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MathemAttic Problems: A Guide

MathemAttic is a child of ***Crux*** and a sibling to ***Mathematical Mayhem***. As such, it has a mathematical problem-solving focus and an active problems section. However, unlike ***Mathematical Mayhem***, ***MathemAttic*** is intended for a much broader audience of high-school students and their teachers. This short article will outline the differences and help serve as guide for people wishing to submit problems to ***MathemAttic***.

MathemAttic is aimed at high-school students and their teachers who are interested in mathematics. This includes, but is not limited to, students who are interested in mathematics contests. Some features, such as *Teaching Problems*, *From the Bookshelf of . . .*, and the new *Mathematics From The Web*, are aimed at that broader audience, while *Problem Solving Vignettes* has the contest participant in mind.

Having said that, the level of problems presented in the ***MathemAttic*** *Problems* section aims to serve our broader audience. That is, we will present *Pre-Olympiad* level problems in the section. Problems suitable for local, regional or national contests at the high school level or lower, such as the *Canadian Open Mathematics Challenge* and the *Canada Jay Mathematical Competition*, will be featured. Readers looking for problems above this level have plenty to choose from in the *Olympiad Corner* and the Problems section of ***Crux***.

Thus we are looking for problems that need no mathematics beyond high-school, as well as none that need Calculus. Problem solving techniques that might be outside the curriculum, but are helpful in high-school mathematics competitions, like things featured in *Problem Solving Vignettes*, would be allowed.

At this point in time, we do not receive enough proposals from the readers that meet our criteria. As such, many of the problems featured are from past math contests, or other problem collections. We would welcome more proposals from readers. Proposals and solutions can be submitted online at:

`https://publications.cms.math.ca/cruxbox`

We welcome and accept material from all the ***MathemAttic*** readers. However, since our focus is high-school students and their teachers, we will look to promote this audience first. That is, we will look to submissions from high-school students first when looking for featured solutions to give this audience a chance to shine. Student and teacher contributors to ***MathemAttic*** and their schools, will be acknowledged, so please make sure to indicate if you are a student or teacher as well as your school and grade on your submissions.

We hope to continue expanding the material in ***MathemAttic*** and we hope to receive more submissions of problems, solutions and articles from our readers. We hope you enjoy the material and the direction we are taking. Any comments are welcome at `crux-editors@cms.math.ca`.

MathemAttic

No. 36

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2022**.

MA176. *Proposed by Rikio Ichishima and Francesca Muntaner-Batle.*

Given the sequence $\{a_n\}$, where $a_1 = 11$, $a_2 = 1111$, $a_3 = 111111$, and

$$a_k = \overbrace{1111\ldots 11}^{2k \text{ ones}},$$

show that no a_n is a perfect square.

MA177. *Proposed by Ed Barbeau.*

Let $ABCD$ be a equilateral trapezoid with $BC \parallel AD$, $AB = CD = 1$, $BC = 2$ and $AD = 3$.

a) Prove that $\angle BAD = \angle CDA$ and that $ABCD$ is concyclic.

b) Determine the radius of the circumcircle of $ABCD$.

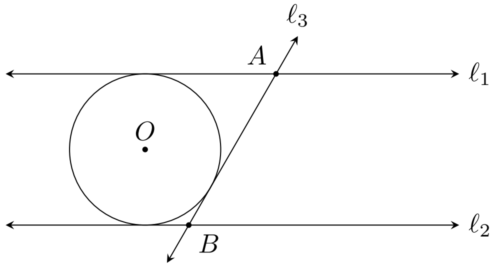
MA178. *Proposed by Vishak Srikanth.*

Let x, y, and z be integers. Find the number of solutions to

$$(xy+1)^2 + (xy-1)^2 + (xz+1)^2 + (xz-1)^2 + (yz+1)^2 + (yz-1)^2 = 2022^2.$$

MA179. *Proposed by Jakob Denes.*

Given two parallel lines, ℓ_1 and ℓ_2 and a transversal, ℓ_3 intersecting ℓ_1 and ℓ_2 at points A and B respectively. A circle with centre O lies between the parallel lines such that it is tangent to all three lines. Show that $\angle AOB$ is a right angle.



MA180. *Proposed by Alex Bloom.*

a) Find $\dfrac{1}{1+2+3+4}+\dfrac{1}{1+2+3+4+5}+\cdots+\dfrac{1}{1+2+3+\cdots+99}$.

b) Find $\dfrac{12}{1^2\cdot 3^2}+\dfrac{24}{3^2\cdot 5^2}+\dfrac{36}{5^2\cdot 7^2}+\dfrac{48}{7^2\cdot 9^2}$.

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2022**.

