

$$a_{12m+r} \equiv 3^m a_r \pmod 7.$$

Since $a_3 \equiv 0 \pmod 7), a_{12m+3} \equiv 0 \pmod 7$.

Since $a_7 = 49 \equiv 0 \pmod 7), a_{12m+7} \equiv 0 \pmod 7$ for $m = 1, 2, \cdots$.

Hence $a_n \equiv 0 \pmod 7$ whenever $n \equiv 3, 7 \pmod{12}$.

By induction, it can be shown that

$$2^k \equiv 4 \pmod{12} \text{ when } k \geq 2 \text{ is even .}$$

$$2^k \equiv 8 \pmod{12} \text{ when } k \geq 3 \text{ is odd .}$$

The result follows.

Comments: (b) can be shown from the fact that, modulo 6, the sequence is

$$1, 2, 1, 5, 4, 5, 1, 2, 1, 5, 4, 5, \cdots ,$$

and, modulo 4, it is $1, 2, 3, 1, 2, 3, \cdots$.

11 *Solution 1* The conclusion is true. It suffices to show that

$$f(n+1) \geq f(n) \text{ for } n = 1, 2, \cdots .$$

If $n \not\equiv 9 \pmod{10}$, then

$$s(n+1) = s(n) + 1,$$

so that

$$(n+1) - s(n+1) = n - s(n)$$

and

$$f(n+1) = f(n - s(n)) + 1 = f(n).$$

Suppose that $n+1 \equiv 0 (\text{mod } 10^k)$, and that $n+1 \not\equiv 0 (\text{mod } 10^{k+1})$.
Then $s(n+1) + 9k - 1 = s(n)$. This implies that

$$\begin{aligned}(n+1) - s(n+1) &= (n+1) - s(n) + 9k - 1 \\ &= (n - s(n)) + 9k \\ &> n - s(n).\end{aligned}$$

Suppose, as an induction hypothesis, we have established

$$f(m) \leq f(n) \text{ whenever } 1 \leq m \leq n \leq r.$$

If $r \not\equiv 9 (\text{mod } 10)$, then $f(r+1) = f(r)$.
If $r \equiv 9 (\text{mod } 10)$, then $(r+1) - s(r+1) > r - s(r)$.
Therefore

$$f((r+1) - s(r+1)) \geq f(r - s(r)),$$

so that

$$f(r+1) \geq f(r).$$

We now obtain the desired result.

12 *Solution 1*
PROPOSITION 1. Let $\phi(n)\sigma(n) = (n-3)(n+1)$, and let the prime p divide n, so that $n = p^a m$ for some m not divisible by p. Then either $a = 1$ or ($p = 3$ and $a = 2$).

PROOF.

$$\begin{aligned}\phi(n) &= \phi(p^a)\phi(m) = p^{a-1}(p-1)\phi(m), \\ \sigma(n) &= \sigma(p^a)\sigma(m) = \left(\frac{p^{a+1}-1}{p-1}\right)\sigma(m),\end{aligned}$$

so

$$\phi(n)\sigma(n) = p^{a-1}(p^{a+1}-1)\phi(m)\sigma(m).$$

Suppose $a \geq 2$. Then

$$0 \equiv \phi(n)\sigma(n) = (n-3)(n+1) \equiv -3 (\text{mod } p^{a-1}).$$

Corollary $n = p_1 p_2 \cdots p_k$ for some primes p_i or $n = 9 q_1 q_2 \cdots q_k$ for distinct primes q_i not divisible by 3.

PROPOSITION 2. If $q = 6, 9$ or prime, then $\phi(q)\sigma(q) > (q-3)(q+1)$.

PROOF.

$$\phi(6)\sigma(6) - (6-3)(6+1) = 2 \times 12 - 3 \times 7 = 3 > 0$$

$$\phi(9)\sigma(9) - (9-3)(9+1) = 6 \times 13 - 6 \times 10 = 18 > 0$$

$$\phi(p)\sigma(p) - (p-3)(p+1) = (p^2-1) - (p^2-2p-3) = 2(p+1) > 0$$

for each prime p.

PROPOSITION 3. Let $n \neq 6, n = pq$ be the product of two primes. Then $(n-3)(n+1) \geq \phi(n)\sigma(n)$ with equality if and only if $|p-q| = 2$.

PROOF.

$$\phi(n) = (p-1)(q-1) = (n+1) - (p+q)$$

and

$$\sigma(n) = (p+1)(q+1) = (n+1) + (p+q).$$

Hence

$$\begin{aligned}
&(n-3)(n+1) - \phi(n)\sigma(n) \\
&= (n^2-2n-3) - (n+1)^2 + (p+q)^2 \\
&= n^2 - 2pq - 3 - n^2 - 2n - 1 + p^2 + 2n + q^2 \\
&= -2pq - 3 - 1 + p^2 + q^2 \ = \ (p^2 - 2pq + q^2) - 4 \\
&= (p-q)^2 - 4 \\
&\geq 0
\end{aligned}$$

and the result follows.

PROPOSITION 4. Suppose p is a prime not dividing m and

$$\phi(m)\sigma(m) \leq (m-3)(m+1).$$

Then

$$\phi(pm)\sigma(pm) < (pm-3)(pm+1).$$

PROOF.

$$\begin{aligned}
&(pm-3)(pm+1) - \phi(pm)\sigma(pm) \\
&= (pm-3)(pm+1) - \phi(p)\sigma(p)\phi(m)\sigma(m) \\
&\geq (p^2m^2 - 2pm - 3) - (p-1)(p+1)(m-3)(m+1) \\
&= (p^2m^2 - 2pm - 3) - (p^2m^2 - 2p^2m - 3p^2 - m^2 + 2m + 3) \\
&= 2m(p^2 - p - 1) + 3p^2 + m^2 - 6 \\
&> 0.
\end{aligned}$$

Corollary If $n = p_1 \cdots p_k$ is the product of $k \geq 3$ distinct primes, then $\phi(n)\sigma(n) < (n-3)(n+1)$.

PROOF (by induction). If $n = p_1p_2p_3$, we can write $n = pm$ where $m \neq 6$ is a product of 2 primes, p is prime and use Propositions 3 and 4. For $k \geq 3$, we can use Proposition 4 on the induction hypothesis.

PROPOSITION 5. Let m be any number not divisible by 3 and suppose that $\phi(m)\sigma(m) \leq (m-3)(m+1)$.

Then $\phi(9m)\sigma(9m) < (9m-3)(9m+1)$.

[Note this implies $m > 3$.]

PROOF.

$$\begin{aligned}
&(9m-3)(9m+1) - \phi(9m)\sigma(9m)\\
&\quad = \quad (9m-3)(9m+1) - \phi(9)\sigma(9)\phi(m)\sigma(m)\\
&\quad \geq \quad (81m^2 - 18m - 3) - (6 \times 13)(m-3)(m+1)\\
&\quad = \quad (81m^2 - 18m - 3) - (78m^2 - 156m - 234)\\
&\quad = \quad 3m^2 + 138m + 231\\
&\quad > \quad 0.
\end{aligned}$$

Corollary If $n = 9q_1q_2\cdots q_k$, then $\phi(n)\sigma(n) < (n-3)(n+1)$.

PROOF. If $k \geq 2$ and q_1 is odd, this follows from Propositions 4 and 5. If $n = 9p$, then

$$\begin{aligned}
(n-3)(n+1) - \phi(n)\sigma(n) &= (9p-3)(9p+1) - 78(p^2-1)\\
&= 3p(p-6) + 75\\
&> 0
\end{aligned}$$

for p prime. If $n = 18p$, then

$$\begin{aligned}
(n-3)(n+1) - \phi(n)\sigma(n) &= (18p-3)(18p+1) - 234(p^2-1)\\
&= 18p(5p-2) + 231\\
&> 0
\end{aligned}$$

for p prime.

From these facts, the desired results follows from Proposition 4.

12 *Solution 2*

PROPOSITION 1. Let $\phi(n)\sigma(n) = (n-3)(n+1)$, and let $p|n$. Indeed, let $n = p^a m$ where $p \not| m$.

Then $a = 1$, and $n = p_1p_2\cdots p_k$ for distinct primes.

PROOF. As before, we have $\phi(n)\sigma(n) = p^{a-1}(p^{a+1}-1)\phi(m)\sigma(m)$ and $n = \prod_1^k p_i$ or $n = 9\prod_1^k q_i$.

If $n = 9$, then $\phi(9)\sigma(9) = 78$, $(9-3)(9+1) = 60$ and the result holds.

If $n = 9q_1q_2\cdots q_k$ $(k \geq 1)$, then

$$\phi(n)\sigma(n) = 78(q_1^2-1)(q_2^2-1)\cdots(q_k^2-1) \equiv 0 (\text{mod } 9)$$

since $q_i^2 \equiv 1 (\text{mod } 3)$ for every prime but 3, while

$$(n-3)(n+1) \equiv -3 (\text{mod } 9).$$

Hence $n = p_1 p_2 \cdots p_k$.

PROPOSITION 2.

$$\begin{aligned}&(u_1-1)(u_2-1)\cdots(u_k-1)\\ =\ & u_1u_2\cdots u_k - \sum_{i=1}^{k}(u_1-1)\cdots(u_{i-1}-1)u_{i+1}\cdots u_k \text{ for } k \geq 1.\end{aligned}$$

PROOF.

$$\begin{aligned}(u_1-1)(u_2-1) &= u_1u_2-(u_1-1)-u_2,\\ (u_1-1)(u_2-1)(u_3-1) &= u_1u_2u_3-(u_1-1)u_3\\ &\quad -u_2u_3-(u_1-1)(u_2-1),\end{aligned}$$

$$\begin{aligned}&(u_1-1)(u_2-1)(u_3-1)(u_4-1)\\ =\ & u_1u_2u_3u_4-(u_1-1)u_3u_4\\ & -u_2u_3u_4-(u_1-1)(u_2-1)u_4\\ & -(u_1-1)(u_2-1)(u_3-1).\end{aligned}$$

The result can be obtained by an induction argument.

PROPOSITION 3. Let $n = p_1p_2\cdots p_k$ where $k \geq 3$, $p_1 < p_2 < \cdots < p_k$. Then $\phi(n)\sigma(n) < (n-3)(n+1)$.

PROOF. Note that $2p_1 < p_2p_3\cdots p_k$.

$$\begin{aligned}&\phi(n)\sigma(n)\\ =\ & \prod_{i=1}^{k}(p_i^2-1)\\ =\ & p_1^2p_2^2\cdots p_k^2\\ & -\sum_{i=1}^{k}(p_1^2-1)\cdots(p_{i-1}^2-1)p_{i+1}^2\cdots p_k^2\\ <\ & n^2-(p_1^2-1)\cdots(p_{k-1}^2-1)-(p_2^2\cdots p_k^2)\\ <\ & n^2-3-2p_1p_2\cdots p_k = n^2-2n-3.\end{aligned}$$

The cases $n = p$ and $n = p_1p_2$ can be handled as in 12 solution 1.

Comment Alternatively, for Proposition 3, we have

$$\begin{aligned}\phi(n)\sigma(n) &= (p_1^2-1)\cdots(p_k^2-1) < (p_1^2-1)p_2^2\cdots p_k^2\\ &= n^2-n\left(\frac{p_2\cdots p_k}{p_1}\right)\\ &< n^2-p_3n \ \leq\ n^2-3n\\ &\leq n^2-2n-3.\end{aligned}$$

PROBLEM SET 2 — Geometry

Definition:

(1) Given a plane figure $ABC \dots XYZ$, the area of the figure is denoted by $[ABC \dots XYZ]$.

Problems

13 Let A, B, C be three distinct points of the plane for which $AB = AC$. Describe the locus of the points P for which $\angle APB = \angle APC$.

14 Let P be a point inside the triangle ABC such that $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ and $\angle ABC = 40°$. Determine $\angle BPC$.

15 The altitude from A of the triangle ABC intersects the side BC in D. A circle touches BC in D, intersects AB in M and N and intersects AC in P and Q. Prove that

$$\frac{AM + AN}{AC} = \frac{AP + AQ}{AB}.$$

16 Let M and O be the orthocentre and the circumcentre, respectively, of triangle ABC. Let N be the mirror reflection of M through O. Prove that the sum of the squares of the sides of the triangles NAB, NBC and NCA are equal.

17 Let $ABCD$ be a parallelogram with $AC/BD = k$. The bisectors of the angles formed by AC and BD intersect the perimeter of $ABCD$ in K, L, M and N. Prove that the ratio of the area of $KLMN$ to that of $ABCD$ is a function of k alone.

18 Let X be a point on the side of BC of triangle ABC and Y be the point where line AX meets the circumcircle of triangle ABC. Prove or disprove: if the length of XY is maximum, then AX lies between the median from A and the bisector of $\angle BAC$.

19 Given three disjoint circles in the plane, construct a point on the plane such that all three circles subtend the same angle at the point.

20 Construct an isosceles triangle, given its circumcircle and orthocentre.

21 Construct a triangle ABC, given the magnitude of the angle A and the lengths of the medians from the vertices B and C.

22 Given n points in the plane, not all on a line, such that the areas of the triangles defined by any three of them are less than 1, prove that the points can be covered by a triangle of area 4.

23 The convex quadrilateral $ABCD$ of area $2t$ has no parallel sides. Locate the point P_1 on CD such that P_1 is on the same side of AB as C and the area of $\triangle ABP_1$ is t. If P_2, P_3 and P_4 are similarly defined for the sides BC, CD, and DA, respectively, prove that P_1, P_2, P_3 and P_4 are collinear.

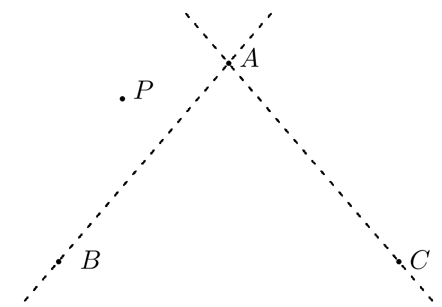
24 Let $ABCD$ be a tetrahedron for which the sides AB, BC and CA have length a, while the sides AD, BD and CD have length b. Suppose that M and N are the midpoints of the sides AB and CD respectively. A plane passing through M and N intersects segments AD and BC in points P and Q.

(a) Prove that $AP : AD = BQ : BC$.

(b) Find the ratio of AP to AD in quadrilateral $MQNP$ is minimum.

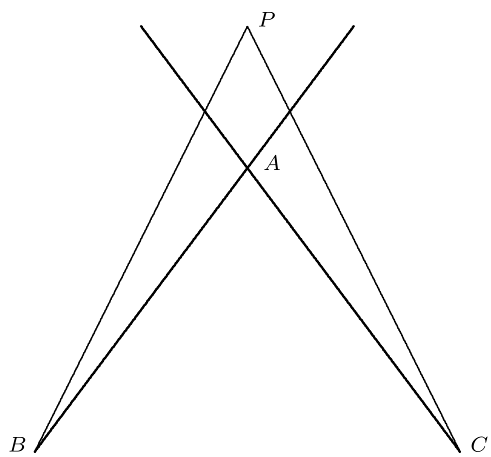
Solutions

13 *Solution 1* Suppose P lies on the opposite side of AB to C and on the same side of AC as B. Then $\angle APB = \angle APC$ if and only if P, B, C are collinear, since in this case, one of the angles is contained in the other.



Similarly, if P is on the opposite side of AC to B and on the same side of AB as C, then $\angle APB = \angle APC$ if and only if B, C, P are collinear.

Suppose that P lies in the angle opposite to $\angle BAC$.



Then $\sin\angle ABP = \dfrac{AP}{AB}\sin\angle APB$ and $\sin\angle ACP = \dfrac{AP}{AC}\sin\angle APC$.

So $\angle APB = \angle APC$ if and only if $\sin\angle ABP = \sin\angle ACP$.

Now, $\angle APB + \angle ACP < \angle CBP + \angle BCP < 180°$,

so $\angle APB = \angle APC$ is equivalent to $\angle ABP = \angle ACP$, which is equivalent to P lying on the right bisector of BC.

Suppose P lies within the angle BAC. Then, as above,

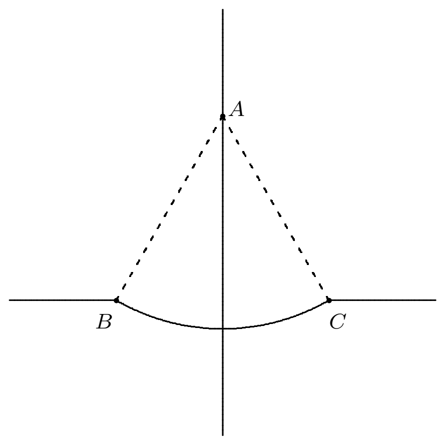
$\angle APB = \angle APC$ is the same as $\sin\angle ABP = \sin\angle ACP$, which is the same as $\angle ABP = \angle ACP$ or $\angle ABP + \angle ACP = 180°$.

If $\angle ABP = \angle ACP$, $\angle APB = \angle APC$, then $\triangle APB \equiv \triangle APC$ (ASA) and $\angle BAP = \angle CAP$, so P lies on the bisector of $\angle BAC$.

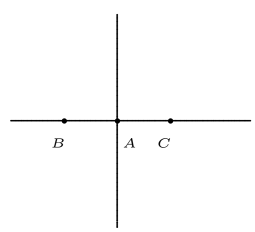
If $\angle ABP + \angle ACP = 180°$, then $ABPC$ is concyclic and P lies on the circumcircle of $\triangle ABC$.

Hence the required locus consists of the union of:

(i) the right bisector of BC,
(ii) the arc BC, not containing A, of the circumcircle of ABC,
(iii) the line BC produced with the interval BC deleted.



A special case is when A is the midpoint of BC. A similar analysis establishes that the locus consists of the following:

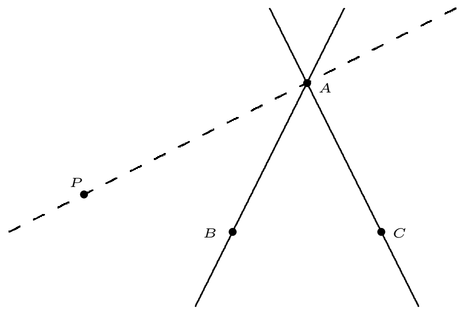


(1) the right bisector of BC along with
(2) that part of line BC lying outside the interior of the segment BC.

Note that the locus does **not** include the interior of the segment BC (if P is between B and A, for example, $\angle APB = 180°, \angle CPB = 0°$). The arc of the circle on the locus when A is off BC tends to the line BC exterior to segment BC as A approaches BC.

13 *Solution 2* The case when A is the midpoint of BC is easy to analyze. Suppose A, B, C are not collinear. We consider two cases for the location of P.

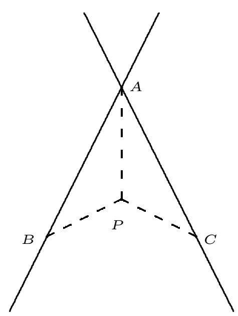
(a) B and C are on the same side of AP produced. In this case, one of the angles APB and APC is contained in the other and they are equal if and only if they coincide (that is, P is collinear with B and C).



(a)

(b) B and C are on opposite sides of AP produced. Consider triangles $\triangle APB$ and $\triangle APC$. If P is on the locus, we have that PA is common, $AB = AC$, and $\angle APB = \angle APC$.

This implies either $\angle PBA = \angle PCA$, so $\triangle APB \equiv \triangle APC$ and $BP = CP$ or $\angle PBA + \angle PCA = 180°$, and so $ABPC$ is concyclic.



(b)

For the converse, if $BP = CP$ or $ABPC$ is concyclic, then $\angle APB = \angle APC$. We can argue to the conclusion as in 13 Solution 1.

13 *Solution 3* It is easy to dispose of the cases in which P is on any of the lines AB, AC or BC.

Suppose otherwise. Applying the Law of Sines to $\triangle PAB$ and $\triangle PAC$ respectively yields

$$\frac{\sin\angle APB}{AB} = \frac{\sin\angle PBA}{AP}$$

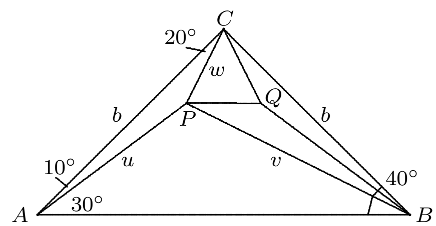
and

$$\frac{\sin\angle APC}{AC} = \frac{\sin\angle PCA}{AP}.$$

Hence, if $\angle APB = \angle APC$, then $\sin\angle PBA = \sin\angle PCA$, so that $\angle PBA = \angle PCA$ or $\angle PBA + \angle PCA = 180°$.

We can now finish as in 13 Solution 2.

14 *Solution 1* Clearly $\triangle ABC$ is isosceles with $AC = BC$. Reflect P in the right bisector (through C) of BA to obtain Q.



$CP = CQ$ implies that $\angle CPQ = \angle CQP$.

Also $\angle ACP = \angle BCQ = 20°$.

Hence $\angle PCQ = 100° - 2 \times 20° = 60°$.

Therefore, $\triangle PCQ$ is equilateral and $PQ = QC$.

$\angle BQC = \angle APC = 150°$ and $\angle CQP = 60°$ together imply that $\angle BQP = 150° = \angle BQC$.

Hence $\triangle BQC \equiv \triangle BQP$ (SAS : $QC = QP$, $\angle BQC = \angle BQP$, BQ is common).

Thus $BP = BC$ and further, $\angle BPC = \angle BCP = 100° - 20° = 80°$.

14 *Solution 2* Let $b = |AC|$. By the Law of Sines,

$$|PC| = \frac{b}{\sin 150°} \cdot \sin 10° = 2b \sin 10°.$$

By the Law of Cosines, using $b = |BC|$,

$$|PB|^2 = |PC|^2 + |CB|^2 - 2|PC||CB| \cos 80°,$$

so that

$$\begin{aligned} |PB| &= \sqrt{4b^2 \sin^2 10° + b^2 - 4b^2 \sin 10° \cos 80°} \\ &= \sqrt{4b^2 \sin^2 10° + b^2 - 4b^2 \sin^2 10°} \\ &= b \ = \ |CB|. \end{aligned}$$

Hence $\angle BPC = \angle BCP = \angle 80°$.