MA176. *Soumis par Rikio Ichishima and Francesca Muntaner-Batle.*

Étant donné la suite $\{a_n\}$, où $a_1 = 11$, $a_2 = 1111$, $a_3 = 111111$ et

$$a_k = \overbrace{1111\ldots 11}^{2k \text{ uns}},$$

montrez qu'aucun a_n n'est un carré parfait.

MA177. *Soumis par Ed Barbeau.*

Soit $ABCD$ un trapèze équilatéral vérifiant $BC \parallel AD$, $AB = CD = 1$, $BC = 2$ et $AD = 3$.

a) Montrez que $\angle BAD = \angle CDA$ et que $ABCD$ est cocyclique.

b) Déterminez le rayon du cercle circonscrit à $ABCD$.

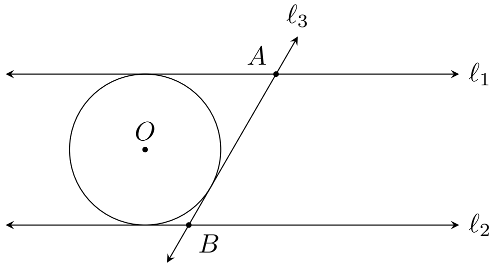
MA178. *Soumis par Vishak Srikanth.*

Soit x, y et z des entiers. Trouvez le nombre de solutions de l'équation suivante :

$$(xy+1)^2+(xy-1)^2+(xz+1)^2+(xz-1)^2+(yz+1)^2+(yz-1)^2 = 2022^2.$$

MA179. *Soumis par Jakob Denes.*

Soit ℓ_1 et ℓ_2 deux droites parallèles et soit ℓ_3 une sécante qui rencontre ℓ_1 et ℓ_2 respectivement en A et B. Un cercle de centre O situé entre deux droites parallèles est tangent à chacune des trois droites. Montrez que $\angle AOB$ est un angle droit.



MA180. *Soumis par Alex Bloom.*

a) Déterminez $\dfrac{1}{1+2+3+4} + \dfrac{1}{1+2+3+4+5} + \cdots + \dfrac{1}{1+2+3+\cdots+99}$.

b) Déterminez $\dfrac{12}{1^2 \cdot 3^2} + \dfrac{24}{3^2 \cdot 5^2} + \dfrac{36}{5^2 \cdot 7^2} + \dfrac{48}{7^2 \cdot 9^2}$.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(1), p. 4–5.

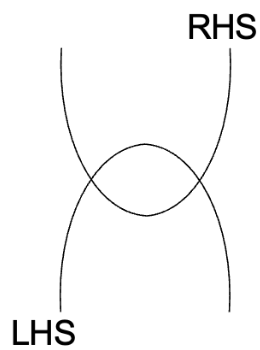
MA151. *Proposed by Mihaela Berindeanu.*

Solve over real numbers:

$$\sqrt{7+x}+\sqrt{18-x}=x^2-11x+25.$$

We received 6 submissions, all correct. We present the solution by Richard Hess.

The only solutions are $x=2$ and $x=9$. The left-hand side of the equation can be written as $\sqrt{12.5+u}+\sqrt{12.5-u}$, where $u=x-5.5$. This expression is even in u with a maximum at $u=0$ and looks a bit like the curve shown below. The right-hand side of the equation is $x^2-11x+25=u^2-5.25$ and looks like the other curve shown below. Clearly there are only two solutions.



MA152. *Proposed by Neculai Stanciu.*

Prove the following cryptarithm, where each letter represents a different digit:

$$DEAD\times REAR<READ\times DEAR.$$

We received 5 submissions, all correct. We present the solution by Prithwijit De.

Let $t=100E+10A$. Then

$$DEAD\times REAR=\{(10^3+1)D+t\}\{(10^3+1)R+t\}=(10^3+1)^2DR+(10^3+1)(R+D)t+t^2$$

and

$$\begin{aligned}READ\times DEAR&=(10^3R+D+t)(10^3D+R+t)\\&=(10^6+1)RD+10^3(R^2+D^2)+(10^3+1)(R+D)t+t^2.\end{aligned}$$

Hence

$$READ \times DEAR - DEAD \times REAR = 10^3(R^2 - 2RD + D^2)$$
$$= 10^3(R - D)^2 > 0.$$

Observe that since D and R are different digits, $(R - D)^2 > 0$ always holds and hence the cryptarithm is correct.

MA153. *Proposed by Roy Barbara.*

Let a, b, x and y be rational numbers so that $x \geq 0$, $a > 0$ and $\sqrt{a}$ is not rational. Suppose further that $\sqrt[3]{\sqrt{a}+b} = \sqrt{x}+y$. Prove that $\sqrt[3]{a-b^2}$ is a rational number.

We received 2 submissions, of which 1 was correct and complete. We present the solution by Corneliu Mănescu-Avram.