14 *Solution 3* Let $u = |PA|$, $v = |PB|$, $w = |PC|$. By the Law of Sines, with $\beta = \angle PBC$

$$\frac{\sin 80°}{v} = \frac{\sin \beta}{w},$$

$$\frac{\sin 10^\circ}{w} = \frac{\sin 20^\circ}{u},$$
$$\frac{\sin(40^\circ - \beta)}{u} = \frac{\sin 30^\circ}{v}.$$

Hence $\sin 80^\circ \sin 10^\circ \sin(40^\circ - \beta) = \sin\beta \sin 20^\circ \sin 30^\circ$.

Since
$$2\sin 80^\circ \sin 10^\circ = 2\cos 10^\circ \sin 10^\circ = \sin 20^\circ$$
and
$$\sin 30^\circ = \frac{1}{2},$$
we have that
$$\sin\beta = \sin(40^\circ - \beta).$$
Since $0 < \beta < 40^\circ$, we see that $\beta = 40^\circ - \beta$. Thus $\beta = 20^\circ$, so that $\angle BPC = 180^\circ - \angle BCP - \angle PBC = 80^\circ$.

14 *Solution 4* [A. Chan] Let $\theta = \angle BPC$. We have $\angle APC = 150^\circ$ and $\angle ABP = \theta - 60^\circ$, so by the Law of Sines

$$\frac{\sin 150^\circ}{b} = \frac{\sin 20^\circ}{u},$$
$$\frac{\sin\theta}{b} = \frac{\sin 80^\circ}{v},$$
$$\frac{\sin(\theta - 60^\circ)}{u} = \frac{\sin 30^\circ}{v}.$$

Hence
$$\sin 150^\circ \sin 80^\circ \sin(\theta - 60^\circ) = \sin 20^\circ \sin 30^\circ \sin\theta.$$
Thus
$$\sin 80^\circ \sin(\theta - 60^\circ) = \sin 20^\circ \sin\theta,$$
and so
$$\cos(140^\circ - \theta) - \cos(20^\circ + \theta) = \cos(20^\circ - \theta) - \cos(20^\circ + \theta),$$
yielding
$$\cos(140^\circ - \theta) = \cos(20^\circ - \theta) = \cos(\theta - 20^\circ).$$
Now
$$\theta = 180^\circ - (\angle BPC + \angle PCB) > 180^\circ - 40^\circ - 100^\circ = 40^\circ$$
and
$$\theta < 180^\circ - \angle BCP = 100^\circ.$$
Hence $20^\circ < \theta < 140^\circ$.

Thus $\cos(140^\circ - \theta) = \cos(\theta - 20^\circ)$, and so $140^\circ - \theta = \theta - 20^\circ$ yielding $\theta = 80^\circ$.

14 *Solution 5* [B. Marthi] By the Law of Sines,

$$\frac{\sin 10^\circ}{w} = \frac{\sin 150^\circ}{b}$$

and

$$\frac{\sin\theta}{b} = \frac{\sin(100^\circ - \theta)}{w}.$$

Thus

$$\sin\theta \sin 10^\circ = \sin 150^\circ \sin(100^\circ - \theta),$$

giving

$$2\sin\theta\sin 10^\circ = \sin(100^\circ - \theta) = \cos(\theta - 10^\circ),$$

which leads to

$$\cos(\theta - 10^\circ) + \cos(\theta - 10^\circ) = \cos(\theta - 10^\circ),$$

and then to

$$\cos(\theta + 10^\circ) = 0.$$

Since $0 < \theta < 180^\circ$, we get $\theta + 10^\circ = 90^\circ$, and finally, that $\theta = 80^\circ$.

14 *Solution 6* By the Law of Sines on $\triangle PAC$,

$$\frac{b}{\sin 150^\circ} = \frac{w}{\sin 10^\circ} = \frac{w}{\cos 80^\circ},$$

so that

$$w = 2b\cos 80^\circ.$$

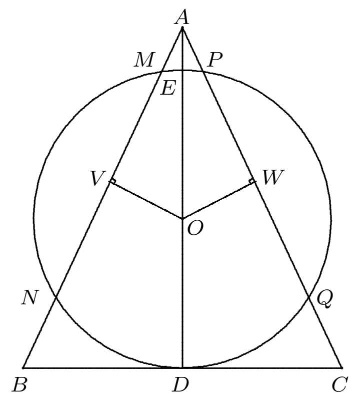
By the Law of Cosines on $\triangle PAC$ (with $\angle BCP = 80^\circ$), we get

$$\begin{aligned} v^2 &= b^2 + w^2 - 2bw\cos 80^\circ \\ &= b^2 + 4b^2\cos^2 80^\circ - 4b^2\cos^2 80^\circ \\ &= b^2, \end{aligned}$$

so that $v = b$.

Hence $\angle BPC = \angle BCP = 80^\circ$ (same as 14 solution 2).

15 *Solution 1*



$$\begin{aligned} AD^2 &= AB^2 - BD^2 = AB^2 - BN \cdot BM \\ &= AB^2 - (AB - AN) \cdot (AB - AM) \\ &= AB \cdot (AM + AN) - AM \cdot AN. \end{aligned}$$

Similarly, $AD^2 = AC \cdot (AP + AQ) - AP \cdot AQ$.

Since $AM \cdot AN = AP \cdot AQ$, we have

$$AB \cdot (AM + AN) = AC \cdot (AP + AQ),$$

from which the result follows.

(The argument is valid even if AB and AC fall on the same side of AD).

15 *Solution 2* Let V and W be the respective midpoints of MN and PQ, so that $OV \perp AB$ and $OW \perp AC$.

Now

$$AM + AN = 2AM + MV + VN = 2AM + 2MV = 2AV.$$

Similarly, $AP + AQ = 2AW$.

$\triangle AVO \,|||\, \triangle ABD$ implies that $\dfrac{AV}{AO} = \dfrac{AD}{AB}$, which in turn implies that

$$AV \cdot AB = AD \cdot AO = (AM + AN) \cdot AB = 2AD \cdot AO.$$

Similarly,

$$(AP + AQ) \cdot AC = 2AD \cdot AO = (AM + AN) \cdot AB,$$

and the result follows.

15 *Solution 3* Let $m = |AM|$, $n = |AN|$, $a = |AO|$ and $r = |OM| = |ON|$ (the radius of the circle).

By the Law of Cosines, m and n are roots of the quadratic equation

$$r^2 = x^2 + a^2 - 2xa\cos(\angle BAD),$$

whence, from the theory of the quadratic, $m + n = 2a\cos(\angle BAD)$.

Let $u = |AB|$ and $d = |AD|$. Then $\cos(\angle BAD) = \frac{d}{u}$.

Hence $m + n = \dfrac{2ad}{u}$, so that $u(m + n) = 2ad$, giving $AB \cdot (AM + AN) = 2AE \cdot AD$.

Similarly, $AC \cdot (AP + AQ) = 2AE \cdot AD$.

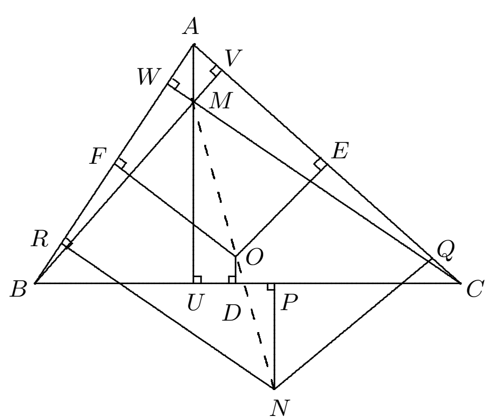
The result follows.

15 *Solution 4*

$$\begin{aligned}
&(AE + AD) \cdot AD \\
&\quad = \quad AE \cdot AD + AD^2 \\
&\quad = \quad AM \cdot AN + AB^2 - BD^2 \\
&\quad = \quad AM \cdot AN + AB^2 - BN \cdot BM \\
&\quad = \quad AM \cdot AN + (AN + NB) \cdot (AM + MB) - BN \cdot BM \\
&\quad = \quad AM \cdot AN + AN \cdot AB + NB \cdot AM \\
&\quad = \quad AM \cdot (AN + NB) + AN \cdot AB \\
&\quad = \quad (AM + AN) \cdot AB.
\end{aligned}$$

Similarly, $(AE + AD) \cdot AD = (AP + AQ) \cdot AC$, and the result follows.

16



Label the feet of the perpendiculars from the points M, O and N as in the diagram.

Basically one has to show that

$$AN^2 + BN^2 + AB^2 = AN^2 + CN^2 + AC^2,$$

or more simply, that

$$BN^2 + AB^2 = CN^2 + AC^2.$$

The remainder follows by a parallel argument to get

$$AN^2 + BN^2 + AB^2 = BN^2 + CN^2 + BC^2.$$

One should play around with the many right triangles to bring the solution home. A key observation is that $DU = PD$, $VE = EQ$, $WF = FR$ (parallel projections of $MO = ON$).

16 *Solution 1* Consider the 180° rotation about O. It carries the line DO to itself and M to N. Since the rotation preserves distances,

$$\textbf{dist}\ (M, \textbf{line}\ OD) = \textbf{dist}\ (N, \textbf{line}\ OD) \ \text{ implies that } \ UD = DP.$$

Since D is the midpoint of BC, we have $BU = PC$ and $BP = UC$.

Therefore,

$$\begin{aligned} BN^2 + AB^2 &= (BP^2 + PN^2) + (AU^2 + BU^2), \\ &\qquad\qquad \text{)by Pythagoras' Theorem)} \\ &= UC^2 + PN^2 + AU^2 + PC^2 \\ &= (UC^2 + AU^2) + (PN^2 + PC^2) \\ &= AC^2 + CN^2. \end{aligned}$$

Hence $AN^2 + BN^2 + AB^2 = AN^2 + CN^2 + AC^2$, as required.

A similar argument shows either side of this equation is equal to

$$BN^2 + CN^2 + BC^2.$$

16 *Solution 2* (sketch) Introduce coordinates, with the origin at U. Make liberal use of the slope condition for perpendicular lines and the coordinates of midpoints of segments to obtain from $A(0, a)$, $B(b, 0)$, $C(c, 0)$, the following $U(0, 0)$, $D\left(\frac{b+c}{2}, 0\right)$, $E\left(\frac{c}{2}, \frac{a}{2}\right)$, $F\left(\frac{b}{2}, \frac{a}{2}\right)$, $M\left(0, -\frac{bc}{a}\right)$, $O\left(\frac{b+c}{2}, \frac{a}{2} + \frac{bc}{2a}\right)$ and $N\left(b + c, a + \frac{2bc}{a}\right)$. Now compute the requisite lengths.

16 *Solution 3* [M. Ting] Let the positions of the vertices A, B, C be given by the vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ respectively with the origin at O.

Let $\underline{m} = \underline{a} + \underline{b} + \underline{c}$.

Then $(\underline{c} - \underline{b}) \cdot (\underline{m} - \underline{a}) = (\underline{c} - \underline{b}) \cdot (\underline{c} + \underline{b}) = \underline{c}^2 - \underline{b}^2 = 0$, etc., so that the orthocentre M is located by the vector $\underline{m}$. Hence $\underline{n} = -(\underline{a} + \underline{b} + \underline{c})$. Thus,

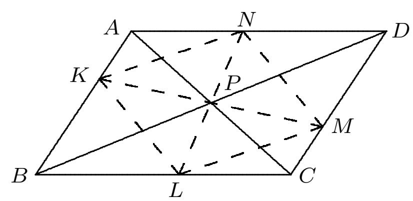
$$\begin{aligned} BN^2 + AB^2 &= (2\underline{b} + \underline{a} + \underline{c})^2 + (\underline{b} - \underline{a})^2 \\ &= 5\underline{b}^2 + 2\underline{a}^2 + \underline{c}^2 + 2\underline{a} \cdot \underline{b} + 2\underline{a} \cdot \underline{c} + 4\underline{b} \cdot \underline{c}. \\ CN^2 + AC^2 &= (2\underline{c} + \underline{a} + \underline{b})^2 + (\underline{c} - \underline{a})^2 \\ &= 5\underline{c}^2 + 2\underline{a}^2 + \underline{b}^2 + 2\underline{a} \cdot \underline{b} + 2\underline{a} \cdot \underline{c} + 4\underline{b} \cdot \underline{c}. \end{aligned}$$

Since $|\underline{a}| = |\underline{b}| = |\underline{c}|$ (the radius of the circumcircle) we have

$$AN^2 + BN^2 + AB^2 = AN^2 + CN^2 + AC^2.$$

17 *Solution 1*

Let P be the centre of the parallelogram.



We have $AP = PC$ and $BP = PD$, so that $\frac{AP}{PD} = \frac{AP}{PB} = k$.

Since PN bisects $\angle APD$, we have $\dfrac{AN}{ND} = \dfrac{AP}{PD} = k$.

Since PK bisects $\angle APB$, we have $\dfrac{AK}{KB} = \dfrac{AP}{PB} = k$.

Hence $KN \parallel BD$ and $\triangle AKN \,|||\, \triangle ABD$.

Now $\dfrac{AN}{AD} = \dfrac{AK}{AB} = \dfrac{KN}{BD} = \dfrac{k}{k+1}$, so that

$$[AKN] = \frac{k^2}{(k+1)^2}[ABD].$$

Similarly, $[CML] = \dfrac{k^2}{(k+1)^2}[CBD]$, $[DNM] = \dfrac{1}{(k+1)^2}[DAC]$, $[BLK] = \dfrac{1}{(k+1)^2}[BCA]$.

Adding these four equations yields

$$\begin{aligned} [ABCD] - [KLMN] &= \frac{k^2}{(k+1)^2}\Big([ABD] + [CBD]\Big) \\ &\quad + \frac{1}{(k+1)^2}\Big([DAC] + [BCA]\Big) \\ &= \frac{k^2+1}{(k+1)^2}[ABCD], \end{aligned}$$

whence

$$[KLMN] = \frac{2k}{(k+1)^2}[ABCD].$$

Thus the required ratio is $\dfrac{2k}{(k+1)^2}$.

17 *Solution 2* Let $|PD| = 1$, so $|AP| = k$. Let $\theta = \angle APD$.
Then $[APD] = \frac{1}{2}k\sin\theta$ implies that $[ABCD] = 2k\sin\theta$.
However, $[APD] = [APN] + [NDP]$ implies that

$$\frac{1}{2}k\sin\theta = \frac{1}{2}k|PN|\sin\frac{\theta}{2} + \frac{1}{2}|PN|\sin\frac{\theta}{2}.$$

This, in turn, implies that

$$2k\sin\frac{\theta}{2}\cos\frac{\theta}{2} = k\sin\theta = (k+1)|PN|\sin\frac{\theta}{2}.$$

Whence $|PN| = \dfrac{2k}{k+1}\cos\dfrac{\theta}{2}$. Similarly $|PL| = \dfrac{2k}{k+1}\cos\dfrac{\theta}{2}$.
A similar argument establishes that

$$|PK| = |PM| = \frac{2k}{k+1}\cos\left(\frac{180^0 - \theta}{2}\right) = \frac{2k}{k+1}\sin\frac{\theta}{2}.$$

Now $KM \perp NL$ and

$$\begin{aligned}
[KLMN] &= 4[PKN] \;=\; 4\cdot\frac{1}{2}\cdot|PK|\cdot|PN| \\
&= 2\left(\frac{2k}{k+1}\sin\frac{\theta}{2}\right)\left(\frac{2k}{k+1}\cos\frac{\theta}{2}\right) \\
&= \frac{4k^2}{(k+1)^2}\sin\theta \\
&= \frac{2k}{(k+1)^2}[ABCD].
\end{aligned}$$

17 *Solution 3* (outline). Establish that

$$KN \parallel BD, \quad NM \parallel AC,$$

$$|KN| = \frac{k}{k+1}|BD|, \quad |NM| = \frac{1}{k+1}|AC|.$$

From this, it follows that $\angle KNM = \angle APD = \theta$.
Now

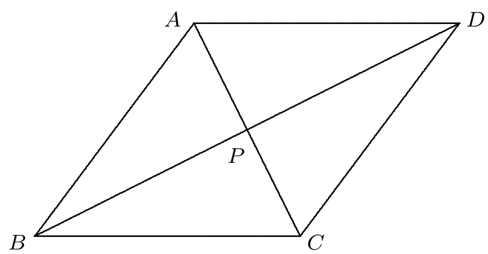
$$\begin{aligned}
[ABCD] &= 4\,[APD] \;=\; 2\,|AP|\cdot|PD|\sin\theta \\
&= \tfrac{1}{2}|AC|\cdot|BD|\sin\theta,
\end{aligned}$$

and

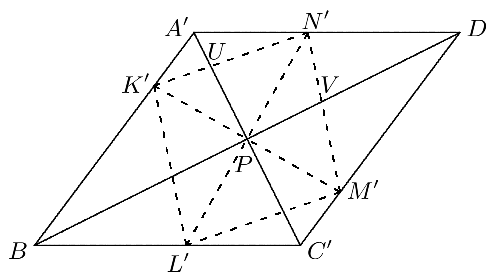
$$[KLMN] = |KN| \cdot |NM| \sin\theta = \frac{k}{(k+1)^2} |BD| \cdot |AC| \sin\theta,$$

whence the result follows.

17 *Solution 4* We transform the parallelogram $ABCD$ as follows:



(1) First, perform a shear which fixes BD and transforms ADC to a line perpendicular to BD: This reduces the ratio $\frac{AC}{BD}$, but preserves the ratio of the two areas in question:



(2) Then, perform an augmentation in the direction perpendicular to BD until A goes to A', C to C' with $AC' = kBD$. This again preserves the ratio of the two areas in question.

We have both $A'C' \perp BD$ and $K'M' \perp L'N'$, so that

$$\frac{[K'L'M'N']}{[A'B'C'D']} = \frac{[N'UPV]}{[PA'D']} = \frac{[PA'D'] - [A'UN'] - [N'VD]}{[PA'D']} = 1 - \frac{k^2}{(k+1)^2} - \frac{1}{(k+1)^2} = \frac{2k}{(k+1)^2}.$$

17 *Solution 5* (Outline)

$$\overrightarrow{PK} = \frac{k\overrightarrow{PB} + \overrightarrow{PA}}{k+1}, \quad \overrightarrow{PN} = \frac{\overrightarrow{PA} + k\overrightarrow{PD}}{k+1} = \frac{\overrightarrow{PA} - k\overrightarrow{PB}}{k+1},$$

$$\overrightarrow{PM} = \frac{k\overrightarrow{PD} + \overrightarrow{PC}}{k+1} = \frac{-k\overrightarrow{PB} - \overrightarrow{PA}}{k+1},$$

$$\overrightarrow{KN} = -\overrightarrow{PK} + \overrightarrow{PN} = -\frac{2k}{k+1}\overrightarrow{PB} = \frac{1}{2}\left(\frac{2k}{k+1}\right)\overrightarrow{BD},$$

$$\overrightarrow{NM} = -\overrightarrow{PN} + \overrightarrow{PM} = \frac{-2}{k+1}\overrightarrow{PA} = \frac{1}{2}\left(\frac{2}{k+1}\right)\overrightarrow{AC},$$

$$[KLMN] = \left|\overrightarrow{KN} \times \overrightarrow{NM}\right| = \frac{k}{(k+1)^2}\left|\overrightarrow{BD} \quad \times \overrightarrow{AC}\right|$$

$$= \left(\textbf{Area} \text{ (parallelogram defined by } \overrightarrow{BD} \ \overrightarrow{AC})\right)\frac{k}{(k+1)^2}$$

$$= \frac{2k}{(k+1)^2}[ABCD].$$

17 *Solution 6* (outline)

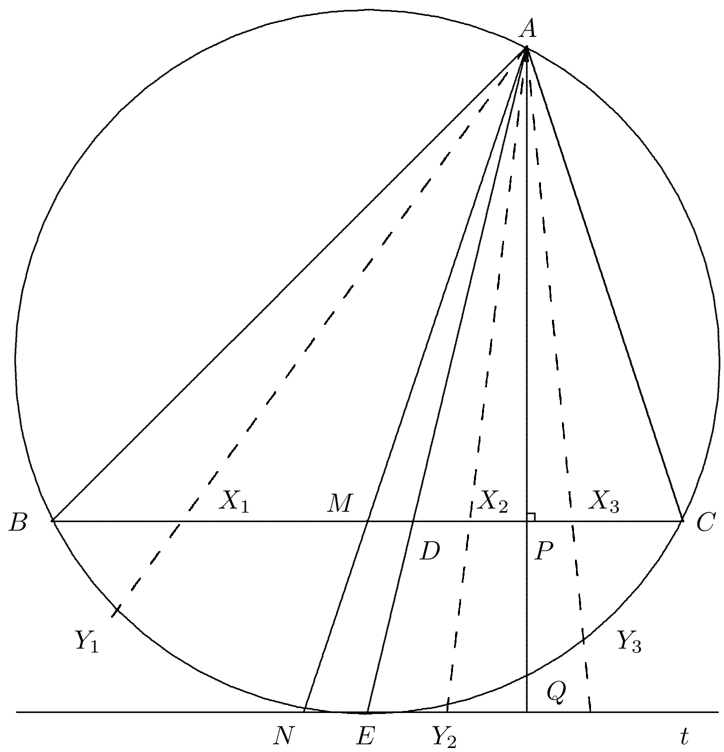
$$\frac{[AKN]}{[PKN]} = \frac{AN}{ND} = k, \qquad \frac{[DMN]}{[PNM]} = \frac{1}{k},$$
$$\frac{[CML]}{[PML]} = k, \qquad \frac{[BLK]}{[PLK]} = \frac{1}{k}.$$

$$[KLMN] = 4[PKN] = 4[PNM] = 4[PML] = 4[PLK].$$

Hence

$$\begin{aligned}\frac{[ABCD]}{[KLMN]} & \\ &= \frac{[AKN]+[DMN]+[CML]+[BLK]+[KLMN]}{[KLMN]} \\ &= \frac{1}{4}\left[k+\frac{1}{k}+k+\frac{1}{k}\right]+1 = \frac{k}{2}+\frac{1}{2k}+1 = \frac{(k+1)^2}{2k}.\end{aligned}$$

18 *Solution 1* In the diagram, the triangle ABC has circumcircle $ABNEQC$.



AM is a median and ADE bisects $\angle BAC$. Since equal angles are subtended by equal arcs, we have arc $BE =$ arc EC. The tangent t to the circle through E is parallel to the chord BC (since the radius joining E to the centre right bisects BC and is perpendicular to t). AP is an altitude.

If $AB = AC$, it is obvious that XY is maximized when $X = M = D$.

Without loss of generality, take $AB > AC$ as in the diagram. There are three cases to consider:

Case (1) $X = X_1$, on the segment BM. Let $Y = Y_1$.

By the Arithmetic-Geometric Mean Inequality,

$$BX_1 \cdot X_1C \leq \frac{1}{4}(BX_1 + X_1C)^2 = \frac{1}{4}BC^2 = \frac{1}{4}(2BM)^2 = BM^2.$$

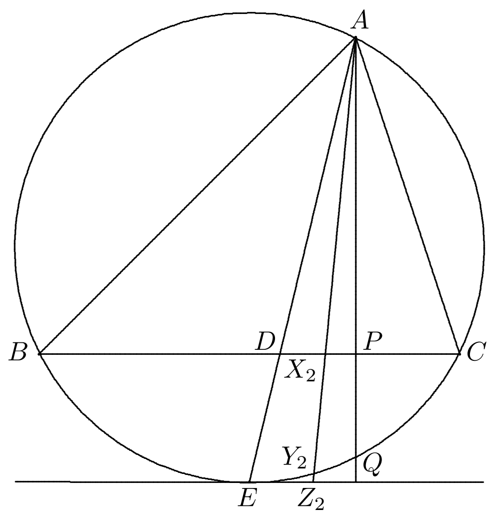
Since $\triangle AX_1M$ has its largest (obtuse) angle at M, we have $AX_1 > AM$.

Hence, $AX_1 \cdot X_1Y_1 = BX_1 \cdot X_1C_1$ and $AM \cdot MN = BM \cdot MC$,

so that $X_1Y_1 = \dfrac{BX_1 \cdot X_1C}{AX_1} < \dfrac{BM^2}{AM} = \dfrac{BM \cdot MC}{AM} = MN$.

Thus XY does not assume its maximum if X is between B and M.

Case (2) $X = X_2$ on the segment DP. Let $Y = Y_2$.



Produce X_2Y_2 to meet the circumcircle in Z_2.

Now, in triangles AX_2D and AZ_2E, the obtuse angle is opposite ADE, so $AD > AX_2$, $AE > AZ_2$. Since the triangles are similar, we have

$$\frac{AD}{AX_2} = \frac{AE}{AZ_2} = \frac{DE}{X_2Z_2}.$$

It follows that $X_2Y_2 < X_2Z_2 < DE$.

Thus, XY does not assume its maximum if X is between D and P.

Case (3) $X = X_3$ on the segment PC. Let $Y = Y_3$.

Argument (i): $X_3Y_3 = \dfrac{BX_3 \cdot X_3C}{AX_3} < \dfrac{BP \cdot PC}{AP} = PQ$

since $BX_3 \cdot X_3C$ increases as X_3 moves towards M and $AX_3 > AP$.

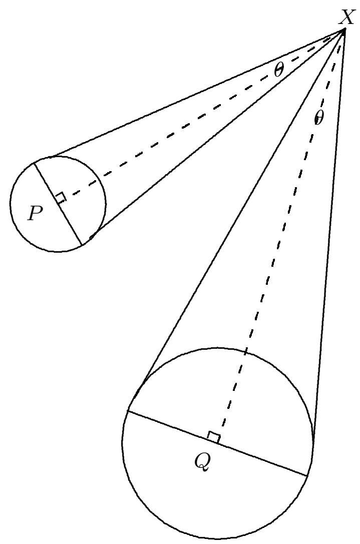
Argument (ii): Reflect AX_3Y_3 to $AX_3'Y_3'$ in APQ.

Then X_3' lies on BP, and Y_3' lies within the circle. Thus $X_3'Y_3'$ can be produced to meet the circle in Y_3''.

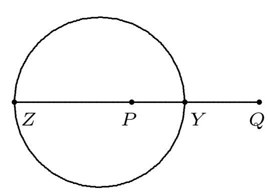
Then $X_3Y_3 = X_3'Y_3' < X_3'Y_3''$ when X_3' lies between B and P.

It follows that XY is maximum when X lies between M and D.

19 *Solution 1*



Analysis Let us examine the locus of points at which two disjoint circles subtend the same angle. The situation is as pictured, where P and Q are the centres of the circles. Lines drawn from X are tangents to the circles. The diagram involving the circle of centre P is the image of the diagram involving the circle of centre Q with respect to a central similarity followed by a rotation with respect to X; thus $XP : XQ$ is equal to the ratio of the diameter of the circles, and therefore constant. If the circles are congruent , the locus of X is the right bisector of P and Q; if they are not, the locus is the Apollonius circle with diameter Y, Z where Y and Z are chosen so that $PY : QY = PZ : QZ$, the ratio of the diameters.

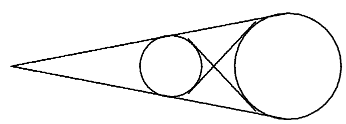


Introducing a third circle, we can find the required point as the intersection of two loci of this type.

Construction If the three circles are congruent, construct the right bisectors of the segments joining the centres of any two pairs from them. The intersection of these right bisectors is the required point. This fails if and only if the three centres of the circles are collinear.

Otherwise, we can find two pairs of noncongruent circles among the three. Let the centres of the circles be P, Q, R and their respective diameters p, q, r where $p \neq r, q \neq r$. On the extended line PR obtain points U, W (distinct) for which $PU : RU$ and $PW : RW$ are each equal to $p : r$. On the extended line QR obtain points V, Z (distinct) for which $QV : RV$ and $QZ : RZ$ are each equal to $q : r$. Construct circles with diameters UW and VZ. If these circles intersect, then the required point can be determined as one of the intersection points.

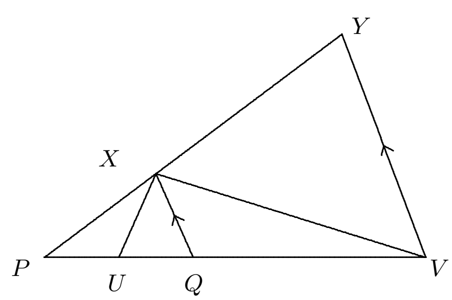
[Alternatively, the end points of the diameter of the Apollonius circle corresponding to a pair of the given circles can be found by taking the intersection points of the two pairs of common tangents to the circle. Evidently, the circles subtend equal angles at each of these points.]



Appendix

APOLLONIUS' THEOREM: Let PQ be a line segment. The locus of a point X for which the ratio of $PX : XQ$ is constant is the right bisector of PQ when $PX = XQ$ and a circle of diameter UV when $PX \neq XQ$, where U and V are points on PQ produced for which $PU : UQ = PV : VQ = PX : XQ$

A synthetic argument



Let X be a point on the locus. Since $PX : XQ = PU : UQ$, we have that XU bisects $\angle PXQ$, that is $\angle PXU = \angle UXQ$.

Construct $VY \parallel QX$ as in the diagram. Then

$$PY : YV = PX : XQ = PV : QV = PY : XY$$

so that $YV = XY$.

Hence $\angle YXV = \angle YVX = \angle QXV$, so that XV bisects QXY.

Hence $\angle UXV = \angle UXQ + \angle QXV = \frac{1}{2}(\angle PXQ + \angle QXY) = 90°$, so X lies on the circle with diameter UV.

On the other hand, suppose X is on the circle with diameter UV, so that $\angle UXV = 90°$. Let Q' be on UV such that $\angle UXQ' = \angle PXU$ and $PX : XQ' = PU : UQ'$.

The point on V' on PQ produced for which

$$PU : UQ' = PV' : Q'V'$$

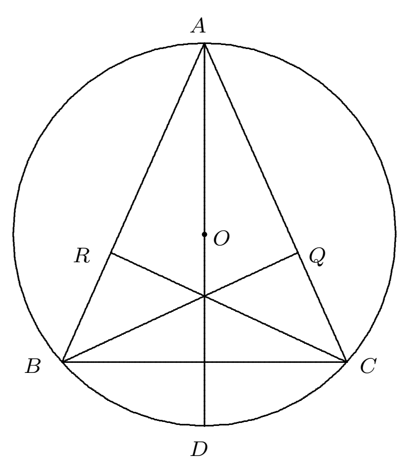
must have $\angle UXV' = 90°$ (by the first part), so that $V = V'$. Thus

$$PU : PV = UQ' : Q'V.$$

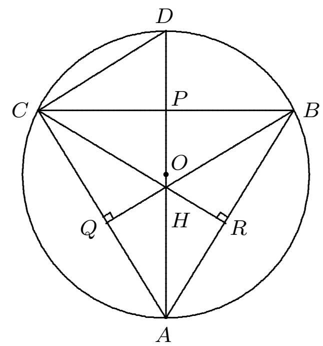
But $PU : PV = UQ : QV$ so $UQ' : Q'V = UQ : QV$ so that $Q = Q'$.

Hence $PX : XQ = PU : UQ$, and so X lies on the desired locus.

20 *Solution 1. Analysis.* There are three possibilities:



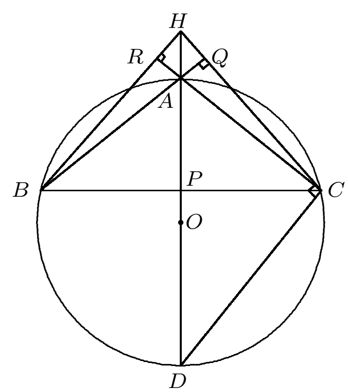
Case (i) The orthocentre H coincides with O, the centre of the circle, whereupon the required triangle will be any equilateral triangle inscribed within the circle.



Case (ii) The orthocentre H is distinct from the circumcentre O, but within the circle. In this case, the two points lie on the right bisector of the base of the triangle and the vertex must be one end of the diameter through OH. There are two possibilities; let A be one of these, and suppose that B and BC have been found (diagram).

Then $BQ \perp AC$, $CD \perp AC$ (since AD is the diameter). Thus $BQ \parallel CD$.

Similarly $CR \perp BD$, so that $HBDC$ is a parallelogram (indeed a rhombus) whose diagonals HD and BC right bisect each other.



Case (iii) The orthocentre H lies outside lies outside the circle. Then if ABC is the required triangle, $HOAD$ are collinear, $BR \perp AC$ (produced) and $CQ \perp BA$. Also $DC \perp AC$ and $DB \perp BA$, so again $BDCH$ is a parallelogram and HD and BC right bisect each other. Since P must lie in the circle, there are two constraints which must be satisified for the construction to work:

(a) A must lie between O and H;

(b) AH cannot exceed DA (thus, the limit of H as $B, C \to A$ is a point for which A bisects the segment HD - is this reasonable for you?)

Construction. Determine the circumcentre O of the circle and denote the orthocentre by H. Let AD be the diameter through O and H: if $O = H$, pick any diameter; if H is within the circle, let A be either end point; if H is on or outside the circle, pick the end point nearest to H for A. Bisect HD at P and choose B, C on the circle so that BPC is the right bisector of HD. This requires the condition that AH cannot exceed DA. Then ABC is the required triangle.

PROOF of Construction. Clearly the given circle is the circumcircle of triangle ABC. Now $CP = PB$, $PD = PH$ and $\angle CPD = \angle BPH$. This implies that $\triangle CPD \equiv \triangle BPH$, and further that $\angle PCD = \angle PBH$ and that $BQ \parallel DC$, so that $BQ \perp AC$ (since $DC \perp AC$). Thus, BQ is an altitude.

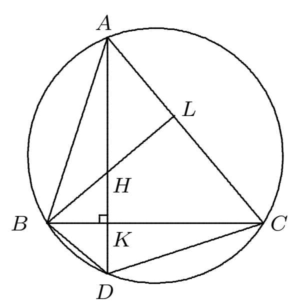
Likewise, CR is an altitude, as is AP. Hence, H is the orthocentre of $\triangle ABC$. Since a reflection in AD carries B to C,we have $AB = AC$ so that $\triangle ABC$ is isosceles.

Remarks. You MUST provide a PROOF of the construction. The analysis is not good enough since the reasoning goes the wrong way; in fact, technically, the analysis is redundant, although it is a good idea to include one. Always try to remark on the uniqueness and feasibility of the construction.

20 *An alternative background proposition.*

LEMMA. Let ABC be a triangle and let D be a point on the circumcircle such that $AD \perp BC$. The point H on AD is the orthocentre of ABC if and only if D and H are on opposite sides of BC, and BC right bisects HD.

PROOF. *Case 1* $\angle A$ is acute.



Suppose $HK = KD$. Then $\triangle BKH \equiv \triangle BKD$ (SAS)

Therefore $\angle LBC = \angle DBC$.

But $\angle DAC = \angle DBC$ implies that $\angle KAC = \angle LBC$. Then

$$\begin{aligned} \angle BLC &= 180^\circ - (\angle BCL + \angle LBC) \\ &= 180^\circ - (\angle KCA + \angle KAC) \\ &= \angle AKC = 90^\circ. \end{aligned}$$

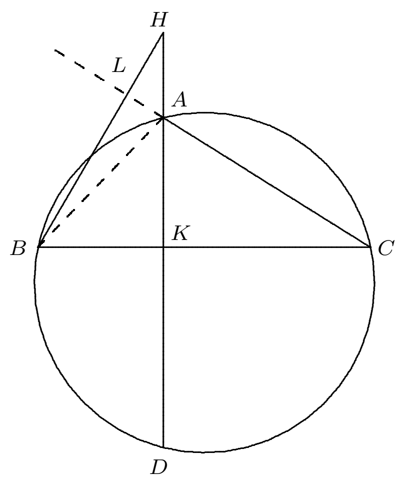
So, BL is an altitude . Hence H is the orthocentre.

On the other hand, suppose $BL \perp AC$, so that H is the orthocentre. Then

$$\angle DBC = \angle DAC = 90^\circ - \angle BCA = \angle LBC.$$

Hence $\triangle BKH \equiv \triangle BKD$ (ASA), so that $KH = DK$.

Case 2 $\angle A$ is obtuse.



Suppose $HK = KD$.

Then $\triangle BKH \equiv \triangle BKD$ so that $\angle LBC = \angle DBC$.

But $\angle LAH = \angle KAC = \angle DAC = \angle DBC = \angle LBC$. Then

$$\begin{aligned}\angle ALH &= 180^\circ - (\angle BHK + \angle LAH) \\ &= 180^\circ - (\angle BHK + \angle HBK) \\ &= \angle HKB \ = \ 90^\circ.\end{aligned}$$

So, BL is an altitude and H is the orthocentre.

On the other hand, let $BL \perp AC$. Then

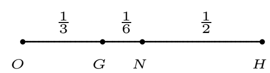
$\angle DBC = \angle DAC = \angle LAH = 90^\circ - \angle LHA = \angle HBK$

so that $\triangle BKH \equiv \triangle BKD$ (ASA), which implies that $KH = DK$.

Case 3 $\angle A$ is right. In this case $A = H$ and BC is a diameter.

20 *Solution 2* [S. Yazdani, O. Lhotak] (sketch)

RECALL: The nine point circle is the circumcircle of the pedal triangle. It contains the feet of the altitudes of the triangle, the midpoints of the sides and the midpoints of the segments joining the orthocentre to the vertices. Its centre N is the midpoint of OH, where O is the circumcentre and H is the orthocentre.

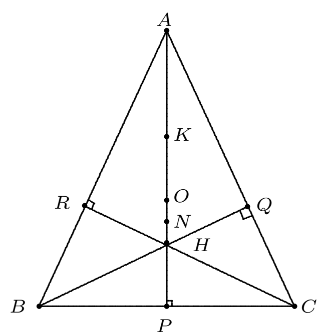


It can be obtained from the circumcircle of the triangle by either of two dilatations:

(a) centre G and factor $-\frac{1}{2}$,
(b) centre H and factor $\frac{1}{2}$.

Thus its radius is $\frac{1}{2}R$, where R is the circumradius.

Analysis: Let ABC be the finished triangle with $AB = AC$.



Let K be the midpoint of AH, let N be the midpoint of OH, the centre of the nine point circle, and let P, Q and R be feet of altitudes as in the diagram. Then K, O, N, H are on the right bisector of BC.

Since $\triangle ARH$ is right, K is equidistant from A, R and H.

Since R is a foot of an altitude, it lies on the nine point circle which passes also through K.

Hence R is the intersection of the circle centre K and radius AK with the circle centre N and radius NK.

Construction Determine A as the intersection of the given circumcircle and line OH (two cases if H is within the circumcircle). Let N be the mid-point of OH and let K be the mid-point of AH.

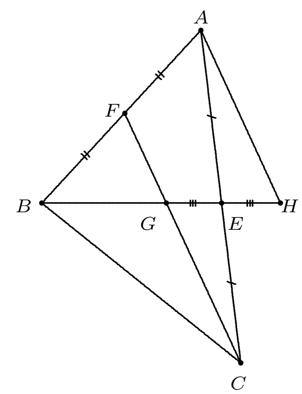
Construct the circle of centre K and radius $AK = AH$, and the circle of centre N and radius NK.

These circles will intersect at R and Q. Produce AR to meet the given circle at B and AQ to meet the given circle at C. Then $\triangle ABC$ is the required triangle.

PROOF. Since A, B, C are on the given circle with centre O, this circle is the circumcircle of $\triangle ABC$ and O is the circumcentre. Since AP contains the diameter of both intersecting circles, we have $AR = AQ$, from which it can be deduced that $AB = AC$. Since Q and R are on the diameter AH, we have $\angle ARH = \angle AQH = 90°$.

Exercise: Show that QH produced passes through B and RH produced passes through C, establishing that H is the orthocentre.

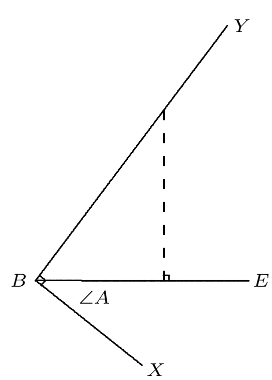
21 *Solution 1 Analysis* Let us consider the finished diagram.



v is the length of median BE.
w is the length of median CF.

(1) A lies on a certain circle with chord BE.
(2) $GE = EH$ implies that $\triangle AEH \equiv \triangle CEG$ (SAS), so that $AH = CG$.
(3) G being the centroid implies that $AH = CG = \frac{2}{3}w$.
(4) A lies on circle with centre H and radius $\frac{2}{3}w$.

Construction Draw $|BE| = v$.



Upon chord BE, construct a circle for which BE subtends an angle equal to $\angle A$ at the circumference. To do this, construct X so that $\angle EBX = \angle A$; let $\angle XBY = 90°$; the centre of the circle is on the intersection of BY and the right bisector of BE.

Construct G and H on BE produced with $BG = \frac{2}{3}v, BH = \frac{4}{3}v = 2BG$ with the order B, G, E, H.

With centre H and radius $\frac{2}{3}w$, construct a circle to meet the first circle at A. Produce AE to C so that $AE = EC$. Then ABC is the required triangle.

PROOF. We have $AE = EC$, $GE = EH$ and $\angle AEH = \angle GEC$, so that $\triangle AEH \equiv \triangle CEQ$ (SAS).

Thus $\angle HAE = \angle ECG$ and $AH = CG$, so that $AH \parallel CG$. Thus $AHCG$ is a parallelogram with $|CG| = |AH| = \frac{2}{3}w$.

Let CG meet AB in F.

Since $GF \parallel HA$ and $BG = GH$, we have $BF = FA$.

Hence $\angle A$ is as required (being on a constructed locus).

BE is a median (by construction of C), $|BE| = v$.

CF is a median.

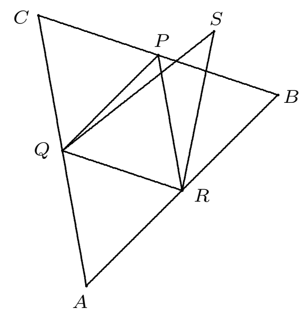
BE and CF meet at G, so that $CF = \frac{3}{2}CG = \frac{3}{2}\left(\frac{2}{3}w\right) = w$.

[*Alternatively:* Given that BE is a median of $\triangle ABC$, we can apply Menelaus' Theorem to $\triangle BAE$ to obtain

$$\frac{AF}{FB} \cdot \frac{BG}{BE} \cdot \frac{AC}{CE} = -1,$$

whence $AF = FB$.]

22 *Solution 1*



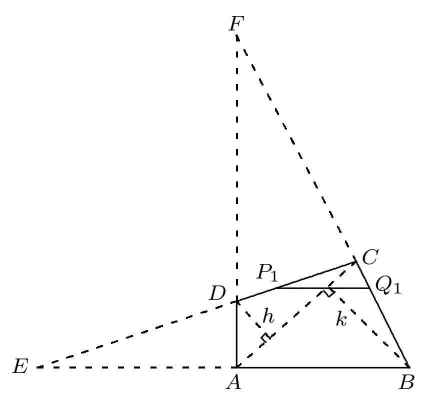
Let $\triangle PQR$ be a triangle of maximum area among those formed by any of the triples. Through each vertex, draw a line parallel to the opposite side to form a triangle ABC for which $[ABC] = 4[PQR]$.

Suppose S lies outside of $\triangle ABC$.

Then S must lie on the opposite side of at least one of AB, BC, CA to $\triangle PQR$. WOLOG, let S and $\triangle PQR$ be on opposite sides of BC. Then **dist** $(S, QR) >$ **dist** (P, QR) so that $[SQR] > [PQR]$.

Hence S cannot be one of the n points, so all n points lie inside $\triangle ABC$. Since $[PQR] < 1$, then $[ABC] < 4$. Find A' so $AA' \perp BC$ and $[A'BC] = 4$ (A between A' and BC). Then $\triangle A'BC$ is the required covering triangle.

23 *Solution 1*



Let h denote the distance from D to diagonal AC and k the distance from B to diagonal AC.

We have $\dfrac{[ACD]}{[ABC]} = \dfrac{h}{k}$, so that $\dfrac{[ABC]}{[ABCD]} = \dfrac{k}{h+k}$.

Determine Q_1 on BC produced so that

$$\frac{|BQ_1|}{|BC|} = \frac{h+k}{2k},$$

and draw $P_1Q_1 \parallel AB$ with P_1 on CD produced. By hypothesis, P_1Q_1 does not lie along a side of the quadrilateral $ABCD$. Now

$$[ABP_1] = [ABQ_1] = \frac{h+k}{2k}[ABC] = \frac{1}{2}[ABCD].$$

With Q_1 on the same side of AB as C, the point P_1 is as described in the problem.

Let E be the intersection point of AB and CD produced. Thus

$$\frac{|EP_1|}{|P_1C|} = \frac{|BQ_1|}{|Q_1C|} = \frac{h+k}{k-h}.$$

[Note that, if $h > k$, C will be between B and Q_1 and between E and P_1].