Claim. Suppose s, t, u and v are rational numbers such that $s, u \geq 0$ and $\sqrt{s}$ is irrational. If

$$\sqrt{s} + t = \sqrt{u} + v,$$

then $s = u$ and $t = v$.

Proof of claim. Suppose that $s \neq u$. Then $\sqrt{s} - \sqrt{u} \neq 0$ and we have

$$\sqrt{s} - \sqrt{u} = v - t,$$
$$\sqrt{s} + \sqrt{u} = \frac{s-u}{\sqrt{s}-\sqrt{u}} = \frac{s-u}{v-t}.$$

Adding these two equalities to solve for $\sqrt{s}$ implies that $\sqrt{s}$ is rational, a contradiction. Thus $s = u$; the fact that $t = v$ follows immediately.

From the hypotheses of the problem we obtain

$$\sqrt{a} + b = (\sqrt{x} + y)^3 = \sqrt{x}(x + 3y^2) + 3xy + y^3.$$

Since a, b, $x(x+3y^2)^2$ and $3xy+y^3$ are rational and $\sqrt{a}$ is irrational, we can apply the claim to conclude that

$$a = x(x + 3y^2)^2$$

and

$$b = 3xy + y^3.$$

Hence

$$a - b^2 = x(x + 3y^2)^2 - (3xy + y^3)^2$$
$$= x^3 + 6x^2y^2 + 9xy^4 - (9x^2y^2 + 6xy^4 + y^6)$$
$$= (x - y^2)^3,$$

which implies that $\sqrt[3]{a - b^2} = x - y^2$ is rational.

MA154. Two bags, Bag A and Bag B, each contain 9 balls. The 9 balls in each bag are numbered from 1 to 9. Suppose one ball is removed randomly from Bag A and another ball from Bag B. If X is the sum of the numbers on the balls left in Bag A and Y is the sum of the numbers of the balls remaining in Bag B, what is the probability that X and Y differ by a multiple of 4?

Originally from The 32nd W.J. Blundon Mathematics Contest, Memorial University of Newfoundland, 2015, problem 10.

We received 6 submissions, five of which were correct. We present the one by Prithwijit De, slightly modified by the editor.

Let a be the number on the ball removed from bag A and b be the number on the ball removed from bag B. Then a and b are random variables with range $\{1, \ldots, 9\}$. Further $X = 45 - a$ and $Y = 45 - b$. Therefore $|X - Y| = |a - b|$. The possible multiples of 4 are 0, 4, and 8.

Case 1: $|a - b| = 0$
$(a, b) \in \{(k, k)|\ k = 1, \ldots, 9\}$. (9 pairs)

Case 2: $|a - b| = 4$
$(a, b) \in \{(k, k + 4)|\ k = 1, \ldots, 5\} \cup \{(k + 4, k)|\ k = 1, \ldots, 5\}$. (10 pairs)

Case 3: $|a - b| = 8$
$(a, b) \in \{(1, 9), (9, 1)\}$. (2 pairs)

The probability that X and Y differ by a multiple of 4 is therefore

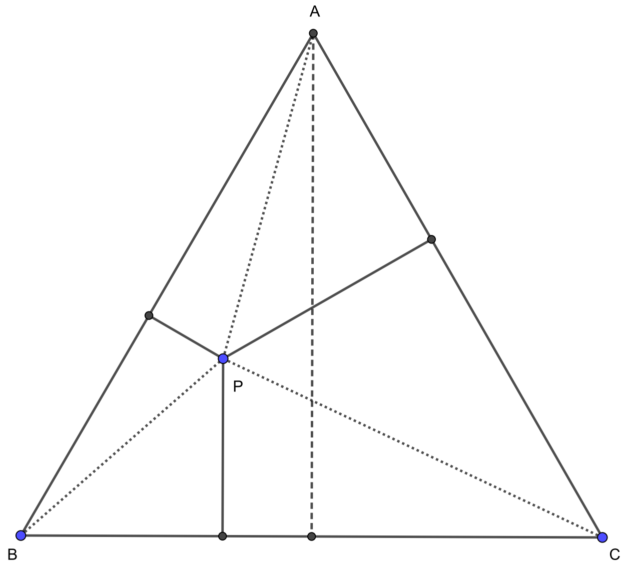
$$\frac{9 + 10 + 2}{9 \times 9} = \frac{21}{81} = \frac{7}{27}.$$

MA155. An arbitrary point is selected inside an equilateral triangle. From this point perpendiculars are dropped to each side of the triangle. Show that the sum of the lengths of these perpendiculars is equal to the length of the altitude of the triangle.

Originally from The 23rd W.J. Blundon Mathematics Contest, Memorial University of Newfoundland, 2006, problem 8.

We received 8 solutions. We present the solution by Henry Ricardo.

Suppose P is an interior point of an equilateral triangle ABC with side length a and altitude h. Let h_1, h_2, h_3 denote the lengths of the perpendiculars dropped from P to sides AB, BC and CA respectively.



Then the area of $\triangle ABC$ is the sum of the areas of triangles APB, BPC and CPA; that is,

$$\frac{ah_1}{2} + \frac{ah_2}{2} + \frac{ah_3}{2} = \frac{ah}{2}.$$

Multiplying by 2 and dividing through by a gives $h_1 + h_2 + h_3 = h$, as desired.

Telescoping Sums

Alex Bloom

In middle and high school, nearly all the series encountered are arithmetic or geometric. However, when they are not, they tend to be much more exciting. Telescoping series will pop up from time to time, usually remaining unsolved by the confused students before a solution is revealed. Despite significant confusion over how to approach them, there are many ways to both recognize and solve problems involving telescoping sums. Generally, the solver reaches a series of main terms, each of which can be transformed into a few simple expressions in such a way that when all of the terms are added together, everything cancels out except for a few terms in the beginning and a few terms at the end. This is similar to what happens when developing the formula for a sum of a geometric series. We start with

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

We then evaluate $S_n - rS_n$; when we gather like terms, we get

$$S_n - rS_n = a - \cancel{ar} + \cancel{ar} - \cancel{ar^2} + \cancel{ar^2} - \cdots - \cancel{ar^{n-1}} + \cancel{ar^{n-1}} - ar^n$$

which leads to our formula $S_n = \dfrac{a(1-r^n)}{1-r}$. This effect of the middle terms colliding into each other and canceling out is in fact where it gets its name: after an old-fashioned collapsible telescope.

In our investigation today, we will cover two of the most common techniques used to create the telescoping effect.

To fit our definition of telescoping sums, a series must eventually reach the form:

$$f(1) - f(1+n) + f(2) - f(2+n) + f(3) - f(3+n) + \cdots$$

for some function f. This series can be infinite, in which case the answer is $f(1) + f(2) + f(3) + \cdots + f(n)$, as the first positive term to cancel out is $f(n+1)$, but it can also be finite, leaving both the first n terms and the last n terms. While these terms must have opposite signs, so that the terms in the middle cancel out, $f(x)$ can be negative, as the first term can have either sign. While n could theoretically be fairly large, for the most part, in practice, the value of n is relatively small, leaving just a few terms at the beginning and end.

In algebra, some likely candidates for a telescoping sums problem are infinite or long non-geometric and non-arithmetic series. Usually, the terms are strictly decreasing or increasing, meaning that after they are split into two terms, the smaller part of one term can cancel out with the larger part of a later term or vice versa. Often, certain numbers in each expression increase arithmetically when moving to the next terms (e.g: $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots$). Sometimes the final terms used to telescope are given, but other times solving it requires simplifying a function and plugging terms in or combining groups of terms into one term before splitting them again to create the telescoping effect. To find the final group of terms, it is usually necessary to modify and move the given expressions around until they fit the criteria mentioned above.

In our investigation today, we will discuss the two most prevalent techniques used to arrive at a sequence of telescoping sums.

The first and most common technique is to reach a series such as

$$\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\cdots$$

which simplifies using partial fractions to

$$\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+2}-\frac{1}{n+3}+\cdots,$$

leaving one term at the beginning and in the case the sum is finite, one term at the end. The same technique can also be seen in

$$\frac{1}{n(n+2)}+\frac{1}{(n+1)(n+3)}+\frac{1}{(n+2)(n+4)}+\cdots=$$
$$\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+2}-\frac{1}{n+4}\cdots\right),$$

as

$$\frac{1}{n}-\frac{1}{n+2}=\frac{(n+2)-n}{n(n+2)}=\frac{2}{n(n+2)}.$$

In this latter case, we have two terms left in the beginning, and if the sum is finite, two at the end.

In general, we want to reach the form $\displaystyle\sum_{n=1}^{y}\frac{b}{n(n+x)}$, with a given y, x, and b, which simplifies to

$$\frac{b}{x}\left(\frac{1}{n}-\frac{1}{n+x}+\frac{1}{n+1}-\frac{1}{n+x+1}+\frac{1}{n+2}-\frac{1}{n+x+2}+\cdots\right),$$

where terms start to cancel out. Given that the sequence is long enough, there will be x terms left at the beginning and if it isn't infinitely long, x terms left at the end. We will now cover some problems of varying difficulty that demonstrate this technique.

Example 1. Given that $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic sequence with a common difference d, find $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k\cdot a_{k+1}}$ in terms of n, a_1, and d.