With any origin O, we have

$$\overrightarrow{OP_1} = \left(\frac{k+h}{2k}\right)\overrightarrow{OC} + \left(\frac{k-h}{2k}\right)\overrightarrow{OE}.$$

Now look at P_2 and side BC. In this case, the role of AB in the first part is played by BC and the role of E is played by F, the intersection of AB and BC produced. We have

$$\overrightarrow{OP_2} = \left(\frac{k+h}{2k}\right)\overrightarrow{OA} + \left(\frac{k-h}{2k}\right)\overrightarrow{OF}.$$

Similarly, we find that

$$\overrightarrow{OP_3} = \left(\frac{k+h}{2h}\right)\overrightarrow{OA} + \left(\frac{h-k}{2h}\right)\overrightarrow{OE}$$

and

$$\overrightarrow{OP_4} = \left(\frac{h+k}{2h}\right)\overrightarrow{OC} + \left(\frac{h-k}{2h}\right)\overrightarrow{OF}.$$

Hence

$$\begin{aligned}\overrightarrow{P_1P_2} &= \left(\frac{k+h}{2h}\right)\overrightarrow{CA} + \left(\frac{k-h}{2k}\right)\overrightarrow{EF} \\ &= \frac{1}{2k}\left[(k+h)\overrightarrow{CA} + (k-h)\overrightarrow{EF}\right]\end{aligned}$$

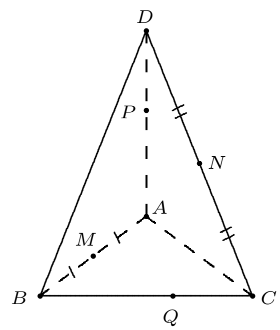
and

$$\begin{aligned}\overrightarrow{P_3P_4} &= \left(\frac{k+h}{2h}\right)\overrightarrow{AC} + \left(\frac{h-k}{2h}\right)\overrightarrow{EF} \\ &= -\frac{1}{2h}\left[(k+h)\overrightarrow{CA} + (k-h)\overrightarrow{EF}\right],\end{aligned}$$

so that $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ are multiples of the same vector (that is, are parallel or coincident).

By going through the same argument for P_2, P_3, P_4, P_1 (still on consecutive sides of the parallelogram) in place of P_1, P_2, P_3, P_4, we have that $\overrightarrow{P_2P_3}$ and $\overrightarrow{P_4P_1}$ are parallel or coincident. Thus either $P_1P_2P_3P_4$ lie along the same line or $P_1P_2P_3P_4$ is a parallelogram. In the latter case, $\overrightarrow{P_1P_2} = \overrightarrow{P_4P_3}$, so that $h = k$ and $\overrightarrow{OP_1} = \overrightarrow{OP_4} = \overrightarrow{OC}$ and $\overrightarrow{OP_2} = \overrightarrow{OP_3} = OA$, from which the result follows.

24 *Solution 1*



Case (a) Let $\overrightarrow{DA} = p\overrightarrow{DP}$ and $\overrightarrow{CB} = qCQ$.

Observe that $\overrightarrow{DB} + \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AD} = \overrightarrow{0}$,

so that $\overrightarrow{DB} + \overrightarrow{CA} = \overrightarrow{DA} + \overrightarrow{CB}$. Then

$$\begin{aligned}
\overrightarrow{NM} &= \overrightarrow{NC} + \overrightarrow{CM} = \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{CB}\right) \\
&= \frac{1}{2}\left(\overrightarrow{DA} + \overrightarrow{CA}\right) = \frac{1}{2}\left[p\overrightarrow{DP} + q\overrightarrow{CQ}\right] \\
&= \frac{1}{2}\left[p\left(\overrightarrow{ND} + \overrightarrow{DP}\right) + q\left(\overrightarrow{NC} + \overrightarrow{CQ}\right) - p\overrightarrow{ND} - q\overrightarrow{NC}\right] \\
&= \frac{1}{2}\left[p\overrightarrow{NP} + q\overrightarrow{NQ} + (p-q)\overrightarrow{NC}\right] \\
&= \left(\frac{p}{2}\right)\overrightarrow{NP} + \left(\frac{q}{2}\right)\overrightarrow{NQ} + \left(\frac{p-q}{4}\right)\overrightarrow{DC}.
\end{aligned}$$

Now $\overrightarrow{NM}$ is a linear combination of $\overrightarrow{NP}$ and $\overrightarrow{NQ}$ (since M, N, P, Q are coplanar).

If $p \neq q$, it would follow that $\overrightarrow{DC}$ is a linear combination of $\overrightarrow{NP}$ and $\overrightarrow{NQ}$, so that N, P, Q, C, D are coplanar. But this is false.

Hence $p = q$. Therefore

$$\frac{|DA|}{|DP|} = \frac{|CB|}{|CQ|} = p$$

and

$$\frac{|AP|}{|AD|} = \frac{|BQ|}{|BC|} = \frac{p-1}{p}.$$

Remarks The special cases that the plane is NCD (that is $P = D$, $Q = C$) or ABN (that is $P = A$, $Q = B$) are easy to deal with. Note that, in this part, the assumptions about equality of certain sides were not used.

Case (b) We use Cartesian coordinates.

Let $a = 1, b = \sqrt{\frac{1}{3} + h^2}, \frac{|AP|}{|AD|} = \frac{|BQ|}{|BC|} = u$. Take

$$\begin{array}{ll} A \sim (0,0,0) & M \sim (\frac{1}{2},0,0) \\ B \sim (1,0,0) & N \sim (\frac{1}{2},\frac{\sqrt{3}}{3},\frac{h}{2}) \\ C \sim (\frac{1}{2},\frac{\sqrt{3}}{2},0) & P \sim (\frac{u}{2},\frac{\sqrt{3}}{6}u,hu) \\ D \sim (\frac{1}{2},\frac{\sqrt{3}}{6},h) & Q \sim (1-\frac{1}{2}u,\frac{\sqrt{3}}{2}u,0). \end{array}$$

Then $\overrightarrow{NP} = \frac{1}{6}\left(3(u-1),\sqrt{3}(u-2),3(2u-1)h\right)$,
$\overrightarrow{NQ} = \frac{1}{6}\left(3(1-u),(3u-2)\sqrt{3},-3h\right)$
$\overrightarrow{MP} = \frac{1}{6}\left(3(u-1),\sqrt{3}u,6hu\right)$,
$\overrightarrow{MQ} = \frac{1}{6}\left(3(1-u),3\sqrt{3}u,0\right)$,

$$\begin{aligned} &\overrightarrow{NP} \times \overrightarrow{NQ} \\ &= \frac{1}{36}\left(3\sqrt{3}h(6u-6u^2),-18h(1-u)^2,12\sqrt{3}(1-u)^2\right) \\ &= \frac{1-u}{36}\left(18\sqrt{3}uh,-18h(1-u),12\sqrt{3}(1-u)\right). \\ &\overrightarrow{MP} \times \overrightarrow{MQ} \\ &= \frac{1}{36}\left(-18\sqrt{3}u^2h,18u(1-u)h,3\sqrt{3}u(1-u)(-3-1)\right) \\ &= -\frac{u}{36}\left(18\sqrt{3}uh,-18(1-u)h,12\sqrt{3}(1-u)\right). \end{aligned}$$

Thus, $\left|\overrightarrow{NP} \times \overrightarrow{NQ}\right| = \frac{1-u}{6}|\overrightarrow{v}|$, and $\left|\overrightarrow{MP} \times \overrightarrow{MQ}\right| = \frac{u}{36}|\overrightarrow{v}|$ where $\overrightarrow{v} = (3\sqrt{3}uh,-3h(1-u),2\sqrt{3}(1-u))$.

Now

$$\begin{aligned} [MQNP] &= [MPQ] + [NPQ] \\ &= \frac{1}{2}\left[\left|\overrightarrow{MP} \times \overrightarrow{NQ}\right|\right] + \frac{1}{2}\left[\left|\overrightarrow{NP} \times \overrightarrow{NQ}\right|\right] \\ &= \frac{1}{12}|\overrightarrow{v}| = \frac{1}{12}\left(27u^2h^2 + (9h^2+12)(1-u)^2\right)^{\frac{1}{2}} \\ &= \frac{1}{4\sqrt{3}}\left[4(3h^2+1)u^2 - (6h^2+8)u + (3h^2+4)\right]^{\frac{1}{2}}. \end{aligned}$$

Thus, we have to minimize

$$q(u) := 4(3h^2+1)u^2 - (6h^2+8)u + (3h^2+4) \quad \text{for} \quad 0 \le u \le 1.$$

Since

$$q(u) = 4(3h^2+1)\left[u - \frac{3h^2+4}{4(3h^2+1)}\right]^2 + \frac{3h^2+4}{4(3h^2+1)}\left[1 - \frac{3h^2+4}{4(3h^2+1)}\right],$$

$q(u)$ is minimized when

$$u = \frac{3h^2+4}{4(3h^2+1} = \frac{1}{4} + \frac{3}{4(3h^2+1)}.$$

Since $\frac{b}{a} = \sqrt{\frac{1}{3}(1+3h^2)}$, the minimizing $u = \frac{|AP|}{|AD|}$ is $\frac{1}{4}\left[1 + \frac{a^2}{b^2}\right]$.

In the case of a regular tetrahedron ($a = b$), $[MNPQ]$ is minimized when P and Q are the midpoints of AD and BC respectively.

Alternatively:

$$\begin{aligned}
[MQNP] &= \frac{\left|\overrightarrow{MN} \times \overrightarrow{PQ}\right|}{2} \\
&= \frac{\left|\left(0, \frac{\sqrt{3}}{3}, \frac{h}{2}\right) \times \left(1-u, \frac{\sqrt{3}}{3}, u, -hu\right)\right|}{2} \\
&= \frac{\left|\left(\frac{-\sqrt{3}}{2}hu, \frac{h(1-u)}{2}, \frac{-\sqrt{3}}{3}(1-u)\right)\right|}{2} \\
&= \frac{\sqrt{(12h^2+4)u^2 + (-6h^2-8)u + (3h^2+4)}}{4\sqrt{3}}.
\end{aligned}$$

24 *Solution 2* [O. Lhotak] Take coordinates
(a)

$$\begin{array}{l} A \sim \left(0, -\frac{a}{2}, 0\right) \\ B \sim \left(\frac{\sqrt{3}a}{2}, 0, 0\right) \\ C \sim \left(0, \frac{a}{2}, 0\right) \\ D \sim \left(\frac{a}{2\sqrt{3}}, 0, \sqrt{b^2 - \frac{a^2}{3}}\right) \\ M \sim \left(\frac{\sqrt{3}a}{4}, -\frac{a}{4}, 0\right) \\ N \sim \left(\frac{a}{4\sqrt{3}}, \frac{a}{4}, \frac{\sqrt{b^2 - \frac{a^2}{3}}}{2}\right) \\ P \sim \left(0, -\frac{a}{2}, 0\right) + t\left(\frac{a}{2\sqrt{3}}, \frac{a}{2}, \sqrt{b^2 - \frac{a^2}{3}}\right) \\ Q \sim \left(\frac{\sqrt{3}a}{2}, 0, 0\right) + s\left(-\frac{\sqrt{3}a}{2}, \frac{a}{2}, 0\right). \end{array}$$

Check that

$$\overrightarrow{NQ} \cdot \left(\overrightarrow{NM} \times \overrightarrow{MP}\right) = \frac{\sqrt{3(9b^2 - 3a^2)}}{24} a^2 (s - t)$$

so that P, N, M, Q are coplanar if and only if $s = t$.
(b)

$$\begin{aligned} [MQNP] &= \frac{1}{2}\left[\left|\overrightarrow{MQ} \times \overrightarrow{NQ}\right| + \left|\overrightarrow{MP} \times \overrightarrow{NP}\right|\right] \\ &= \frac{a^2b^2}{8}\left[t^2 - \frac{(a^2 + b^2)}{2b^2}t + \frac{a^2 + b^2}{4b^2}\right] \\ &= \frac{a^2b^2}{8}\left[\left(t - \frac{a^2 + b^2}{2b^2}\right)^2 + \text{constant}\right] \end{aligned}$$

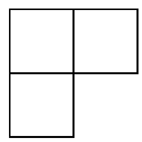
is minimized when $t = \frac{a^2+b^2}{b^2}$.

PROBLEM SET 3 — Combinatorics

Problems

25 Two evenly matched teams are engaged in a best four-of-seven series of games with each other. Is it more likely for the series to end in six games than in seven games?

26 One square is deleted from a square "checkerboard" with 2^{2n} squares. Show that the remaining $2^{2n} - 1$ squares can always be tiled with shapes of the form



which cover three squares

27 None of the nine participants in a scientific symposium speaks more than three languages. Two of any three participants speak a common language. Show that there is a language spoken by at least three participants.

28 There are nine people in a room. Two of any three know each other. Show that four people can be found in the room such that any two of them know each other.

29 Place 32 white and 32 black checkers on a standard 8×8 checkerboard. Two checkers of different colours will be said to form a "related pair" if they are placed either in the same row or the same column.

Determine the maximum and the minimum number of related pairs (over all possible arrangements of the checkers).

30 Let E be a system of $n^2 + 1$ intervals of the real line. Show that E has either a subsystem consisting of $n + 1$ intervals for which, given any pair, one is contained in the other, or else a subsystem consisting of $n + 1$ intervals, none of which contains any other member of the subsystem.

31 Last Friday, many students visited the school library. Each arrived and left only once. However, for any three of them, two of the three were present at the same time (that is, their stays in the library overlapped). Prove that there were two instants during the day such that each student was present for at least one of them.

32 Between one and two dozen people were present at a party. Each pair of strangers had two common acquaintances, and each pair of acquaintances had no common acquaintances. How many people were present?

33 Find the number of ways of choosing k numbers from $\{1, 2, \cdots, n\}$ so that no three consecutive numbers appear in any choice.

34 A function f, defined on the subsets of a given set S to themselves, has the property that, for any subsets A and B of S, $f(A)$ is a subset of $f(B)$ whenever A is a subset of B. Prove that there is a subset C of S for which $f(C) = C$.

35 Let n be a positive integer exceeding 5. Given are n coplanar points for which no two of the distances between pairs are equal. Suppose that each point is connected to the point nearest to it with a line segment. Prove that no point is connected to more than five others.

36 A subset of the plane is *closed* if it contains all of its boundary points. The *diameter* of a closed set is equal to the maximum distance between any pair of its points. Thus, the *diameter* of a square unit is $\sqrt{2}$.

(a) Prove that a closed unit square can be covered by three sets of diameter not exceeding $\sqrt{65}/8$.

(b) Prove that a closed unit square cannot be covered by three sets, all of which have diameter less than $\sqrt{65}/8$.

Solutions

25 *Solution 1* The series is equally likely to end in six games or in seven. If the series goes beyond the fifth game, then at the end of the fifth game one team must be leading the other 3 to 2. Each is equally likely to win the sixth game, leading to a 4 to 2 series for a win in six games or a 3 to 3 tie and the need for a seventh game.

25 *Solution 2* Let A denote a win for one team, B a win for the other. A six-game win needs four A's and two B's with an A last, or four B's and two A's with a B last. The probability of this is

$$\binom{5}{3}\binom{2}{2}\frac{1}{2^6}+\binom{5}{3}\binom{2}{2}\frac{1}{2^6}=\frac{5}{16}.$$

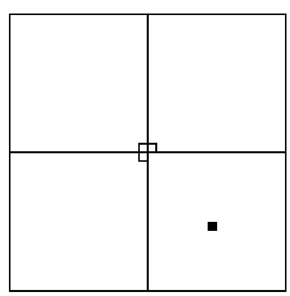
A seventh-game win needs four A's and three B's with an A last, or four B's and three A's with a B last. The probability of this is

$$\binom{6}{3}\binom{3}{3}\frac{1}{2^7}+\binom{6}{3}\binom{3}{3}\frac{1}{2^7}=\frac{5}{16}.$$

The result follows.

26 *Solution 1* The proof is by induction on n.
If $n=1$, the region to be tiled has the same shape as the tile and the result follows.

Suppose the result holds for $n=k$.



Given a checkerboard with $2^{2(k+1)}=4\times 2^{2k}$ squares, partition it into 4 checkerboards with 2^{2k} squares as indicated in the diagram. One of the 4 checkerboards involves the deleted square. Place one tile at the centre of the large checkerboard so that it covers one square of each of the remaining 3 smaller checkerboards (as indicated).

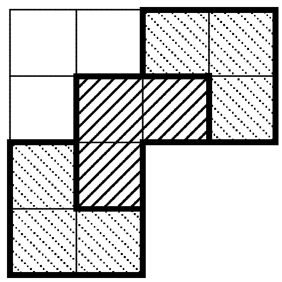
By the induction hypothesis, the remaining squares of the 4 smaller checkerboards can be tiled as specified and the result follows.

Question: Does the result hold for any even sided checkerboard?

26 *Solution 2* The proof is by induction on n. The case for $n = 1$ is clear. Suppose the result holds for $n = k$

A checkerboard with $2^{2(k+1)}$ 1×1 squares can be regarded as a 2^{2k} board with 2×2 squares.

Let S be the 2×2 square containing the deleted square. Then the rest of the board (excluding S), by the induction hypothesis can be covered by L-shaped conglomerations of tiles like this:



each made up of 4 of the basic tiles. S without the deleted square is another such tile. The result follows.

27 *Solution 1* There are two cases to consider.

Case (i): Suppose every pair of participants has a language common to both members.

There are $\binom{9}{2} = 36$ different pairs and at most $3 \times 9 = 27$ languages involved.

By the Pigeonhole Principle, there must be a language spoken by at least two distinct pairs. But these pairs comprise at least three individuals and the results follows.

Case (ii): Suppose there are two participants A and B with no language in common. Between them, they know $k \leq 6$ languages. Let the other participants be $P_1, P_2, \cdots, P_7$.

For each i, two of the trio $\{A, B, P_i\}$ have a language in common. They cannot be A and B. Hence, P_i speaks one of the k languages spoken by A or B.

Hence, each of the 7 participants P_i knows at least one of the k languages. By the Pigeonhole Principle, there are two participants, P_j and P_l with one of the k languages in common. But this language is also known by A or B and the result follows.

27 *Solution 2* Suppose that no three have a language in common. Then A has a language in common with at most three people and no language in common with at least five, say B, C, D, E, F. By considering $\{A, B, C\}$, we see that B and C have a language in common.

Similarly, B has a language in common with D, E, F. But then B has a language in common with four people, leading to a contradiction, since two of the three must have the **same** language in common with B.

28 *Solution 1* Suppose someone, say A, does not know four people, say B, C, D, E. Among A, B, and C, two know each other, so B must know C, say.

Similarly, any pair from B, C, D, E must know each other and the result must follow in this case.

Otherwise, no one fails to know more than three people and everyone must know at least five people.

Let $[XY \cdots Z]$ denote that each pair in the indicated set knows each other.

With each pair that knows each other counted twice (once for each individual), there must be at least 9×5 possibilities. But this number must be even. Hence some individual A must know at least six others, say B, C, D, E, F, G.

If B fails to know three of the remaining five, say E, F, G, then $[EF]$, $[EG]$ and $[FG]$ (from condition on triples involving B) which would give $[AEFG]$ and a solution to this problem.

Suppose then B knows at least three of the remaining five, say $[BC]$, $[BD]$ and $[BE]$.

If no two of C, D, E know each other, then we contradict the hypothesis for the triple $\{C, D, E\}$. Suppose, say $[CD]$.

Then, $[ABCD]$ and we have a solution to the problem.

28 *Solution 2* As above, we can deduce that either someone does not know four people or someone knows six people. Of these six people, it can be shown (Ramsey's Theorem) that either three do not know each other (which cannot occur by hypothesis), or three know each other. These three along with the person knowing the six, constitute the desired quartet.

29 *Solution 1*

Maximum number of related pairs.

Suppose that a row (or column) contains x white and $8-x$ black pieces; then there are $x(8-x) = 16-(x-4)^2$ related pairs; this number never exceeds 16 and equals 16 if and only if $x = 4$.

Since there are 8 rows and 8 columns, the total number of related pairs cannot exceed 256, and equals 256 for any configuration in which each row and column contains 4 white pieces.

The standard checkerboard configuration indicates such a configuration. The maximum is 256.

Minimum number of related pairs.

Observe that the number of related pairs is independent of the order in which the rows or the columns appear. Thus, interchanging a pair of rows or a pair of columns does not change the number of related pairs.

Suppose a configuration is given with N_0 related pairs. Rearrange the columns so that the number of white pieces in a column decreases from left to right. This does not alter the number of related pairs. Suppose that a white piece is to the right of a black piece in the same row.

Let the white piece be in a column with r white pieces, the black in a column with s white pieces ; we have $r \leq s$. Interchange the white and black pieces. The number of row-related pairs remains unchanged while the number of column related pairs is reduced by

$$\begin{aligned}
&[r(8-r)+s(8-s)]-[(r-1)(9-r)+(s+1)(7-s)] \\
&= \quad -r^2+8r-s^2+8s+r^2-10r+9+s^2-6s-7 \\
&= \quad 2(s+1-r) \ > \ 0.
\end{aligned}$$

We get a new configuration with fewer related pairs.

By repeating this process whenever a black piece is to the left of a white piece, we can achieve a situation in which the number of white pieces in a column decreases from left to right and in each row, there is a block of white pieces (possibly void) followed by a block of black pieces. Interchange the rows until the number of white pieces decreases from top to bottom. There are now N_1 related pairs with $N_1 \leq N_0$.

Suppose that the i^{th} row contains x_i white pieces and the j^{th} column contains y_j white pieces. Then

(a) $8 \geq x_1 \geq x_2 \geq \cdots \geq x_8 \geq 0; \quad 8 \geq y_1 \geq y_2 \geq \cdots \geq y_8 \geq 0;$

(b) $x_1 + x_2 + \cdots + x_8 = y_1 + y_2 + \cdots + y_8 = 32;$

(c) with the convention that $x_9 = 0, x_i - x_{i+1}$ is the number of y_i equal to i.