We first notice that since (a_n) is an arithmetic sequence, the two factors in the denominator have a constant difference d. Furthermore, to reach the next term, we are replacing the a_k with a_{k+1} and the a_{k+1} with a_{k+2}. Both of these should be indicators that the solution likely contains telescoping. Acknowledging that, we can rewrite it as

$$\sum_{k=1}^{n} \frac{1}{a_k \cdot a_{k+1}} = \sum_{k=1}^{n} \left(\frac{1}{d} \cdot \frac{d}{a_k \cdot a_{k+1}} \right) = \sum_{k=1}^{n} \left(\frac{1}{d} \cdot \frac{a_{k+1} - a_k}{a_k \cdot a_{k+1}} \right) = \sum_{k=1}^{n} \frac{1}{d} \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right).$$

Now, expanding it out, we get:

$$\sum_{k=1}^{n} \frac{1}{d} \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \cdots + \frac{1}{a_n} - \frac{1}{a_{n+1}} \right) = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_{n+1}} \right).$$

Since $a_{n+1} = a_1 + nd$, we find that the desired sum is equal to

$$\frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_1 + nd} \right) = \frac{n}{a_1(a_1 + nd)}.$$

Example 2. [1] Find $\displaystyle\sum_{n=2}^{63} (\log_n 2)(\log_{n+1} 2) \log_2 \left(1 + \frac{1}{n}\right)$.

Converting all logarithms to base 2 using the exchange formula, we obtain:

$$\frac{\log_2 \left(\frac{n+1}{n}\right)}{\log_2 n \log_2 (n+1)} = \frac{\log_2 (n+1) - \log_2 n}{\log_2 n \log_2 (n+1)} = \frac{1}{\log_2 n} - \frac{1}{\log_2 (n+1)}.$$

Therefore, the sum becomes:

$$\frac{1}{\log_2 2} - \frac{1}{\log_2 3} + \frac{1}{\log_2 3} - \frac{1}{\log_2 4} + \cdots + \frac{1}{\log_2 63} - \frac{1}{\log_2 64} = \frac{1}{\log_2 2} - \frac{1}{\log_2 64} = 1 - \frac{1}{6} = \frac{5}{6}.$$

Example 3. Let (a_n) be a sequence of numbers with the property that

$$a_1 + a_2 + \cdots + a_n = \frac{n(n+1)(n+2)}{6}$$

for all $n \geq 1$. Evaluate $\displaystyle\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{50}}$.

We start by finding a formula for the general term a_n. We know that:

$$a_1 + a_2 + \cdots + a_n = \frac{n(n+1)(n+2)}{6} \quad \text{and} \quad a_1 + a_2 + \cdots + a_{n-1} = \frac{(n-1)(n)(n+1)}{6}.$$

By subtracting, we find that

$$a_n = \frac{n(n+1)(n+2)}{6} - \frac{(n-1)(n)(n+1)}{6} = \frac{n(n+1)(3)}{6} = \frac{n(n+1)}{2}.$$

Thus, $\dfrac{1}{a_n} = \dfrac{2}{n(n+1)} = 2\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$ and $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_{50}}$ becomes:

$$2\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{50} - \frac{1}{51}\right) = 2\left(1 - \frac{1}{51}\right) = \frac{100}{51}\,.$$

Example 4. [3] Let $f(n) = \dfrac{1}{2^n} + \dfrac{1}{3^n} + \dfrac{1}{4^n} + \cdots$. Find $\displaystyle\sum_{n=2}^{\infty} f(n)$.

By writing out the first few terms, we get

$$\begin{aligned} f(2) &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ f(3) &= \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \\ f(4) &= \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \\ &\ \ \vdots \end{aligned}$$

Notice that there does not appear to be any obvious way to find the sum of each individual row $f(n)$. However, for each $n \geq 1$, the n^{th} column is a decreasing geometric sequence with first term $\dfrac{1}{(n+1)^2}$ and common ratio $\dfrac{1}{(n+1)}$. The sum of each geometric sequence is equal to

$$\frac{\dfrac{1}{(n+1)^2}}{1 - \dfrac{1}{n+1}} = \frac{1}{(n+1)^2\left(\dfrac{n}{n+1}\right)} = \frac{1}{(n+1)(n)}\,.$$

Therefore, our sum can be written as

$$\begin{aligned} \sum_{n=2}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1. \end{aligned}$$

Our second technique for problems recognized to likely involve telescoping is to rationalize or simplify the denominator of each term of a sequence of fractions to create a common denominator. While it can involve factoring or other manipulation of the denominator or numerator, the crucial part is to rationalize the denominator, making it possible to spot the telescoping pattern. While it can sometimes take more manipulation of the numerator, the telescoping terms mostly appear soon after the denominator is simplified.

Our first problem with this technique is the following:

Example 5. Find

$$S = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Note that the terms of S form a decreasing sequence that is neither arithmetic nor geometric. In order to see if S is telescoping, we rationalize the denominators:

$$S = \frac{\sqrt{2}-\sqrt{1}}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \frac{\sqrt{4}-\sqrt{3}}{4-3} + \cdots + \frac{\sqrt{100}-\sqrt{99}}{100-99}$$

Since the denominators are all 1, we obtain:

$$S = \sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\cdots+\sqrt{100}-\sqrt{99} = \sqrt{100}-\sqrt{1} = 9.$$

Example 6. [2] Find

$$\frac{1}{2\sqrt{1}+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + \cdots \frac{1}{100\sqrt{99}+99\sqrt{100}}.$$

Proceeding as in the previous example, we realize that the sum is telescoping after rationalizing the denominators:

$$\begin{aligned} &\frac{2\sqrt{1}-1\sqrt{2}}{2} + \frac{3\sqrt{2}-2\sqrt{3}}{6} + \frac{4\sqrt{3}-3\sqrt{4}}{12} + \cdots + \frac{100\sqrt{99}-99\sqrt{100}}{9900} \\ =&\frac{\sqrt{1}}{1} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4} + \cdots + \frac{\sqrt{99}}{99} - \frac{\sqrt{100}}{100} \\ =&\frac{\sqrt{1}}{1} - \frac{\sqrt{100}}{100} = \frac{9}{10} \end{aligned}$$

Example 7. Find $\displaystyle\sum_{n=0}^{287} \frac{6(\sqrt{n+1}+\sqrt{n+2})}{\sqrt{n^2+n}+\sqrt{n^2+2n}+\sqrt{n^2+4n+3}+\sqrt{n^2+5n+6}}$.

We first simplify the denominator by factoring each of the quadratic expressions under the square roots:

$$\frac{6(\sqrt{n+1}+\sqrt{n+2})}{\sqrt{n(n+1)}+\sqrt{n(n+2)}+\sqrt{(n+1)(n+3)}+\sqrt{(n+3)(n+2)}}$$

Using factoring by grouping, we get

$$\begin{aligned} &\frac{6(\sqrt{n+1}+\sqrt{n+2})}{\sqrt{n}(\sqrt{n+1}+\sqrt{n+2})+\sqrt{n+3}(\sqrt{n+1}+\sqrt{n+2})} \\ &= \frac{6(\sqrt{n+1}+\sqrt{n+2})}{(\sqrt{n+1}+\sqrt{n+2})(\sqrt{n+3}+\sqrt{n})} \\ &= \frac{6}{\sqrt{n+3}+\sqrt{n}}. \end{aligned}$$

Now rationalizing the denominator yields:

$$\frac{6(\sqrt{n+3}-\sqrt{n})}{(n+3)-n} = \frac{6(\sqrt{n+3}-\sqrt{n})}{3} = 2\sqrt{n+3}-2\sqrt{n}.$$

Hence our original sum becomes $\sum_{n=0}^{287}(2\sqrt{n+3}-2\sqrt{n})$, which expands to:

$$2(\sqrt{3}-\sqrt{0}+\sqrt{4}-\sqrt{1}+\sqrt{5}-\sqrt{2}+\sqrt{6}-\sqrt{3}\cdots+\sqrt{288}-\sqrt{285}+\sqrt{289}-\sqrt{286}+\sqrt{290}-\sqrt{287}),$$

with three terms left at both the beginning and the end, giving a final answer of

$$\begin{aligned}2\left(-\sqrt{0}-\sqrt{1}-\sqrt{2}+\sqrt{288}+\sqrt{289}+\sqrt{290}\right) &= 2\left(0-1-\sqrt{2}+12\sqrt{2}+17+\sqrt{290}\right)\\ &= 32+22\sqrt{2}+2\sqrt{290}.\end{aligned}$$

I will leave the following problems for experimentation with these methods.

1. If $S = \dfrac{1}{1+2+3+4} + \dfrac{1}{1+2+3+4+5} + \cdots + \dfrac{1}{1+2+3+\cdots+99}$, find S. (see MA 180)
2. Find $\displaystyle\sum_{n=1}^{99} \frac{1}{(\sqrt{n}+\sqrt{n+1})\sqrt{n(n+1)}}$.
3. Find $\displaystyle\sum_{n=1}^{\infty} \frac{8\cdot 3^n}{3^{2n+2}+3^{n+2}+3^n+1}$.
4. Find $\dfrac{12}{3^2} + \dfrac{24}{3^2\cdot 5^2} + \dfrac{36}{5^2\cdot 7^2} + \dfrac{48}{7^2\cdot 9^2}$. (see MA 180)

Acknowledgements

I would like to thank Dr. Lucian Sega for the helpful discussions.