The total number of related pairs is

$$\begin{aligned}
&\sum_{i=1}^{8} x_i(8-x_i) + \sum_{i=1}^{8} y_i(8-y_i) \\
&= \quad 8\sum_{i=1}^{8} x_i + 8\sum_{i=1}^{8} y_i - \sum_{i=1}^{8} x_i^2 - \sum_{i=1}^{8} y_i^2
\end{aligned}$$

$$\begin{aligned}
&= 8\times 32 + 8\times 32 - (x_1^2+x_2^2+\cdots+x_8^2) - \sum_{i=1}^{8}(x_i - x_{i+1})i^2\\
&= 512 - [(x_1^2+x_2^2+\cdots+x_8^2)\\
&\quad +(x_1-x_2)+4(x_2-x_3)+9(x_3-x_4)+\cdots+64x_8]\\
&= 512 - [x_1^2+x_2^2+\cdots+x_8^2+x_1+3x_2+5x_3+\cdots+15x_8]\\
&= 512 - [(x_1^2+x_2^2+2x_2+x_3^2+4x_3+\cdots+x_8^2+14x_8)\\
&\quad +(x_1+x_2+\cdots+x_8)]\\
&= 512 - [x_1^2+(x_2+1)^2+\cdots+(x_8+7)^2-140+32]\\
&= 620 - [x_1^2+(x_2+1)^2+\cdots+(x_8+7)^2].
\end{aligned}$$

We have to maximize $x_1^2+(x_2+1)^2+\cdots+(x_8+7)^2$ subject to conditions (a) and (b). Since there are finitely many possibilities, the maximum can be achieved.

Now, we have that

$$[(u-1)^2+(v+1)^2]-[u^2+v^2] = 2(v+1-u) > 0,$$

which is equivalent to

$$v+1>u.$$

If we have a maximizing set $(x_1, x_2, \cdots, x_8)$, then it must never happen that $x_i \le 7$ and $x_i+i+1 > x_j+j$ (for then we could add 1 to x_i and subtract 1 from x_j and get a higher sum).

Hence, for all i, j, if $x_i \le 7$, then $x_i - x_j \le j-i-1$. Hence, either

$x_1 = 8$ or $x_1 \le x_2$, that is $x_1 = x_2$;

$x_2 = 8$ or $x_2 = x_3$, that is $x_1 = x_2 = 8$ or $x_2 = x_3$;

$x_3 = 8$ or $x_3 \le x_4$, that is $x_1 = x_2 = x_3 = 8$ or $x_3 = x_4$;

$x_4 = 8$ or $x_4 \le x_5$, that is $x_1 = x_2 = x_3 = x_4 = 8$ or $x_4 = x_5$.

Since $x_i \ne 8$ for $i \ge 5$, we must have $x_5 = x_6 = x_7 = x_8$.

Thus, either all x_i are equal or

$x_1 = 8, x_2 = x_3 = \cdots = x_8$;

$x_1 = x_2 = 8$, $x_3 = x_4 \cdots = x_8$;

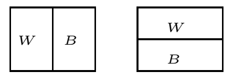
$x_1 = x_2 = x_3 = 8, x_4 = x_5 = \cdots = x_8$; or

$x_1 = x_2 = x_3 = x_4 = 8, x_5 = x_6 = x_7 = x_8 = 0$.

The only possibilities for $(x_1, \cdots, x_8)$ are

$$(4,4,4,\cdots,4) \quad \text{or} \quad (8,8,8,8,0,0,0,0).$$

This yields the two configurations, each with 128 related pairs.



29 *Solution 2* *Minimum number of related pairs*
Suppose that a given row contains k black checkers.

If $k \leq 4$, there are $k(8-k)$ related pairs in the same row involving these checkers, and this is at least $4k$.

If $k \geq 4$, there are at least $32-8(8-k)$ related pairs involving these checkers in the same column and $k(8-k)$ related pairs in the same row. Thus these k checkers are responsible for at least

$$32-(8-k)^2 = 4k+(8-k)(k-4) \geq 4k$$

related pairs.

The number of related pairs involving the black checkers in any row is at least 4 times that number.

Hence, the number of related pairs is at least $4 \times 32 = 128$.

30 *Solution 1* A *chain* is a subfamily of intervals $\{I_1, I_2, \cdots, I_k\}$ for which $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k$. For each positive integer k, we say that an interval I is of *rank* k if and only if it is the largest interval in a chain of length k, but of no longer chain.

[Thus rank 1 intervals contain no other intervals. Note that a rank k interval can be the largest of many chains of various lengths, but none exceeding k.]

Let I and J be two intervals of rank k. If $I \supseteq J$, then I must be the largest in a chain of $k+1$ intervals of which the k^{th} largest is J. But this contradicts the rank of I. Hence the class of rank k intervals has the property that no one of any pair contains the other. Thus, if there are at least $n+1$ intervals of any rank, the problem is solved.

Suppose on the other hand, there are at most n intervals at any rank. Then the intervals of rank not exceeding n account for at most n^2 intervals. Hence, there is an interval of rank exceeding $n+1$, and it is the largest in a chain of at least $n+1$ intervals (in which one of any pair contains the other). Again, the problem is solved.

30 *Solution 2* Let the intervals be $[x_i, y_i] \subseteq \mathbb{R}$ where the x_i are indexed so that $x_0 \leq x_1 \leq x_3 \leq \cdots \leq x_{n^2}$. If $x_i = x_{i+1}$, let $y_i > y_{i+1}$. Consider the sequence $\{y_0, y_1, \cdots, y_{n^2}\}$. This sequence has a monotone subsequence with at least $n+1$ elements. If this monotone sequence is decreasing, we obtain a nested sequence of intervals. If it is increasing, we get a sequence of intervals for which no one of any pair contains the other.

30 *Solution 3* [D. Robbins] Arrange the intervals in order of decreasing length. Form classes Si of intervals as follows. At the i^{th} stage, go through the

intervals not already selected in order; put an interval in S_i if and only if it is not contained in any interval already selected for S_i.

Suppose $i \geq 2$ and $I \in S_i$. Since $I \notin S_{i-1}, I$ must have been contained in some interval in S_{i-1}.

If $S_{n+1} \neq \emptyset$, then each interval in S_{n+1} is contained in some interval in S_n, which is contained in some interval of S_{n-1}, and so on, until we obtain a chain of $n+1$.

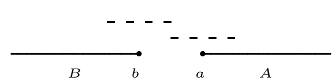
If $S_{n+1} = \emptyset$, then $S_1 \cup S_2 \cup \cdots \cup S_n$ contains all the intervals, so there exists an i such that S_i contains $n+1$ intervals. No one of these contains any other.

31 *Solution 1* Suppose student A was the last to arrive at instant a, and that student B was the first to leave at instant b. Any third student X arrived no later that a and left no sooner than b.

If a was no later than b, then A, B and X were all present for the whole interval from a to b.

If b was earlier than a, then A and B did not overlap, and X overlapped either A or B. If X overlapped A, then X was present at instant a. If X overlapped B, then X was present at instant b.

In any case, each student was present at one of the instants a and b.

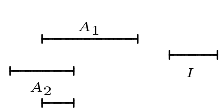


31 *Solution 2* [W.L. Yee] We can represent the visits of the students by intervals on a real time line. Suppose I is one of the intervals. Let $\mathcal{A}_0$ be the set of intervals not intersecting I. Then, if $A, B \in \mathcal{A}_0$, then $A \cap B \neq \emptyset$. Thus any two intervals in $\mathcal{A}_0$ intersect each other.

Let $x = \max\{a : [a,b] \in \mathcal{A}_0\}$, $y = \min\{b : [a,b] \in \mathcal{A}_0\}$. Then $x \leq y$; otherwise there are two intervals in $\mathcal{A}_0$ that fail to intersect.

Suppose $A_0 = \cap\{A : A \in \mathcal{A}_0\} = [x,y]$. $A_0 = A_1 \cap A_2$ where A_1 and A_2 are in $\mathcal{A}_0$.

Suppose U, $V \notin \mathcal{A}_0\}$. If $U \cap V = \emptyset$, WOLOG, suppose A_1, A_2 and I are configured as in the diagram. Consider $\{U, V, A_2\}$. One of U and V has a nonvoid intersection with A_2. Since it also intersects I, it must intersect A_0. Augment to $\mathcal{A}_1$ by including it.



By applying the argument above to $\mathcal{A}_1$, we determine that we have $A_2 = \cap\{A : A \in \mathcal{A}_1\}$, a nonvoid set.

Continue on in this way, obtaining a family $\mathcal{A}_r$ with nonvoid intersection until any pair of sets not in $\mathcal{A}_r$ have nonvoid intersection. Then we can apply the argument above to the set of intervals not in $\mathcal{A}_r$ (including I) to get a common intersection for all these intervals.

32 *Solution 1* Specify a certain person p.

Let S be the set of p's acquaintances and $m = \#S$ (the number of members of S).

Let T be the set of people not acquainted with p.

For $t \in T$, let A_t be the pair with whom p and t are acquainted. Clearly $A_t \subseteq S$. Suppose, if possible, $u \in T$ and $A_t = A_u$. Then the members of $A_t = A_u$ have three common acquaintances (p, t, u), which is impossible.

On the other hand, suppose $\{a, b\} \subseteq S$. Since a and b both know p, they are strangers. Hence they have exactly one other common acquaintance t. Since p, t have common acquaintances, $t \in T$ and $A_t = \{a, b\}$.

Thus, $t \to A_t$ is a one-one map of T onto pairs of elements of S. Thus,

$$\#T = \binom{m}{2} = \frac{m^2 - m}{2}.$$

Let n be the number of those present. Then $n = 1 + m + \binom{m}{2}$, and so $m^2 + m + 2(1 - n) = 0$. Thus

$$m = \frac{-1 + \sqrt{8n - 7}}{2}$$

(the second root is negative and extraneous). Therefore $8n - 7$ is a perfect square, so that $n = 16$ or $n = 22$. Note that the number of acquaintances for each person is the same.
Suppose $n = 16$. Then $m = 5$. The following table shows $n = 16$ is possible.

Person	***List of acquaintances***	***Person***	***List of acquaintances***
1	2, 3, 4, 5, 6	9	2, 5, 11, 13, 15
2	1, 7, 8, 9, 10	10	2, 6, 11, 12, 14
3	1, 7, 11, 12, 13	11	3, 4, 9, 10, 16
4	1, 8, 11, 14, 15	12	3, 5, 8, 10, 15
5	1, 9, 12, 14, 16	13	3, 6, 8, 9, 14
6	1, 10, 13, 15, 16	14	4, 5, 7, 10, 13
7	2, 3, 14, 15, 16	15	4, 6, 7, 9, 12
8	2, 4, 12, 13, 16	16	5, 6, 7, 8, 11

Suppose $n = 22$. Then $m = 6$.

Let person 1 know 2, 3, 4, 5, 6, 7. Each pair of these six are not acquainted.

Suppose 8 and 1 are acquaintances of each of 2 and 3.

Then 8 knows 2 and 3 and four other people, each of whom is acquainted with two of $\{2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ (since these four are not acquainted with 1, and the pair $(1,8)$ of strangers has two common acquaintances). However, none of the four knows 2 or 3 (since none of the four can have a common acquaintance with 8).

Hence 8 knows four of the common acquaintances of the six pairs $(4,5)$, $(4,6)$, $(4,7)$, $(5,6)$, $(5,7)$, $(6,7)$. Thus 8 is a stranger to one of the common acquaintances, say 9, who knows 4 and 5.

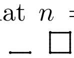
Now, 8 and 9 have two common acquaintances. But 9 does not know 2 or 3 (since 1 and 9 know 4 and 5) nor the common acquaintances of $(4,5)$, $(4,6)$, $(4,7)$, $(5,6)$, $(5,7)$.

Hence the only common acquaintance of 8 and 9 is the common acquaintance of 6 and 7.

This is untenable. Hence the case $n = 22$ is not possible. Thus $n = 16$.

Remark: From the first part, the candidates for n are

$1+1=2, 1+2+1=4, 1+3+3=7, 1+4+6=11,$
$1+5+10=16, 1+6+15=22, 1+7+21=29$, etc.

Note that $n = 2$ and $n = 4$ are possible (with respective acquaintances graphs ) but 7 is not. It would be interesting to characterize the possible n.

32 *Solution 2* [B. Marthi] Example for $n = 16$
Denote the sixteen individuals by $P_{i,j}$ $[0 \le i, j \le 3]$.

Suppose $P_{i,j}$ knows only $P_{i,j+1}, P_{i,j-1}, P_{i-1,j}, P_{i+2,j+2}$ where addition and subtraction are taken **modulo** 4.

This satisfies the conditions.

32 *Solution 3* [A. Chan] Proof that each person has the same number of acquaintances.
Suppose A and B are acquainted. Let B have r acquaintances.

Let C be acquainted to B. Then A and C have two common acquaintances, namely B and someone else, say D. The mapping $C \to D$ thus maps acquaintances of B to acquaintances of A.

This mapping is one-one. (Otherwise, suppose $C \to E, E \to D$ for two acquaintances C and E of B. Then B and D have three common acquaintances, namely A, C, E, a contradiction).

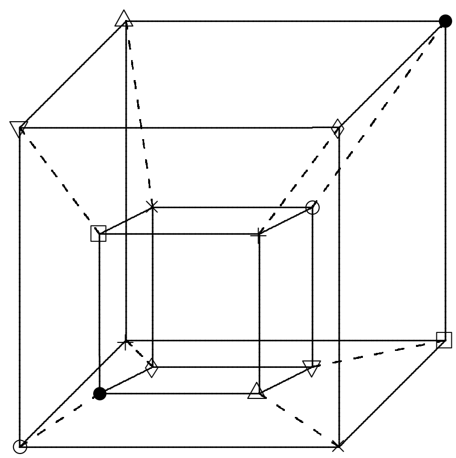
Thus B has no more acquaintances than A.

Reversing the roles of B and A, we see that A has no more acquaintances than A.

Hence A and B have the same number of acquaintances.

Suppose now that A and B are not acquainted. Then they have a common acquaintance C, and so A, C and B have the same number of acquaintances.

32 *Solution 4* [D. Robbins] Example for $n = 16$.
The graph is that of a 4 dimensional hypercube with additional edges joining each opposite pairs of vertices.



33 *Solution 1* Each choice of k numbers corresponds to an ordered n–tuple $(x_1, x_2, x_3, \cdots, x_n)$ with k 1's corresponding to indices chosen and $(n-k)$ 0's corresponding to indices not chosen.

An ordered n–tuple is acceptable if and only if it does not have three 1's in a row. We need to count the number of acceptable n–tuples.

Let a be the number of appearances of pairs 11, b be the number of appearances of singleton 1.

Then $2a + b = k$. We can arrange the a pairs and b singletons in order in $\binom{a+b}{a}$ distinct ways. Now we have to insert the 0's between and possibly at the ends of the designated pairs and singletons. Begin by placing $(a+b-1)$ 0's to make sure the pairs and singletons are separated. There are $((n-k) - (a+b-1)) = ((n+1) - (k+a+b))$ 0's left to be distributed into $a+b+1$ positions $[a+b-1$ between blocks of 1 plus 2 at the end$]$, provided of course $a + b + k = 3a + 2b \leq n + 1$.

To put r zeros in s positions, indicate $r+s-1$ slots, select $s-1$ of them to be position barriers and r of them to be zeros. This can be done in $\begin{pmatrix} r+s-1 \\ s-1 \end{pmatrix}$ ways. In this case, the $(n+1) - (k+a+b)$ zeros are put into $a+b+1$ positions in $\begin{pmatrix} n-k+1 \\ a+b \end{pmatrix}$ ways.

Hence the total number of acceptable n–tuples is

$$\begin{aligned} f(n,k) \quad &:= \sum_{\substack{a,\,b\,\geq\,0 \\ 2a\,+\,b\,=\,k \\ 3a\,+\,2b\,\leq\,n\,+\,1}} \binom{a+b}{a}\binom{n-k+1}{a+b} \\ &= \sum_{\substack{a\,\geq\,0 \\ 2a\,\leq\,k \\ a\,\geq\,2k\,-\,(n\,+\,1)}} \binom{k-a}{a}\binom{n-k+1}{k-a}. \end{aligned}$$

[This assumes $k \leq n$. The answer is 0 if $n \leq k$.]

Check

$$\begin{aligned} f(1,1) \quad &= \quad \binom{1}{0}\binom{1}{1} = 1; \\ f(n,1) \quad &= \quad \binom{1}{0}\binom{n}{1} = n; \\ f(2,2) \quad &= \quad \binom{1}{1}\binom{1}{1} \\ &= \quad 1 \;=\; \binom{2}{2}; \\ f(n,2) \quad &= \quad \binom{2}{0}\binom{n-1}{2} + \binom{1}{1}\binom{n-1}{1} \\ &= \quad \binom{n-1}{2} + (n-1) \;=\; \binom{n}{2} \text{ for } n \geq 3; \\ f(3,3) \quad &= \quad 0 \text{ since the sum requires } 2a \leq 3 \text{ and } a \geq 2; \\ f(4,3) \quad &= \quad \binom{2}{1}\binom{2}{2} = 2 \text{ since the sum requires } 2a \leq 3 \text{ and } a \geq 1; \\ f(n,3) \quad &= \quad \binom{3}{0}\binom{n-2}{3} + \binom{2}{1}\binom{n-2}{2} \\ &= \quad \frac{(n+2)(n-2)(n-3)}{6} \;=\; \binom{n}{3} - (n-2) \text{ for } n \geq 5. \end{aligned}$$

33 *Solution 2* [S. Cautis; C. Percival] As in solution 1, we need to find all arrangements of n 0's and 1's so that at most two ones are adjacent. This is equivalent to selecting

$$\left.\begin{array}{c} i \text{ blocks of } 110 \\ k-2i \text{ blocks of } 10 \\ n-2k+i+1 \text{ blocks of } 0 \end{array}\right\} \text{a total of } n-k+1 \text{ blocks,}$$

for a total of $n+1$ symbols and then neglecting the last symbol where $2i \leq k, i \geq 2k-1-n$. The answer is

$$\sum_{\substack{1 \le i \le \lfloor \frac{k}{2} \rfloor \\ i \ge 2k+1-n}} \frac{(n-k+1)!}{i!(k-2i)!(n-2k+i+1)!}.$$

34 *Solution 1* Let $\mathcal{A} = \{X : f(X) \subseteq X \subseteq S\}$. $\mathcal{A}$ is a nonvoid class of sets, since $S \in \mathcal{A}$.

Let $Z = \cap\{X : X \in \mathcal{A}\}$.

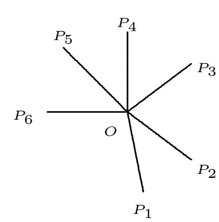
Observe that, if $X \in \mathcal{A}$, then $f(X) \in \mathcal{A}$ since $f(X) \subseteq X$ implies that $f(f(X)) \subseteq f(X)$ by the condition on f.

Observe also that, if $f(Y) \subseteq Y$, then $Y \in \mathcal{A}$, so $Z \subseteq Y$ (thus Z will turn out to be the miniumum set satisfying the condition).

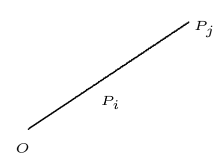
For each $X \in \mathcal{A}$ and $Z \subseteq X$, we have $f(Z) \subseteq f(X) \subseteq X$. Thus $f(Z) \subseteq \cap\{X : X \in \mathcal{A}\} = Z$. It follows that $Z \in \mathcal{A}$, and further that $f(Z) \in \mathcal{A}$. Therefore $Z \subseteq f(Z)$.

Hence $Z = f(Z)$ and this is the required set C.

35 *Solution 1* Suppose, if possible, that one point O is connected to six others $P_1, P_2, \cdots, P_6$, labelled so that the segments $OP_1, OP_2, \cdots, OP_6$ occur in counterclockwise order (as indicated in the diagram).



First, none of the angles $\angle P_iOP_{i+1}$ $(1 \le i \le 6,\ P_7 \equiv P_1)$ is zero. (Otherwise, if P_i is on the segment OP_j, then P_i is closer to O than P_j and P_i is closer to P_j than O, and OP_j would not have been joined.)



Consider the three points O, P_i, P_{i+1}

Either $\angle P_iOP_{i+1} > 180°$, or OP_iP_{i+1} is a triangle with $\angle P_iOP_{i+1} < 180°$. Suppose the latter.

If O is the closest point to both P_i and P_{i+1}, then $P_iP_{i+1} > P_iO$ and $P_iP_{i+1} > P_{i+1}O$. On the other hand, if say, P_i is the closest point to O, then O is the closest point to P_{i+1} and $OP_i < OP_{i+1} < P_iP_{i+1}$. In either case, P_iP_{i+1} is the longest side of $\triangle OP_iP_{i+1}$, so that $\angle P_iOP_{i+1}$ is the largest angle. Hence

$$\angle P_iOP_{i+1} > \frac{1}{3}(\angle P_iOP_{i+1} + \angle OP_iP_{i+1} + \angle OP_{i+1}P_i) = 60^0.$$

Hence $\sum_{i=1}^{6} P_iOP_{i+1} > 360°$ which is a contradiction.

Comments: Do not neglect to consider the case that $\angle P_iOP_{i+1}$ could be 0 or reflex. The argument does not preclude the possibility that O could be joined to more than six others.

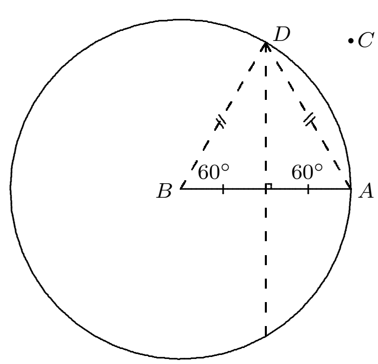
35 *Solution 2* Carry out the above argument for $i = 1, 2, \cdots, 5$, and get to

$$\sum_{i=1}^{5} P_iOP_{i+1} > 300°$$

Then P_6P_1 is not the longest side of $\triangle P_6OP_1$. Hence, $P_6P_1 < OP_1$ or $P_6P_1 < OP_6$. If $P_6P_1 < OP_1$, say, then P_1 must be the closest point to O (O is not closest to P_1)

Therefore $P_6P_1 < OP_1 < OP_6$ so that O is not closest to P_6, which implies that OP_6 should not have been joined, a contradiction.

35 *Solution 3* Suppose there exists a point A such that it is the closest point to each of two points B and C. Note that C must lie on the same side of right bisector of AB as A. (Otherwise B is closer to C than A).



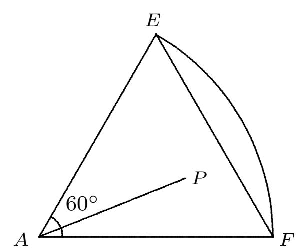
Also, C must lie outside the circle with centre B and radius AB. (Otherwise C is closer to B than A).

Hence $\angle CAB > \angle DAB = 60^0$.

Thus, there are at most five points for which A is the closest point. There is exactly one point closest to A, say P.