References

1. *Alcumus*, Art of Problem Solving, `https://artofproblemsolving.com/alcumus`

2. National Team of Canada Training as cited in *Lecture Notes On Mathematical Olympiad Courses* for Junior Section Vol 2 by Xu Jiagu, page 5, Example 9.

3. 2017 Hong Kong IMO Preliminary Selection Test, `https://www.hkage.org.hk/file/competitions/3560/Prelim2017.pdf`

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TEACHING PROBLEMS

No.17

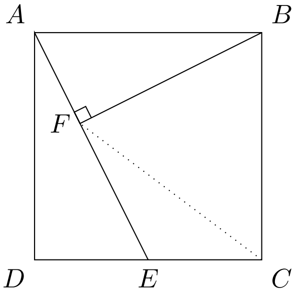
John McLoughlin & Elias Brettler

A Geometry Problem: Inviting Numerous Forms of Proof

This problem has a story. The original version appeared in an ATOM publication prepared by Jim Totten. Subsequently an occasional request was made to one of the authors (John) to provide a problem that could be a filler in a journal, *For the Learning of Mathematics* (FLM). This geometry problem seemed to suit that role with the figure adding to its value. The problem was inserted on a page following the completion of an article. Curiously Elias Brettler saw this problem and shared it with students in a first-year undergraduate course at York University. The course was called *Problems, Conjectures and Proofs.* That is where it emerged as a teaching problem in that solutions of many forms were offered by students.

Here a glimpse into the problem is provided. The statement of the problem is followed by a brief description of some directions that were pursued by students in that class. The details in these proofs are not provided in full here. Rather they will appear in the next issue of *Teaching Problems.* Meanwhile readers are invited to play with some of the ideas and directions to gain greater appreciation for the problem's potential as well as perhaps learning more about proof.

> Given a square $ABCD$ with E the mid-point of the side CD. Join A to E and drop a perpendicular from B to AE at F. Prove $CF = CD$.



At a 2010 meeting of the Canadian Mathematics Education Study Group, Elias Brettler led a short session discussing the problem and his experience with the class at York. The remainder of this feature is mainly excerpted from the summary that appears in the proceedings of that meeting. A detailed discussion of various solutions will follow in a future issue this autumn. For now, it is starting points or directions only that are offered. Engage and see if some of these appeal to you. Maybe you have another approach altogether. Students in the course were to look for two or more different ways to prove that $CF = CD$. Generally these proofs used measurement or the ideas of congruence and/or similarity.

The simplest measurement proof is obtained by using coordinates such as $A(0,10)$, $B(10,10)$, $C(10,0)$, and $D(0,0)$. Then F has coordinates $(2,6)$ and the distance formula gives the length of CF as 10. Another proof based on measurement extends AE and BC to meet at G and uses the law of cosines in ΔCGF. Further, the observation that the points E, F, B, and C lie on a circle gives a third measurement proof based on the idea that chords which subtend equal arcs are equal in length. A proof based on classical construction ideas and the fact that an angle inscribed in a semi-circle is a right angle was given.

Another solution used the idea of showing that C lies on the perpendicular bisector of FB. As the non-right angles which appear in the diagram are either equal or complementary, there are proofs which exploit this in order to use congruence or similarity.

This problem is simple to present making it accessible. Virtually any reasonable form of attack yields results. These two features distinguish this as a teaching problem. Many forms of proof turn out to be viable. What remains unclear however is to identify what it is about this problem that supports such variation. Can such qualities be seen in advance with particular problems or is it necessary to have the experience with a class that brings to light the relative merits of one problem or another?

References

Brettler, E. (2010). What my students taught me about Jim Totten's geometry problem in FLM. In Liljedahl, P., Oesterle, S. & Allan, D. (Eds.), *Proceedings of the 2010 Canadian Mathematical Education Study Group Meeting, May 25 -29, 2010*, p. 213, 214. Available at:
`https://www.cmesg.org/wp-content/uploads/2015/01/CMESG2010.pdf`

Grant McLoughlin, J. (2010). *For the Learning of Mathematics*, 30(1), 14.

Totten, J. (2007). *A taste of mathematics VII: Problems of the week.* Ottawa, ON: CMS. Canadian Mathematical Society.

Nerdle: A Variant of WORDLE for Math Enthusiasts

Doddy Kastanya

Since the beginning of 2022, the **WORDLE** word puzzle has enjoyed a world-wide popularity. It is a straight-forward game in which a player has six attempts to guess a five-letter word. As each guess is entered, the game will provide color-coded feedback to the player informing them if any of the letters match or occupy the correct position. The popularity of this game has inspired Richard Mann, a British data scientist, to launch a numerical version of this game called **NERDLE**.