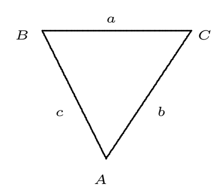
If the result fails, then there is a point A for which it closest to exactly five points and a point P closest to A. Let E be one of the five points for which $\angle PAE < 60°$. Form a triangle AEF as in the diagram with $\angle EAF = 60°, AE = AF$.

Since $AP < AE$, we obtain that P must lie in the sector AEF.



Thus, P must be closer to E than A, giving a contradiction.

35 *Solution 4* [W.L. Yee] Suppose there is a point A connected to six others. Then there are two points B, C of these six for which $\angle BAC < 60^0$. Let $AC > AB$, so $b > c$.

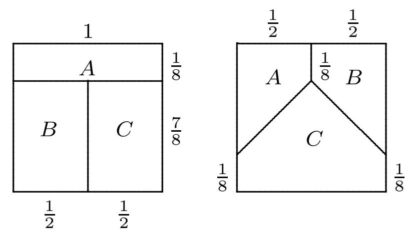


$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &< b^2 + c^2 - 2c^2\cos A \\ &= b^2 + c^2(1 - 2\cos A) \ < \ b^2, \end{aligned}$$

since $1 < 2\cos A$.

Thus $a < b$, $c < b$, so neither of A and C is closest to the other.
The result follows.

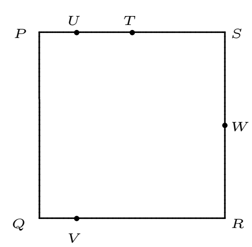
36 *Solution 1* (a)



(b) Let P, Q, R, S be the vertices of the unit square. Suppose it is possible that there are three sets that cover the square, all of diameter less than $\frac{\sqrt{65}}{8}$. By the Pigeonhole Principle, one set must cover two of the vertices, say P and Q, and these must be adjacent (since the distance between opposite vertices exceeds $\frac{\sqrt{65}}{8}$).

Let U, V, W be points indicated in the diagram, where

$$|PU| = \frac{1}{8} \quad |QV| = \frac{1}{8} \quad |SW| = \frac{1}{2} \quad |RW| = \frac{1}{2}.$$



Then $|PV| = |QU| = \frac{\sqrt{65}}{8}$ and $|UW| = |VW| = \frac{\sqrt{65}}{8}$.

The set which contains P and Q contains none of U, V and W. Hence, U, V and W must be contained in the union of the other two sets.

Since U and W, and since V and W, cannot belong to the same sets, U and V must both be in a second set and W in a third.

Since $|UR| > \frac{\sqrt{65}}{8}, |VS| > \frac{\sqrt{65}}{8}$, we obtain that R and S do not belong to the second set and hence must be in the third.

Consider the midpoint T of PS.

$$|TQ| > |TV| > \frac{\sqrt{65}}{8} \text{ and } |TR| > \frac{\sqrt{65}}{8}.$$

Thus, T cannot belong to any of the sets and we obtain a contradiction.

PROBLEM SET 4 — Miscellaneous

Problems

37 Determine a such that the roots of the equation

$$x^2 - (3a+1)x + (2a^2 - 3a - 2) = 0$$

are real and the sum of their squares is minimum.

38 Find a necessary and sufficient condition on the coefficients in the equation $ax^2 + bx + c = 0$ such that the square of one root is equal to the other root.

39 Find all possible finite sequences $(w_0, w_1, w_2, \cdots, w_n)$ of nonnegative integers with the property that, for each $i = 0, 1, 2, \cdots, n$, the integer i appears w_i times and no other integer appears.

40 A function $f(x)$ is *periodic with period* p if and only if $f(x+p) = f(x)$ for each x. Prove that $\sin x^2$ is not periodic with any non-zero period.

41 (a) A set S contains a finite number of elements. For any pair a, b of elements in S, an operation $*$ is defined such that $a * b$ is in S. This operation is associative (that is, $a*(b*c) = (a*b)*c$). Prove that S contains an element u for which $u * u = u$.
(b) Give an example to show that (a) fails if we drop the condition that S is finite.

42 The real-valued function f satisifies

$$f(\tan 2x) = \tan^4 x + \cot^4 x$$

for all real x. Prove that

$$f(\sin x) + f(\cos x) \geq 196$$

for all real x.

43 The *centre of gravity* of points $(x_1, y_1), (x_2, y_2), \cdots, (x_k, y_k)$ in the plane is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{k}\sum_{i=1}^{k} x_i \qquad \text{and} \qquad \overline{y} = \frac{1}{k}\sum_{i=1}^{k} y_i.$$

A *lattice point* in the plane is a point (x, y) both of whose coordinates are integers.
(a) For any positive integer n, find n lattice points in the plane for which the centre of gravity of any nonempty subset of these points is a lattice point.

(b) Is it possible to find an *infinite* set of lattice points in the plane for which the centre of gravity of any finite subset of them is a lattice point?

44 Determine all solutions in *integer* triples (x, y, z) of the equation

$$\sqrt[3]{x+\sqrt{y}}+\sqrt[3]{x-\sqrt{y}}=z.$$

45 Determine necessary and sufficient conditions on the real numbers a, b, c such that the system of equations

$$ax+by+cz=0$$
$$a\sqrt{1-x^2}+b\sqrt{1-y^2}+c\sqrt{1-z^2}=0$$

admits a real solution (x, y, z).

46 Find all integer triples (x, y, z) that satisfy the system

$$3=x+y+z=x^3+y^3+z^3.$$

47 Let f be a function defined on the positive reals with the following properties:

$$(1)\ f(1)=1;\quad (2)\ f(x+1)=xf(x);\quad (3)\ f(x)=10^{g(x)};$$

where $g(x)$ is a function defined on the reals satisfying

$$g(ty+(1-t)z)\le tg(y)+(1-t)g(z)$$

for all y and z and for $0\le t\le 1$.

(a) Prove that

$$t\ [g(n)-g(n-1)]\le g(n+t)-g(n)\le t\ [g(n+1)-g(n)]$$

where n is an integer and $0\le t\le 1$.

(b) Prove that $\frac{4}{3}\le f\left(\frac{1}{2}\right)\le\frac{4}{3}\sqrt{2}$.

48 Let $x_1, x_2, \cdots, x_n$ be positive real numbers. Prove that

$$\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\cdots+\frac{x_{n-1}^2}{x_{n-1}+x_n}+\frac{x_n^2}{x_n+x_1}$$
$$\ge\ \tfrac{1}{2}(x_1+x_2+\cdots+x_n).$$

Solutions

37 *Solution 1* The roots of $x^2-(3a+1)x+(2a^2-3a-2)=0$ are real if and only if $(3a+1)^2-4(2a^2-3a-2)\geq 0$,

that is, if $a^2+18a+9\geq 0$ or $(a+9)^2\geq 72$.

This occurs when $a\leq -9-6\sqrt{2}$ and $a\geq -9+6\sqrt{2}$.

Let r, s be the roots of the polynomial. Then

$$\begin{aligned} r^2+s^2 &= (r+s)^2-2rs \\ &= (3a+1)^2-2(2a^2-3a-2) \\ &= 5a^2+12a+5 \\ &= 5\left(a+\frac{6}{5}\right)^2-\frac{11}{5}. \end{aligned}$$

The overall minimum of r^2+s^2 occurs at $a=-\frac{6}{5}$ and the minimum, subject to the constraint, occurs for the value of a closest to $-\frac{6}{5}$ for which $a\leq -9-6\sqrt{2}$ or $a\geq -9+6\sqrt{2}$, namely

$$a=-9+6\sqrt{2}.$$

38 *Solution 1* Let the roots of $ax^2+bx+c=0$ be r and s. One root is the square of the other if and only if

$$0=(r-s^2)(r^2-s)=r^3+s^3-rs-r^2s^2.$$

This is equivalent to

$$\begin{aligned} 0 &= \left[\left(-\frac{b}{a}\right)^3-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)\right]-\frac{c}{a}-\frac{c^2}{a^2} \\ &= -\frac{1}{a^3}[a^2c+ac^2+b^3-3abc], \end{aligned}$$

or

$$0=b^3+a^2c+ac^2-3abc.$$

38 *Solution 2* Suppose the roots of $ax^2+bx+c=0$ are u and u^2. Then $u+u^2=-\frac{b}{a}$, so $au^2+au+b=0$. Thus, u is a common root of the two equations

$$ax^2+bx+c=0,$$

$$ax^2+ax+b=0,$$

and so $(b-a)u=b-c$.

If $b=a$, then $b=c$ and u is a root of $x^2+x+1=0$ (that is u and u^2 are $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$ respectively).

Otherwise, $u = \frac{b-c}{b-a}$, so that

$$0 = au^2 + au + b = a\left[\left(\frac{b-c}{b-a}\right)^2 + \left(\frac{b-c}{b-a}\right)\right] + b.$$

Thus

$$0 = \frac{1}{(b-a)^2}\left[b^3 + ac^2 + a^2c - 3abc\right],$$

which implies that

$$b^3 + a^2c + ac^2 - 3abc = 0.$$

(Note this is satisified when $a = b = c$.)

On the other hand, let u and v be the roots of $ax^2 + bx + c = 0$, where $b^3 + a^2c + ac^2 - 3abc = 0$. Then $b = -a(u+v)$, $c = auv$, so that

$$\begin{aligned} 0 &= -(u+v)^3 + uv + u^2v^2 + 3uv(u+v) \\ &= uv + u^2v^2 - u^3 - v^3 \\ &= (u - v^2)(v - u^2). \end{aligned}$$

Hence, either $u = v^2$ or $v = u^2$.

39 *Solution 1* The sum of all the numbers is the total of the frequencies of the numbers, which is $n+1$. Hence

$$\begin{aligned} n+1 &= w_0 + w_1 + w_2 + \cdots + w_n \\ &= 0.w_0 + 1.w_1 + \cdots + n.w_n, \end{aligned}$$

whence $w_0 = w_2 + 2w_3 + 3w_4 + \cdots + (n-1)w_n$.
Consider the possible values of w_0.

(i) $w_0 = 0$. This is impossible (since $w_0 = 0$ implies that the number of 0's exceeds 0, so that $w_0 > 0$).

(ii) $w_0 = 1$. This implies that $w_2 = 1$ and $w_i = 0$ $(2 \le i \le n)$. From this we get that $w_1 = 2$, $n = 3$ and so we have (1 2 1 0).

(iii) $w_0 = 2$. This implies that $w_2 \ge 1$.
Thus $w_3 = w_4 = \cdots = w_n = 0$ (since $2w_3 + \cdots + (n-1)w_n \le 1$). Therefore $w_2 = w_0 = 2$.
There are two possibilities: (2 0 2 0), (2 1 2 0 0).

(iv) $w_0 \ge 3$. Let $w_0 = k$. Then $k \ge (k-1)w_k$. This implies that $w_k \le \frac{k}{k-1} = 1 + \frac{1}{k-1} < 2$, so that $w_k = 1$.
Thus

$$\begin{aligned} k - (k-1) &= 1 \\ &= w_2 + 2w_3 + \cdots + (k-2)w_{k-1} + kw_{k+1} \\ &\qquad + \cdots + (n-1)w_n. \end{aligned}$$

Therefore $w_2 = 1$ and $w_3 = w_4 = \cdots = w_n = 0$ implying that $w_1 = 2$. Thus, there are $n-3$ zero, (namely all of $w_3, w_4, \cdots, w_n$ except w_k) so that $k = n-3$ yielding that $n \geq 6$.

Hence the remaining possibilities are:

$$(n{-}3\ 2\ 1\ 0\ 0\ 0\ \ldots\ 0\ 1\ 0\ 0\ 0) \quad (n \geq 6).$$

For example, we have

$$(3\ 2\ 1\ 1\ 0\ 0\ 0),\ (4\ 2\ 1\ 0\ 1\ 0\ 0\ 0),\ (5\ 2\ 1\ 0\ 0\ 1\ 0\ 0\ 0),\ \text{ etc.}$$

There are no sequences with $n = 0, 1, 2, 5$.

39 *Solution 2* Let $S = \{i : w_i = 0\}$. Then $0 \notin S$ and

$$w_0 + \sum\{w_j : j \notin S, j \geq 1\} = \sum_{i=0}^{n} w_i = n+1.$$

Therefore

$$\sum\{w_j : j \notin S, j \geq 1\} = n+1-w_0.$$

Now S has w_0 elements, so that

$$\{w_j : j \notin S, j \geq 1\}$$

has

$$(n+1)-(w_0+1) = (n+1-w_0)-1$$

elements, all of which are positive.

The only way that $(n+1-w_0)-1$ elements can add up to $n+1-w_0$ is for there to be $(n+1-w_0-2)$ "1's" and 1 "2".

Hence $w_i = 0$ when $i \neq w_0, i \geq 3$.

Consider cases:

(i) $w_1 = 0$. This implies that $w_2 \neq 1$, so that $w_2 = 2$ and that $w_0 = 2$, and so we get (2 0 2 0).

(ii) $w_1 = 1$. This implies that $w_2 \neq 1$, so that $w_2 = 2$ and so we get (2 1 2 0 0).

(iii) $w_1 = 2$ and $w_0 = 1$. This implies that $w_2 = 1$, and so we get (1 2 1 0 0).

(iv) $w_1 = 2$ and $w_0 = k \geq 2$. This implies that $w_2 = w_k = 1$, so that $k \neq 2$, and so we get

$$(k\ 2\ 1\ \underbrace{0\ 0\ \ldots\ 0\ 0\ 0}_{k-3\ \text{0's}}\ 1\ 0\ 0\ 0).$$

40 *Solution 1* Let $f(x) = \sin(x^2)$, and suppose if possible there exists a $p > 0$ with $f(x+p) = f(x)$ for all x. Then $f(2p) = f(p) = f(0)$, so $p = \sqrt{n\pi}$ for some positive integer n.

Now, $f(x) = 0$ for $0 \le x \le 2p$. This occurs exactly when

$$x = 0, \sqrt{\pi}, \sqrt{2\pi}, \cdots \sqrt{4n\pi}.$$

Thus $f(x) = 0$ has $n-1$ solutions when $0 < x < p$, and $4n - n - 1 = 3n - 1$ solutions when $p < x < 2p$.

But this contradicts the periodicity of f (which would force the same number of solutions in both intervals).

40 *Solution 2* Let p be a period for f. For all x, we have

$$0 = \sin((x+p)^2) - \sin((x-p)^2) = 2\cos(p^2 + x^2)\sin 2px.$$

Consider any interval $I = \{x : a \le x \le b\}$. Now $\cos(p^2 + x^2)$ vanishes in I for at most finitely many values of x. Hence $\sin 2px = 0$ for infinitely many values of x. But this occurs only if $p = 0$. Hence f cannot have a nonzero period.

40 *Solution 3* [A. Tang] Note that $\sin x^2$ has a relative minimum value of 0 exactly once at $x = 0$. If $\sin x^2$ were periodic, this same relative minimum value would occur infinitely often (once in each period). Hence, $\sin x^2$ cannot be periodic.

41 *Solution 1* (a) Let v be any element of S and define

$$v^m = ((((v * v) * v) * v) * v) \text{ with } m\ v\text{'s}.$$

Because of associatively, it does not matter how the brackets are arranged and we get $v^{m+n} = v^m v^n$ for any pair of positive integers m, n.

Consider the set $\{v, v^2, v^3, \cdots, v^r, \cdots\}$ of all "powers" of v. Since S is finite, not all are distinct. Let v^p be the first power equal to another power, say v^q $(p < q)$. Then

$$\begin{aligned} v^p &= v^q = v^p v^{q-p} = v^q v^{q-p} \\ &= v^{2q-p} = v^p v^{2q-2p} = v^{3q-2p} = \cdots \\ &= v^{(k+1)q-kp} (k = 1, 2, \cdots). \end{aligned}$$

Choose k sufficiently large to make $\dfrac{q}{p} > \dfrac{k+2}{k+1}$ so that

$$x = (k+1)q - (k+2)p > 0.$$

Thus $2p + x = (k+1)q - kp$. Hence

$$\begin{aligned} (v^{p+x})^2 &= v^{2p+2x} \\ &= v^{(k+1)q-kp+x} = v^{(k+1)q-kp} v^x \\ &= v^p v^x = v^{p+x}. \end{aligned}$$

Hence $u = v^{p+x}$ is the required element.

A counterexample to show that the result could fail when S is infinite is $S = \{1, 2, 3, 4, \cdots, n, \cdots\}$ with the operation of addition.

41 *Solution 2* [S. Cautis] Determine v^p, v^q as above. Let $w = v^p$. Then

$$w = v^p = v^{pq-(p-1)p} = (v^p)^{q-p+1} = w^{q-p+1} = w^{r+1}$$

where $r = q - p$.

Then $w^r = \left(w^{r+1}\right)^r = \left(w^{r+1}\right)\left(w^{r+1}\right)^{r-1} = w^{r+1}w^{r-1} = w^{2r}$ since $w^{r+1} = w$.

$u = w^r$ is the required element.

41 *Solution 3* [D. Kisman] Two powers among $v, v^2, v^4, v^8, \cdots$ must be equal. Let them be v^p, v^q where $p < q$ and p, q are both powers of 2. Then $q \geq 2p$. Let $u = v^{q-p}$.

Then $u^2 = v^{2q-2p} = v^q v^{q-2p} = v^p v^{q-2p} = v^{q-p} = u$.

41 *Solution 4* [A. Chan] As in 41 Solution 1, select v, p, q such that $p < q$ and $v^p = v^q$.

Then $v^{r+p} = v^{r+q}$ $(r \geq 0)$. It follows that $v^n = v^n v^{q-p}$ for every positive integer $n \geq p$. Multiplying by v^{q-p}, we find that

$$v^n = v^n v^{q-p} = v^n v^{2(q-p)} = \cdots = v^n v^{s(q-p)} \ (s \geq 1).$$

Choose s sufficiently large that $s(q-p) \geq p$ and set $n = s(q-p)$. Then $u = v^{s(q-p)}$ is the desired element.

42 *Solution 1* Let $u = \tan 2x$ and $v = \tan x$. Then $u = \dfrac{2v}{1-v^2}$, so that

$$v^2 + \left(\frac{2}{u}\right) v - 1 = 0.$$

For each u, v may assume one of two values v_1 and v_2, where $v_1 v_2 = -1$ and $v_1 + v_2 = -\frac{2}{u}$.

For $i = 1, 2$, we have that

$$\begin{aligned} v_i^2 &= 1 - \left(\frac{2}{u}\right) v_i \\ v_i^3 &= v_i - \left(\frac{2}{u}\right) v_i^2 \\ v_i^4 &= v_i^2 - \left(\frac{2}{u}\right) v_i^3 \end{aligned}$$

whence

$$\begin{aligned}
v_1^2 + v_2^2 &= 2 - \frac{2}{u}(v_1 + v_2) \ = \ 2 + \frac{4}{u^2}, \\
v_1^3 + v_2^3 &= (v_1 + v_2) - \left(\frac{2}{u}\right)(v_1^2 + v_2^2) \\
&= -\frac{2}{u}\left(3 + \frac{4}{u^2}\right), \\
v_1^4 + v_2^4 &= (v_1^2 + v_2^2) - \left(\frac{2}{u}\right)(v_1^3 + v_2^3) \\
&= \left(2 + \frac{4}{u^2}\right) + \frac{4}{u^2}\left(3 + \frac{4}{u^2}\right) \\
&= \frac{2u^4 + 16u^2 + 16}{u^4}.
\end{aligned}$$

Thus $f(u) = \dfrac{2u^4 + 16u^2 + 16}{u^4}$ is well defined when $u \neq 0$.
(When $u = 0$, $\tan 2x = 0$ so x is a multiple of $\frac{\pi}{2}$. If we take $f(0) = \infty$, then the result is trivially obvious.)

Suppose $y = \sin x, z = \cos x$. Then $y^2 + z^2 = 1$, and

$$\begin{aligned}
f(y) + f(z) &= \left(2 + \frac{16}{y^2} + \frac{16}{y^4}\right) + \left(2 + \frac{16}{z^2} + \frac{16}{z^4}\right) \\
&= 4 + \frac{16}{y^2z^2} + \frac{16(y^4 + z^4)}{y^4z^4} \\
&= 4 + \frac{16}{y^2z^2} + \frac{16[(y^2 + z^2)^2 - 2y^2z^2]}{y^4z^4} \\
&= 4 - \frac{16}{y^2z^2} + \frac{16}{y^4z^4} \\
&= 4\left[\frac{2}{y^2z^2} - 1\right]^2.
\end{aligned}$$

By the arithmetic-geometric mean inequality, we have

$$2|yz| \leq (y^2 + z^2)^2 = 1,$$

so that $|yz| \leq \dfrac{1}{2}$.

Therefore $y^2z^2 \leq \dfrac{1}{4}$, and further, $\dfrac{2}{y^2x^2} - 1 \geq 8 - 1 = 7$.

Hence $f(\sin x) + f(\cos x) \geq 4 \cdot 7^2 = 196$ with equality if and only if $\sin x = \cos x = \pm\dfrac{1}{\sqrt{2}}$.

Alternatively:

$$\begin{aligned} f(\sin x)+f(\cos x) &= 4\left[\frac{2}{\sin^2 x\cos^2 x}-1\right]^2 \\ &= 4\left[\frac{8}{\sin^2 2x}-1\right]^2 \;\geq\; 4\cdot 7^2 \;=\; 196. \end{aligned}$$

42 *Solution 2* [S. Yazdani] With u and v as above

$$\begin{aligned} f(u) &= v^4+\frac{1}{v^4} \;=\; \left(v-\frac{1}{v}\right)^4+4\left(v-\frac{1}{v}\right)^2+2 \\ &= \left(\frac{v^2-1}{v}\right)^4+4\left(\frac{v^2-1}{v}\right)^2+2 \\ &= \frac{16}{u^4}+\frac{16}{u^2}+2. \end{aligned}$$

Complete as above.

43 *Solution 1* It is required to find a pair of finite sequences.

$$\{x_1,x_2,\cdots,x_n\} \text{ and } \{y_1,y_2,\cdots,y_n\}$$

such that, if K is any subset of $\{1,2,\cdots,n\}$ with k members

$$\sum_{i\in K} x_i \equiv \sum_{j\in K} y_j \equiv 0 (\text{mod } k).$$

This can be achieved by arranging, for example, that each x_i and y_j is divisible by each k with $1\leq k\leq n$. Accordingly, let $x_i=a_i n!$, $y_i=b_i n!$ where (a_i,b_i) are distinct pairs of integers. The required n lattice points are

$$(x_i,y_i) \qquad (1\leq i\leq n).$$

We show by contradiction that it is not possible to find an infinite set of distinct lattice points for which the centre of gravity of every finite subset is a lattice point. Suppose, if possible, such a set exists and let (r,s) and (u,v) be any pair of points in the set.

Let k be any positive integer. Choose $k-1$ other points from the set, say $(x_1,y_1),\cdots,(x_{k-1},y_{k-1})$. Then

$$\begin{aligned} r+x_1+\cdots+x_{k-1} &\equiv 0(\text{mod } k), \\ u+x_1+\cdots+x_{k-1} &\equiv 0(\text{mod } k), \end{aligned}$$

whence $r\equiv u(\text{mod } k)$. Similarly $s\equiv v(\text{mod } k)$.

Thus $r-u$ and $s-v$ are each divisible by each integer k, so $r=u$ and $s=v$. But this contradicts the fact that (r,s) and (u,v) are supposed to be distinct.

43 *Solution 2* [C. Percival] Follow the above argument, taking $k \geq 1 + \textbf{dist}\ [(r,s),(u,v)]$, and observe that $\text{dist}\ [(r,s),(u,v)] < k$ is not possible when $r-u, s-v$ are multiples of k of which at least one is not zero.

44 We first note that the equation is satisfied by $(x,y,z) = (0,y,0)$ for any integer y. In what follows, we exclude this case.

44 *Solution 1* Let $u = \sqrt[3]{x+\sqrt{y}}, v = \sqrt[3]{x-\sqrt{y}}$. Then

(1) $u + v = z$.

(2) $u^3 + v^3 + 3uv(u+v) = z^3$. Therefore

$$3uvz = z^3 - u^3 - v^3 = z^3 - 2x$$

so that uv is a rational number.

[We have excluded $z = z^3 - 2x = 0$.] Let $uv = \frac{p}{q}$ in lowest terms.

(3) $u^3v^3 = (x+\sqrt{y})(x-\sqrt{y}) = x^2 - y$, which is an integer. Therefore $p^3 = (x^2-y)q^3$. Now any prime dividing q must divide p. Since $\gcd(p,q) = 1$, we have $q = 1$. Hence $uv = p$, which is an integer.

Thus u and v are roots of the quadratic expression $t^2 - zt + p$. Hence $2u = z + \sqrt{w}$, $2v = z - \sqrt{w}$ where $w = z^2 - 4p$, which can be positive or negative.

[If $w \geq 0$ the choice of signs is determined by $u \geq v$; if $w < 0$ we choose by convention.]

Hence

$$\begin{aligned} 8(x+\sqrt{y}) &= 8u^3 = (z^3 + 3zw) + (3z^2 + w)\sqrt{w}, \\ 8(x-\sqrt{y}) &= 8v^3 = (z^3 + 3zw) - (3z^2 + w)\sqrt{w}, \end{aligned}$$

so that $8x = z^3 + 3zw = z(z^2 + 3w)$ and $8\sqrt{y} = (3z^2 + w)\sqrt{w}$.

Thus $64y = (3z^2 + w)^2 w$.

Thus, if there is a solution, we must have

$(*)$ $\quad (x,y,z) = \left(\frac{1}{8}z(z^2+3w), \frac{1}{64}w(3z^2+w)^2, z\right)$ for suitable w, z.

On the other hand, $(*)$ yields solutions since

$$\begin{aligned} \sqrt[3]{x+\sqrt{y}} + \sqrt[3]{x-\sqrt{y}} &= \frac{1}{2}\left[\sqrt[3]{(z^3+3wz) + (3z^2+w)\sqrt{w}}\right. \\ &\qquad \left. + \sqrt[3]{(z^3+3wz) - (3z^2+w)\sqrt{w}}\right] \\ &= \frac{1}{2}\left[\sqrt[3]{(z+\sqrt{w})^3} + \sqrt[3]{(z-\sqrt{w})^3}\right] \\ &= \frac{1}{2}\left[(z+\sqrt{w}) + (z-\sqrt{w})\right] \\ &= z. \end{aligned}$$

We impose conditions to ensure that x, y, z are integers:

Suppose that z is even. Let $z = 2s$.

Then $0 \equiv (3z^2 + w)^2 w = (12s^2 + w)^2 w \equiv 0 (\text{mod } 64)$. Therefore w is even. Let $w = 2r$, so that $(6s^2 + r)^2 r \equiv 0 (\text{mod } 8)$. Thus r is even, yielding that $4|w$, so that $w = 4t$.

Suppose now that z is odd.

Then $z(z^2 + 3w) \equiv 0 (\text{mod } 8)$, yielding that

$1 + 3w = z^2 + 3w \equiv 0 (\text{mod } 8)$, so that $w \equiv 5 (\text{mod } 8)$.

We find that the integer solutions are given by $(*)$ where

$$(z, w) = (2s, 4t) \quad \text{or} \quad (z, w) = (2k + 1, 8l + 5)$$

for integers s, t, k, l.

Examples: (i) $(z, w) = (1, 5)$. Then $(x, y, z) = (2, 5, 1)$, and $u = \frac{1}{2}(1 + \sqrt{5})$, $u^3 = 2 + \sqrt{5}$.

(ii) $w = 0$. This implies that $(x, y, z) = (s^3, 0, 2s)$.

(iii) $(z, w) = (6, -4)$. This leads to $u = 3 + i, v = 3 - i$, and further to $(x, y, z) = (18, -676, 6)$.

44 *Solution 2*

$$\sqrt[3]{x + \sqrt{y}} + \sqrt[3]{x - \sqrt{y}} \quad = \quad z. \tag{1}$$

Therefore

$$2x + 3z\sqrt[3]{x^2 - y} \quad = \quad z^3.$$

This is equivalent to

$$\sqrt[3]{x^2 - y} \quad = \quad \frac{z^3 - 2x}{3z},$$

which is equivalent to

$$y \quad = \quad x^2 - \left(\frac{z^3 - 2x}{3z}\right)^3. \tag{2}$$

To solve the given equation, we need to solve (2). This can be done by selecting any integer values of x and z for which $z^3 - 2x$ is a multiple of $3z$. It is convenient to look at case of z modulo 6. A summary of the result follows.

(i) $z = 6a$. Since $z^3 \equiv 0 \ (\text{mod } 3z)$, we require that $2x \equiv 0 \ (\text{mod } 18a)$, or that $x = 9ab$ for some integer b. This yields the solution

$$\begin{aligned} (x, y, z) &= (9ab, 81a^2b^2 - (12a^2 - b)^3, 6a) \\ &= (9ab, (b - 3a^2)(24a^2 + b)^2, 6a). \end{aligned}$$

(ii) $z = 6a + 1$. We require that $2x \equiv z^3 \equiv 1 \pmod 3$ and $x \equiv 0 \pmod{6a+1}$, so that $x = (6a+1)(3b+2)$. Thus

$$\begin{aligned}(x, y, z) &= ((6a+1)(3b+2),\\ &\qquad (18ab + 12a + 3b + 2)^2 - (12a^2 + 4a - 2b - 1)^3,\\ &\qquad\quad 6a+1)\\ &= ((6a+1)(3b+2),\\ &\qquad (8b + 5 - 4a - 12a^2)(12a^2 + 4a + b + 1)^2,\\ &\qquad\quad 6a+1).\end{aligned}$$

For example, $(a, b) = (0, 0)$ yields $(x, y, z) = (2, 5, 1)$, while $(a, b) = (1, 1)$ yields $(35, -972, 7)$.

(iii) $z = 6a + 2$. Then $2x \equiv z^3 \equiv 2 \pmod 3$ and $x \equiv 0 \pmod{3a+1}$, so that

$$\begin{aligned}(x, y, z) &= ((3a+1)(3b+1),\\ &\qquad (9ab + 3a + 3b + 1)^2 - (12a^2 + 8a + 1 - b)^3,\\ &\qquad\quad 2(3a+1))\\ &= ((3a+1)(3b+1),\\ &\qquad (b - 3a^2 - 2a)(24a^2 + 16a + 3 + b)^2,\\ &\qquad\quad 2(3a+1)).\end{aligned}$$

For example, $(a, b) = (0, 0)$ yields $(x, y, z) = (1, 0, 2)$, $(a, b) = (0, 1)$ yields $(x, y, z) = (4, 16, 2)$, and $(a, b) = (1, 3)$ yields $(x, y, z) = (40, -4232, 8)$.

(iv) $z = 6a + 3$. Then $2x \equiv 0 \pmod{3z}$, so that

$$\begin{aligned}(x, y, z) &= (9b(2a+1),\\ &\qquad 81(2ab + b)^2 - (12a^2 + 12a + 3 - 2b)^3,\\ &\qquad\quad 3(2a+1))\\ &= (9b(2a+1),\\ &\qquad (8b - 12a^2 - 12a - 3)(12a^2 + 12a + 3 + b)^2,\\ &\qquad\quad 3(2a+1)).\end{aligned}$$

For example, $(a, b) = (0, 1)$ yields $(x, y, z) = (9, 80, 3)$.

(v) $z = 6a + 4$. Then $2x \equiv 1 \pmod 3$ and $x \equiv 0 \pmod{3a+2}$, so that

$$\begin{aligned}(x, y, z) &= ((3a+2)(3b+1),\\ &\qquad (9ab + 3a + 6b + 2)^2 - (12a^2 + 16a + 5 - b)^3,\\ &\qquad\quad 2(3a+2))\\ &= ((3a+2)(3b+1),\\ &\qquad (b - 1 - 3a^2 - 4a)(24a^2 + 32a + 11 + b)^2,\\ &\qquad\quad 2(3a+2)).\end{aligned}$$

For example, $(a, b) = (0, 0)$ yields $(x, y, z) = (2, -121, 4)$, and $(a, b) = (0, 2)$ yields $(x, y, z) = (14, 169, 4)$.

(vi) $z = 6a + 5$. Then $2x \equiv 2 \pmod 3$ and $x \equiv 0 \pmod{6a+5}$, so that

$$\begin{aligned}
&(x, y, z)\\
&\quad= ((6a+5)(3b+2),\\
&\qquad\quad (18ab+12a+15b+10)^2-(12a^2+20a+7-2b)^3,\\
&\qquad\quad\ 6a+5)\\
&\quad= ((6a+5)(3b+2),\\
&\qquad\quad (8b-3-12a^2-20a)(12a^2+20a+9+b)^2,\\
&\qquad\quad\ 6a+5).
\end{aligned}$$

For example, $(a, b) = (0, 1)$ yields $(x, y, z) = (25, 500, 5)$.

Comments: A little care is needed with the equivalence of equations (1) and (2). Setting $x = -1$, $z = 1$ in (2) leads to $y = 0$, but substituting $x = -1$, $y = 0$ in the left side of (1) apparently leads to $z = -2$. However, (1) and (2) can be reconciled if we are careful with our choice of cube roots. When $y > 0$, both terms of the left side of (1) have real cube roots and there is only one choice of cube roots to make the left side real. In this case, (1) and (2) are manifestly equivalent.

When $y < 0$, there are three choices of the cube root pairs (all non-real) that make the left side real, while if $y > 0$, we also have different possible choices of cube root. Thus, for example,

$$(x, y, z) \ = \ (-1, 0, 1)$$

is a solution of (1) if we make the choices $\frac{1}{2}(1+i\sqrt{3})$ and $\frac{1}{2}(1-i\sqrt{3})$ for the cube roots of -1.

Let us specialize to the case $z = 1$ and consider the relationship between equations (1) and (2).

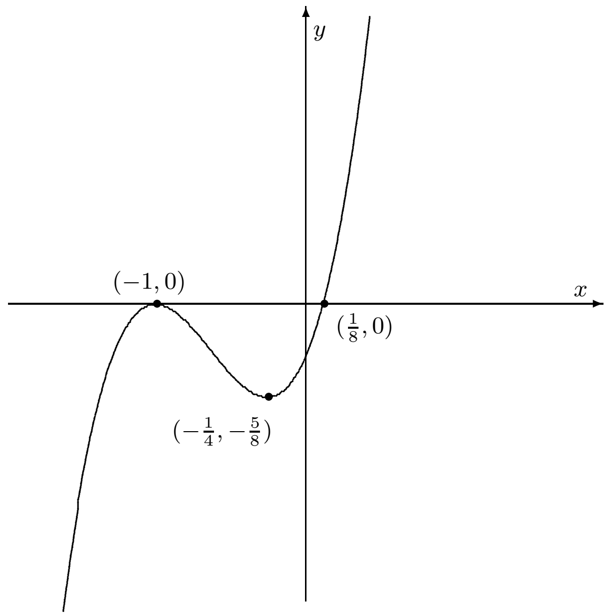
Equation (2) becomes:

$$\begin{aligned}
y &= x^2 - \frac{1}{27}(1-2x)^3\\
&= \frac{1}{27}(27x^2 + 8x^3 - 12x^2 + 6x - 1)\\
&= \frac{1}{27}(8x^3 + 15x^2 + 6x - 1)\\
&= \frac{1}{27}(x+1)^2(8x-1),
\end{aligned}$$

giving

$$\begin{aligned}
y' &= \frac{1}{27}(24x^2 + 30x + 6)\\
&= \frac{2}{9}(4x^2 + 5x + 1)\\
&= \frac{2}{9}(4x+1)(x+1).
\end{aligned}$$

The graph of equation (2) is shown in the diagram on page 94.



If $y > 0$, then $x + \sqrt{y}, x - \sqrt{y}$, $\sqrt[3]{x + \sqrt{y}} + \sqrt[3]{x - \sqrt{y}}$ are all real. Using the fact that cubing is a one-one function on the reals, we see that equations (1) and (2) are equivalent. Thus, for example,

$$(x, y) = (2, 5),\ (5, 52) \quad \text{or} \quad (8, 189)$$

satisfies both equations.

However $(x, y) = (-1, 0)$ satisfies equation (2) but not equation (1). To obtain equation (1) with $(x, y) = (-1, 0)$, we need to take imaginary cube roots. Thus

$$\sqrt[3]{-1} + \sqrt[3]{-1} = (-1)(\omega) + (-1)(\omega^2) = (-1)(\omega + \omega^2) = 1,$$

where ω is an imaginary cube root of 1.

In general, if $z = k$, a real, then the equation

$$\sqrt[3]{x + \sqrt{y}} + \sqrt[3]{x - \sqrt{y}} = k$$

is equivalent to

$$\sqrt[3]{\left(\frac{x}{k^3}\right) + \sqrt{\frac{y}{k^6}}} + \sqrt[3]{\left(\frac{x}{k^3}\right) - \sqrt{\frac{y}{k^6}}} = 1$$

when x and y are real.

When $y > 0$, as before, this is equivalent to

$$y = x^2 - \left(\frac{k^3 - 2x}{3k}\right)^3,$$

and so to

$$\frac{y}{k^6} = \left(\frac{x}{k^3}\right)^2 - \left(\frac{1 - 2(x/k^3)}{3k}\right)^3.$$

Thus the mapping $(x, y) \to \left(\frac{x}{k^3}, \frac{y}{k^6}\right)$ changing horizontal and vertical scales takes the locus to the above diagram.

When $x = \dfrac{z^3}{8}$, we find

$$y = \frac{z^6}{64} - \left(\frac{z^3 - z^3/4}{3z}\right) = \frac{z^6}{64} - \left(\frac{z^2}{4}\right)^3 = 0,$$

and indeed we have

$$\sqrt[3]{x + \sqrt{y}} + \sqrt[3]{x - \sqrt{y}} = \frac{z}{2} + \frac{z}{2} = z$$

as desired.

Let $y = -r^2 \leq 0$ for some $r > 0$.
Suppose that $\cos 3\theta = \dfrac{x}{p}$, $\sin 3\theta = \dfrac{r}{p}$ where $0 \leq \theta \leq 60°$ and $p = \sqrt{x^2 + r^2}$.
Then

$$\begin{aligned} &\sqrt[3]{x + \sqrt{y}} \\ &= \left\{ \begin{array}{l} p^{1/3}(\cos\theta + i\sin\theta), \\ p^{1/3}(\cos(\theta + 120°) + i\sin(\theta + 120°)), \\ p^{1/3}(\cos(\theta + 240°) + i\sin(\theta + 240°)). \end{array} \right. \\ &\sqrt[3]{x - \sqrt{y}} \\ &= \left\{ \begin{array}{l} p^{1/3}(\cos\theta - i\sin\theta), \\ p^{1/3}(\cos(\theta + 120°) - i\sin(\theta + 120°), \\ p^{1/3}(\cos(\theta + 240°) - i\sin(\theta + 240°). \end{array} \right. \end{aligned}$$

If $\sqrt[3]{x + \sqrt{y}} + \sqrt[3]{x - \sqrt{y}}$ is to be real, its value is one of

$$2p^{1/3}\cos\theta,\ 2p^{1/3}\cos(\theta + 120°),\ 2p(\cos\theta + 240°).$$

Suppose that $z = 2p^{1/3}\cos\theta$. Then

$$\begin{aligned} z^3 - 2x &= 8p\cos^3\theta - 2p\cos 3\theta \\ &= 8p\cos^3\theta - 2p(4\cos^3\theta - 3\cos\theta) \\ &= 6p\cos\theta. \end{aligned}$$

Therefore $\dfrac{z^3-2x}{3z} = \dfrac{6p\cos\theta}{6p^{1/3}\cos\theta} = p^{2/3}$, and further

$$x^2 - \left(\frac{z^3-2x}{3z}\right)^3 = p^2\cos^2 3\theta - p^2 = -p^2\sin^2 3\theta = -r^2 = y.$$

Suppose that $z = 2p^{1/3}\cos(\theta \pm 120^\circ)$. Then

$$\begin{aligned} z^3 - 2x &= 8p\cos^3(\theta\pm 120^\circ) - 2p\cos 3\theta \\ &= 2p[\cos 3(\theta\pm 120^\circ) + 3\cos(\theta\pm 120^\circ)] - 2p\cos 3\theta \\ &= 6p\cos(\theta\pm 120^\circ). \end{aligned}$$

Therefore $x^2 - \left(\dfrac{z^3-2x}{3z}\right)^3 = p^2\cos^2 3\theta - p^2 = -r^2 = y$.

When $y \le 0$, the choice of the pair (x, y) gives rise to three possible real values of z, depending on which choice of cube root is taken. Taking *this* value of z will lead back to the correct value of y.

For example, let $x = -1, y = r = 0$, so that $\theta = 60^\circ$ and $p = 1$. Then $z = 1, -2, 1$.

With $(x, z) = (1, 1)$, we have $x^2 - \left(\frac{z^3-2x}{3z}\right) = 1 - \left(\frac{1+2}{3}\right) = 0$.

With $(x, z) = (-1, -2)$, we have $x^2 - \left(\frac{z^3-2x}{3z}\right) = 1 - \left(\frac{-6}{-6}\right) = 0$.

When $y > 0$, then we have that $u = x + \sqrt{y}$ and $v = x - \sqrt{y}$ are real and distinct. The possible values of

$$\sqrt[3]{x+\sqrt{y}} + \sqrt[3]{x-\sqrt{y}}$$

are

$$\sqrt[3]{u} + \sqrt[3]{v}, (\sqrt[3]{u} + \sqrt[3]{v})\omega, (\sqrt[3]{u} + \sqrt[3]{v})\omega^2,$$
$$\sqrt[3]{u} + (\sqrt[3]{v})\omega, \sqrt[3]{u} + (\sqrt[3]{v})\omega^2, (\sqrt[3]{u})\omega + (\sqrt[3]{v})\omega^2, \cdots,$$

where $\omega^3 = 1,\ \omega \neq 1$, that is, a linear combination of two roots of unity (with coefficients $\sqrt[3]{u}$, $\sqrt[3]{v}$, the real cube roots of u and v). The only such real linear combination is $\sqrt[3]{u}$ and $\sqrt[3]{v}$, and there is no ambiguity.

Solutions 1 and 2 appear to be quite different and should be reconciled. We will indicate how to do this for solutions of equation (2) in the particular case that $z = 6a$ and $x = 9ab$. If this is to fall into the form of the general Solution to equation (1) given in solution 1, then w must satisfy

$$\tfrac{1}{8}(6a)(36a^2 + 3w) \;=\; 9ab.$$

This gives $w = 4(b - 3a^2)$, whereupon

$$\begin{aligned} \frac{1}{64}w(3z^2+w)^2 &= \frac{1}{16}(b-3a)^2\left(108a^2 + 4(b-3a)^2\right)^2 \\ &= (b-3a^2)\left(24a^2+b\right)^2, \end{aligned}$$

so that the values of y given in the two solutions agree.

45 *Solution 1* We will suppose that all quantities involved are real and that "$\sqrt{}$" refers to the non-negative square root. Thus, we have $|x| \leq 1, |y| \leq 1, |z| \leq 1$.

Let us clear away some preliminary situations.

(1) $a = b = c = 0$. The system is trivally solvable.

(2) $a = b$, $c \neq 0$. The system is not solvable.

(3) $a = 0$, $bc \neq 0$. The system becomes

$$by + cx = b\sqrt{1-y^2} + c\sqrt{1-z^2} = 0$$

or

$$by = -cz, b\sqrt{1-y^2} = -c\sqrt{1-z^2}.$$

Suppose that a solution exists. Then, from the second equation, b and c have opposite signs and $bc < 0$. Also,

$$b^2y^2 = c^2z^2, b^2(1-y^2) = c^2(1-z^2).$$

Therefore $b^2 = c^2$. Hence $b = -c$.

On the other hand, if $b = -c$, the system is

$$y - z = \sqrt{1-y^2} - \sqrt{1-z^2} = 0,$$

which is solvable by, say $(x, y, z) = (1, 1, 1)$.

Hence if $a = 0$, the system is solvable if and only if $b + c = 0$.

More generally, if $abc = 0$, the system is solvable if and only if

$$a + b + c = 0.$$

Henceforth, assume $abc \neq 0$. Let there be a solution

$$(x, y, z) = (\cos A, \cos B, \cos C) \text{ where } 0 \leq A, B, C \leq \pi.$$

Then

$$a\cos A + b\cos B + c\cos C = a\sin A + b\sin B + c\sin C = 0, \tag{1}$$

whence, for all θ, we have

$$\begin{aligned} & a\cos(A+\theta) + b\cos(B+\theta) + c\cos(C+\theta) \\ = \;& \cos\theta[a\cos A + b\cos B + c\cos C] \\ & \quad - \sin\theta[a\sin A + b\sin B + c\sin C] \\ = \;& 0. \end{aligned}$$

Similarly, for all θ, we have

$$a\sin(A+\theta) + b\sin(B+\theta) + c\sin(C+\theta) = 0.$$

Take $\theta = -A$. We then have

$$a + b\cos(B - A) + c\cos(C - A) = 0. \tag{2}$$

Therefore $a = -b\cos(B - A) - c\cos(C - A)$, so that

$$|a| \leq |b| + |c|. \tag{3}$$

Similarly,

$$|b| \leq |a| + |c| \quad \text{and} \quad |c| \leq |a| + |b|. \tag{3}$$

Furthermore, since $a\sqrt{1-x^2} + b\sqrt{1-y^2} + c\sqrt{1-z^2} = 0$, we obtain that a, b, c cannot have all the same sign. Thus, either

$$\left.\begin{array}{l} a > 0, b < 0, c < 0 \text{ and } a + b + c < 0 < a + b - c, a - b + c \\ \text{or} \\ a > 0, b > 0, c > 0 \text{ and } a + b - c, a - b + c < 0 < a + b + c. \end{array}\right\} \tag{4}$$

up to permutation of a, b, c.

Now suppose the conditions (3) and (4) are satisfied. We have to find angles A, B, C to satisfy (1). Thus we have to solve

$$a\cos A = -(b\cos B + c\cos C) \quad , a\sin A = -(b\sin B + c\sin C),$$

so that

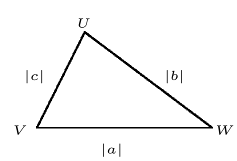
$$a^2 = a^2\cos^2 A + a^2\sin^2 A = b^2 + c^2 + 2bc\cos(B - C).$$

Similarly

$$b^2 = a^2 + c^2 + 2ac\cos(A - C), \quad \text{and} \quad c^2 = a^2 + b^2 + 2bc\cos(A - B).$$

Suppose $bc > 0$, $ab < 0$, $ac < 0$.

Form the triangle with sides $|a|, |b|, |c|$ which is possible by (3). Let U, V, W be the respective angles opposite the sides.



We have to arrange that

$$\cos U = -\cos(B - C), \cos V = \cos(A - C), \cos W = \cos(A - B).$$

that is, either

$$A - C = V, B - A = W, B - C = \pi - U$$

(which implies $B \geq A \geq C$), or

$$C - A = V, A - B = W, C - B = \pi - U$$

(which implies $C \geq A \geq B$).

[The Law of Sines gives $|a| \sin W = |c| \sin U$ and $|a| \sin V = |b| \sin U$ which suggest looking at

$$(A, B, C) = (W, 0, \pi - U), (\pi - W, \pi, U), (V, \pi - U, 0), (\pi - V, U, 0),$$

each of which yields

$$a \sin A + b \sin B + c \sin C = 0.$$

Now $(A, B, C) = (W, O, \pi - U)$, so that

$$\begin{aligned} &a \cos A + b \cos B + c \cos C \\ &\quad = \quad a \cos W + b - c \cos U \\ &\quad = \quad a \left[\frac{c^2 - a^2 - b^2}{-2ab} \right] + b - c \left[\frac{a^2 - b^2 - c^2}{2bc} \right] \\ &\quad = \quad \frac{-c^2 + a^2 + b^2 + 2b^2 - a^2 + b^2 + c^2}{2b} \;=\; 2b. \end{aligned}$$

Also $(A, B, C) = (\pi - W, \pi, U)$ implies that

$$a \cos A + b \cos B + c \cos C = -a \cos W - b + c \cos U = -2b.$$

Further, $(A, B, C) = (\pi - V, U, \pi)$ implies that

$$a \cos A + b \cos B + c \cos C = -2c,$$

and $(A, B, C) = (V, \pi - U, O)$ implies that

$$a \cos A + b \cos B + c \cos C = 2c.$$

None of these is what we want, but we try to combine them in some way. Note that b and c have the same sign and that $2b$ and $-2b$ have opposite signs.]

We construct functions $f(t), g(t), h(t)$ $(0 \leq t \leq 1)$ which are continuous and satisfy

$$\begin{aligned} f(0) &= W, & f(1) &= \pi - W; \\ g(0) &= 0, & g(1) &= \pi; \\ h(0) &= \pi - U, & h(1) &= U; \end{aligned}$$

and $\quad a \sin f(t) + b \sin g(t) + c \sin h(t) = 0. \qquad (5)$

Note that equation (5) holds for $t = 0$ and $t = 1$.

Choose t_1 with $0 < t_1 < 1$ and let $f(t_1) = \frac{\pi}{2}$. As $\sin f(t)$ increases from $\sin f(0) = \sin W$ to $\sin f(t_1) = 1$, we hold $h(t)$ constant and let $g(t)$ increase so that equation (5) holds. Since $|a| \le |b| + |c|$, it may happen that at t_2 ($0 < t_2 < t_1$), we have $g(t_2) = \frac{\pi}{2}$. Then between t_2 and t_1, hold $g(t)$ constant and let $h(t)$ vary to maintain equation (5). Choose t_3 with $t_1 < t_3 < 1$. Let $f(t)$ be constant at $\frac{\pi}{2}$, allow $g(t)$ to vary and $h(t)$ to vary to $\frac{\pi}{2}$ maintaining equation (5). Finally, let $f(t)$ move to $\pi - W$ with $g(t)$ and $h(t)$ varying to maintain equation (5).

Let $\phi(t) = a\cos f(t) + b\cos g(t) + c\cos h(t)$. Then ϕ is continuous and $\phi(0)\phi(1) = -4b^2 < 0$, and so, by the intermediate value theorem, there exists t_0 with $\phi(t_0) = 0$.

The desired solution is $(x, y, z) = (\cos f(t_0), \cos g(t_0), \cos h(t_0))$.

45 *Solution 2* Consider, for example the case that

$$ab > 0, ac < 0, bc < 0.$$

Dividing through by c, we reduce the system to

$$\sqrt{1 - z^2} = p\sqrt{1 - x^2} + q\sqrt{1 - y^2}, z = px + qy$$

(where $p \ge q \ge 0$, say). Squaring and manipulating leads to

$$xy + \sqrt{(1 - x^2)(1 - y^2)} = \frac{1 - p^2 - q^2}{2pq}.$$

We have

$$\begin{aligned} -1 &\le xy + \sqrt{(1 - x^2)(1 - y^2)} \\ &\le \frac{x^2 + y^2}{2} + \frac{(1 - x^2) + (1 - y^2)}{2} \\ &= 1 \end{aligned}$$

when $|x| \le 1$, $|y| \le 1$ with equality on the left when $(x, y) = (1, -1)$ and on the right when $(x, y) = (1, 1)$.

Because $xy + \sqrt{(1 - x^2)(1 - y^2)}$ is continuous in x and y, it assumes all values between -1 and 1 for $-1 \le x, y \le 1$. Thus, if there is a solution, we must have

$$-1 \le \frac{1 - p^2 - q^2}{2pq} \le 1.$$

This is equivalent to

$$-2pq \le 1 - p^2 - q^2 \le 2pq,$$

which is the same as

$$(p - q)^2 \le 1 \le (p + q)^2,$$

or

$$p - q \le 1 \le p + q.$$

If $a > 0, b > 0, c < 0$, we have $p = \frac{a}{-c}$, $q = \frac{b}{-c}$, so that $a + b + c \geq 0 \geq a - b + c$. The imposed condition $p \geq q$ corresponds to $a \geq b$, so that $0 \geq -a + b + c$ as well.

If $a < 0$, $b < 0$, $c > 0$, we have $p = \frac{-a}{c}$, $q = \frac{-b}{c}$, so that $a + b + c \leq 0 \leq a - b + c$.

Now suppose that the conditions on p and q are satisified. We find x, y to make

$$xy + \sqrt{(1 - x^2)(1 - y^2)} = \frac{1 - p^2 - q^2}{2pq}$$

and let $z = px + qy$. Then

$$\begin{aligned}
&\left(p\sqrt{1-x^2} + q\sqrt{1-y^2}\right)^2 \\
&= p^2(1-x^2) + q^2(1-y^2) + 2pq\sqrt{(1-x^2)(1-y^2)} \\
&= p^2 + q^2 + 2pq\left(\frac{1-p^2-q^2}{2pq} - xy\right) - p^2x^2 - q^2y^2 \\
&= 1 - p^2x^2 - q^2y - 2pq(xy) \\
&= 1 - z^2.
\end{aligned}$$

Hence $\sqrt{1-z^2} = p\sqrt{1-x^2} + q\sqrt{1-y^2}$.

45 *Solution 3* [C. Percival] We can handle the case $abc = 0$ as in 45 Solution 1. Suppose $abc \neq 0$. Then

$$a\sqrt{1-x^2} + b\sqrt{1-y^2} + c\sqrt{1-z^2}$$

cannot vanish if all of a, b, c have the same sign. WOLOG, suppose that $a < 0 < c$. Let

$$(x, y, z) = \left(\frac{c^2 - a^2 - b^2}{2ab}, 1, \frac{a^2 - b^2 - c^2}{2bc}\right).$$

Then

$$ax + by + cz = \frac{c^2 - a^2}{2b} - \frac{b}{2} + b + \frac{a^2 - c^2}{2b} - \frac{b}{2} = 0.$$

Thus

$$\begin{aligned}
\sqrt{1-x^2} &= \sqrt{1 - \left(\frac{a^4 + b^4 + c^4 - 2a^2c^2 - 2c^2b^2 + 2a^2b^2}{4a^2b^2}\right)} \\
&= \frac{1}{|2ab|}\sqrt{-a^4 - b^4 - c^4 + 2a^2c^2 + 2b^2c^2 + 2a^2b^2} \\
&= -\frac{1}{2a|b|}\sqrt{-a^4 - b^4 - c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2)}
\end{aligned}$$

since $a < 0$.

Similarly

$$\begin{aligned}\sqrt{1-z^2} &= \frac{1}{|2bc|}\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}\\ &= \frac{1}{2bc}\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}.\end{aligned}$$

Hence

$$\begin{aligned}&a\sqrt{1-x^2}+b\sqrt{1-y^2}+c\sqrt{1-z^2}\\ &= -\frac{1}{2b}\sqrt{\cdots}+0+\frac{1}{2b}\sqrt{\cdots}\\ &= 0.\end{aligned}$$

46 *Solution 1* Plugging $z = 3-(x+y)$ into $x^3+y^3+z^3 = 3$ yields

$$x^3+y^3+27-27(x+y)+9(x+y)^2-(x+y)^3 = 3.$$

This is equivalent to

$$\begin{aligned}&24\\ &= (x+y)\left[-(x^2-xy+y^2)+27-9x-9y+(x^2+2xy+y^2)\right]\\ &= 3(x+y)(9-3x-3y+xy)\\ &= 3(x+y)(3-x)(3-y),\end{aligned}$$

or $8 = (x+y)(3-x)(3-y)$.

Without loss of generally, we can assume that $xy \geq 0$.
If $x \leq 0, y \leq 0$, then the right side would be nonpositive giving a contradiction.
Hence $x+y = 1$, 2, 4 or 8.

If $x+y = 8$, then $(3-x)(3-y) = 1$, yielding $x = y = 4$, and we get $(x,y,z) = (4,4,-5)$.

If $x+y = 4$, then $(3-x)(3-y) = 2$. But these are incompatible.

If $x+y = 2$, then $x = y = 1$ is the only possibility and we get $(x,y,z) = (1,1,1)$. Finally, $x+y = 1$ is not a possibility.

By symmetry, we have also $(x,y,z) = (4,-5,4), (-5,4,4)$.

46 *Solution 2* Suppose

$$x+y+z = x^3+y^3+z^3 = 3$$

and

$$xy+xz+yz = a.$$

Since

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-xz-yz),$$

we obtain that

$$1 - xyz = (x + y + z)^2 - 3(xy + xz + yz) = 9 - 3a,$$

so that

$$xyz = 3a - 8.$$

Hence x, y, z are roots of the cubic equation $t^3 - 3t^2 + at - (3a - 8) = 0$. Solving this for a, we have $a = -t^2 + \frac{8}{3-t}$.

Since a must be an integer, we have that $3 - t$ must divde 8. Checking out $t = -5, -1, 1, 2, 4, 5, 7, 11$, we find that only $t = -5, 1, 4$ work.

The solutions are $(x, y, z) = (1, 1, 1), (4, 4, -5), (4, -5, 4), (-5, 4, 4)$.

46 *Solution 3* [S. Yazdani] $x + y = 3 - z$ and $x^3 + y^3 = 3 - z^3$ imply that $(x + y)^3 - 3(x + y)xy = 3 - z^3$. This further implies that $(3 - z)^3 - 3(3 - z)xy = 3 - z^3$, and so that $xy = \frac{8-9z+3z^2}{3-z} = \frac{8}{3-z} - 3z$.

Hence $(3 - z)|8$.

Similarly $(3 - x)|8$ and $(3 - y)|8$. Thus x, y, z must be chosen from among $\{11, 7, 5, 4, 2, 1, -1, -5\}$. We can now check out the possibilities.

46 *Solution 4* [A. Chan] The equation $3 = x + y + z = x^3 + y^3 + z^3$ implies that

$$\begin{aligned} 27 &= (x + y + z)^3 \\ &= (x^3 + y^3 + z^3) + 3(x + y + z)(xy + xz + yz) - 3xyz. \end{aligned}$$

This leads to $xyz = 3(xy + xz + yz) - 8 \equiv 1 (\text{mod } 3)$.

Hence, modulo 3, we have

$$(x, y, z) \equiv (1, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).$$

But we also have that $x + y + z \equiv 0 (\text{mod } 3)$.

Hence, the only possibility is $(x, y, z) \equiv (1, 1, 1) (\text{mod } 3)$.

Let $x = 3u + 1, y = 3v + 1, z = 3w + 1$.

Then $0 = x + y + z - 3 = 3(u + v + w)$, and so

$$\begin{aligned} 0 &= x^3 + y^3 + z^3 - 3 \;=\; 3(u + v + w) \\ &= (3u + 1)^3 + (3v + 1)^3 + (3w + 1)^3 - 3 \\ &= 27(u^3 + v^3 + w^3) + 27(u^2 + v^2 + w^2) + 9(u + v + w). \end{aligned}$$

Hence

(1) $u + v + w = 0$.

(2) $(u^3 + v^3 + w^3) + (u^2 + v^2 + w^2) = 0$.

so that

(3) $uv + uw + vw = \frac{1}{2}[(u + v + w)^2 - (u^2 + v^2 + w^2)] = -\frac{s}{2}$

where $s = u^2 + v^2 + w^2$.

Since $xyz = 3(xy + xz + yz) - 8$, we have

$$\begin{aligned} &27uvw + 9(uv + uw + vw) + 3(u + v + w) + 1 \\ &= \quad 3[9(uv + uw + vw) + 6(u + v + w) + 3] - 8. \end{aligned}$$

so that $uvw = \dfrac{2}{3}(uv + uw + vw) = -\dfrac{s}{3}$.

The case $s = 0$ corresponds to $u = v = w = 0$, which yields the solution

$$(x, y, z) - (1, 1, 1).$$

Suppose $s \neq 0$. Since $\dfrac{s}{2}$ and $\dfrac{s}{3}$ are integers, we know that $s = 6r$ for some positive integer r. Since u, v, w are roots of the equation $t^3 - 3rt + 2r = 0$, we have that $(3t - 2)r = t^3$.

Any prime divisor of $3t - 2$ must divide t^3, and so must divide t (where t is u, v or w).

Hence 2 is the only prime dividing $3t - 2$. Let $3t - 2 = 2^k$

Then $t = 2\left(\dfrac{2^{k-1} + 1}{3}\right)$ where $\dfrac{2^{k-1} + 1}{3}$ is odd.

Since $2^k | t^3$, we get $k \leq 3$. We consider the possibilities:

$3t-2 = 1$. This implies that $t = 1$, and further, $r = 1$, so that $0 = t^3-3t+2 = (t - 1)^2(t + 2)$. Therefore $(u, v, w) = (1, 1, -2)$, and so $(x, y, z) = (4, 4, -5)$ in some order.

$3t - 2 = 2$. This is not possible.

$3t-2 = 4$. This implies that $t = 2$, and futher, $r = 2$, so that $0 = t^3-6t+4 = (t - 2)(t^2 + 2t - 2)$, which does not have three integer roots.

$3t - 2 = 8$. This is not possible.

46 *Solution 5* [B. Marthi] As in Solution 1, we have

$$8 = (x + y)\,(9 - 3(x + y) + xy)\,.$$

Let $x + y = w$. Then $xy = \dfrac{8}{w} + 3w - 9$ so that

$$(x - y)^2 = w^2 - 4\left(\frac{8}{w} + 3w - 9\right).$$

Using the fact that $w|8$ and that $(x - y)^2$ is a perfect square, we arrive at the desired solutions.

Remark: [S. Cautis] $x + y + z = x^3 + y^3 + z^3 = n$ leads to

$$(n + 1)n(n - 1) = 3(n - x)(n - y)(n - z).$$

47 *Solution 1*
(a) When $t = 0$, the required inequality is trivial.
When $t = 1$, since $2g(n) \leq g(n + 1) + g(n - 1)$, we have that

$$g(n) - g(n - 1) \leq g(n + 1) - g(n).$$

For $0 < t < 1$, the substitution $y = n + 1, t = n$ yields

$$g(n + t) \leq tg(n + 1) + (1 - t)g(n),$$

which implies that

$$g(n + t) - g(n) \leq t[g(n + 1) - g(n)],$$

while the substitution $y = n - 1, z = n + t, t \to \frac{t}{1+t}$ yields

$$\begin{aligned} g(n) &= g\left(\frac{t}{t+1}(n-1) + \frac{1}{1+t}(n+t)\right) \\ &\leq \left(\frac{t}{1+t}\right) g(n-1) + \left(\frac{1}{1+t}\right) g(n+t). \end{aligned}$$

Therefore $g(n) + tg(n) \leq tg(n - 1) + g(n + t)$, implying that

$$t[g(n) - g(n - 1)] \leq g(n + t) - g(n).$$

(b) Writing the inequality in (a) in terms of f yields

$$(n-1)^t = \left[\frac{f(n)}{f(n-1)}\right]^t \leq \frac{f(n+t)}{f(n)} \leq \left[\frac{f(n+1)}{f(n)}\right]^t = n^t.$$

Since $f(1) = 1$, we have $f(2) = 1$, so that when $n = 2, t = 1/2$, we have

$$1 \leq f\left(\tfrac{5}{2}\right) = \tfrac{3}{2} f\left(\tfrac{3}{2}\right) = \tfrac{3}{4} f\left(\tfrac{1}{2}\right) \leq \sqrt{2},$$

giving that

$$\tfrac{4}{3} \leq f\left(\tfrac{1}{2}\right) \leq \tfrac{4}{3}\sqrt{2} \text{ as desired .}$$

48 *Solution 1* Let $x_{n+1} = x_1$. Observe that

$$\sum_{i=1}^{n} \frac{x_i^2 - x_{i+1}^2}{x_i + x_{i+1}} = \sum_{i=1}^{n} x_i - x_{i+1} = 0,$$

so that

$$\sum_{i=1}^{n}\frac{x_i^2}{x_i+x_{i+1}}=\sum_{i=1}^{n}\frac{x_{i+1}^2}{x_i+x_{i+1}}=\frac{1}{2}\left[\sum_{i=1}^{n}\frac{x_i^2+x_{i+1}^2}{x_i+x_{i+1}}\right].$$

Now

$$\sqrt{\frac{x_i^2+x_{i+1}^2}{2}}\geq\frac{x_i+x_{i+1}}{2},$$

implying that

$$x_i^2+x_{i+1}^2\geq\frac{(x_i+x_{i+1})^2}{2}.$$

Hence

$$\sum_{i=1}^{n}\frac{x_i^2}{x_i+x_{i+1}}\geq\frac{1}{2}\sum_{i=1}^{n}\frac{x_i+x_{i+1}}{2}=\frac{1}{2}(x_1+x_2+\cdots+x_n),$$

as desired.

48 *Solution 2* Let $x_{n+1}=x_1$. Then $\sqrt{x_ix_{i+1}}\leq\frac{x_i+x_{i+1}}{2}$ implies that $\frac{x_ix_{i+1}}{x_i+x_{i+1}}\leq\frac{x_i+x_{i+1}}{4}$ $(i=1,2,\cdots,n)$.

Hence

$$\begin{aligned}\sum_{i=1}^{n}\frac{x_i^2}{x_i+x_{i+1}} &= \sum_{i=1}^{n}\left(x_i-\frac{x_ix_{i+1}}{x_i+x_{i+1}}\right)\\ &= \sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{x_ix_{i+1}}{x_i+x_{i+1}}\\ &\geq \sum_{i=1}^{n}x_i-\frac{1}{4}\left(\sum_{i=1}^{n}(x_i+x_{i+1})\right)\\ &= \sum_{i=1}^{n}x_i-\frac{1}{2}\sum_{i=1}^{n}x_i\\ &= \frac{1}{2}\sum_{i=1}^{n}x_i.\end{aligned}$$

ATOM

A Taste Of Mathematics / Aime-T-On les Mathématiques

1. Edward J. Barbeau *Mathematical Olympiads' Correspondence Program (1995-1996)*
2. Bruce L.R. Shawyer *Algebra — Intermediate Methods*
3. Peter I. Booth, John Grant McLoughlin, and Bruce L.R. Shawyer *Problems for Mathematics Leagues*
4. Edward J. Barbeau, and Bruce L.R. Shawyer *Inequalities*
5. Richard Hoshino, and John Grant McLoughlin *Combinatorial Explorations*
6. Peter I. Booth, John Grant McLoughlin, and Bruce L.R. Shawyer *Problems for Mathematics Leagues — II*

Cost per volume (including shipping and handling charges):
Regular Rate $12.00 — CMS Member's Rate $9.00 (excluding taxes).
For non-Canadian shipping addresses the rates quoted must be paid in US funds.

Coût par volume (frais y compris d'expédition et de manutention) :
Taux régulier 12,00 $ — Tarif d'membre de la SMC 9,00 $ (avant taxes).
Pour une adresse à l'étranger, paiement requis en devises américaines.

For more information and to order:
Pour plus d'informations et pour compléter une commande :

www.cms.math.ca/Publications/books

or contact:
ou contacter :

CMS/SMC Publications
577 King Edward, Ottawa (Ontario) Canada K1N 6N5

Email: publications@cms.math.ca
Courriel : publications@smc.math.ca

Canadian Mathematical Society / Société mathématique du Canada
577 King Edward, POB/CP 450-A
Ottawa, Ontario, Canada K1N 6N5
www.cms.math.ca