Similar to **WORDLE**, the player needs to guess the **NERDLE** in six attempts by guessing the right combinations of numbers and operations for the desired mathematical expression. However, **NERDLE** involves eight 'tiles' instead of just five in **WORDLE**. Before discussing the mathematical aspects of this game, we should start by introducing the basic rules for this game:

- As mentioned above, there are eight 'tiles' to fill up in each attempt. Each tile is one of "0 1 2 3 4 5 6 7 8 9 + − ∗ / =".
- Each attempt must be a calculation that is mathematically correct. This means that each attempt must include one "=" sign.
- While the left side of the "=" sign involves mathematical operations, the right side of the "=" sign involves just a number.
- The standard order of operations applies. In other words, "∗" and "/" will be executed before "+" and "−".
- A number is expressed in a standard fashion and no leading zero is allowed.
- The game starts with six rows of eight gray tiles. After each attempt, the game will provide one of the following color-coded feedbacks for each tile:
  - Green : the guessed number or operation is a part of the solution and occupies the correct tile location.
  - Purple : the guessed number or operation is a part of the solution but does not occupy the correct tile location.
  - Black : the guessed number or operation is not a part of the solution.

Now that the general rules of the game have been covered, it is curious to define the "best" starting guesses and a reasonable strategy for a successful completion of this game. Some of them, based on my experience and observations, are discussed herein.

Placement of the "=" sign

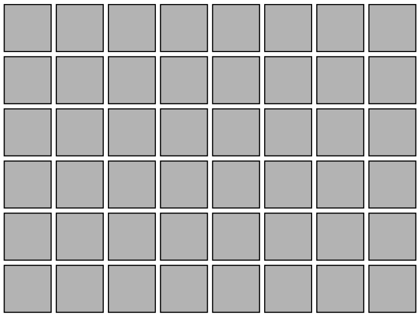


Figure 1. Empty tiles for the six attempts.

At the beginning of the game, the "game board" resembles the one shown in Figure 1. Could the "=" sign be placed in any one of them? No. The simplest operation will be an operation involving two single-digit numbers. So, for each attempt, the first possible tile to place the "=" sign is the fourth tile from the left. In this scenario, the result of the operation will be a four-digit number. This is not a feasible scenario, since the operation that will give the highest number involving two single-digit numbers is 9×9 which will give us 81, a two-digit number. The next available tile to place the "=" sign is the fifth tile from the left. This is a feasible tile since there is at least one scenario which satisfies the requirement, namely $9 \times 99 = 891$. The sixth tile from the left is also possible since there are many operations involving an operation between two two-digit numbers which will result in another two-digit number that could be used as examples. Placing the "=" sign in the seventh tile from the left is also possible. An example for this is $100 - 99 = 1$. The "=" sign cannot be placed in the eighth tile from the left since it will not be a correct mathematical expression. So, now we know that we could only put the "=" sign in the 5^{th}, 6^{th}, or 7^{th} tile from the left.

Starting Guesses

For each attempt, we would like to get as much information as possible related to the numbers and operations involved in the desired expression. Figure 2 shows an example of a reasonable starting guess since it will confirm whether or not the addition operation is used as well as checking the usability of six out of ten numbers. So, it is important to maximize the usage of unique numbers in the guess.

3	4	+	5	6	=	9	0

Figure 2. A reasonable starting guess.

Another strategy is to have a sequence of two or three guesses which will give us a clue as for what operations and what numbers are involved in the desired expression. Figure 3 shows an example of three guesses that could be useful in deciphering the puzzle. For the fourth guess, we will already know where the location of the "=" sign is and what numbers/operations are involved.

1	0	5	−	9	8	=	7
2	3	+	4	6	=	6	9
9	/	3	*	6	=	1	8

Figure 3. An example of three guesses to reveal required numbers and operations.

Rearranging the known components

After making three reasonable guesses, we usually already know the components required to solve the puzzle. However, there is one more twist that we need to consider. A number could be used more than once. Nevertheless, there are other clues that we could use to find the answer based on the information that we have at this point. Various possible outcomes are provided below along with examples.

- If the "=" sign is on the fifth tile from the left, then we have two possible scenarios involving the following on the left side of the "=" sign:
  - Two numbers (one two-digit and one one-digit) with:
    - One multiplication.

5	7	*	8	=	4	5	6

    - One addition.

9	9	+	2	=	1	0	1

- If the "=" sign is on the sixth tile from the left, then we have eleven possible scenarios involving the following on the left side of the "=" sign:
  - Two numbers (one three-digit and one one-digit) with:
    - One subtraction. Moreover, we know that the first tile will be "1" and the seventh tile will be "9".

1	0	3	−	9	=	9	4

    - One division.

1	2	5	/	5	=	2	5

  - Two numbers (both two-digit) with:
    - One addition.

1	1	+	7	7	=	8	8

    - One subtraction.

6	9	−	1	5	=	5	4

- Three numbers (all one-digit) with:
  - Two additions.

8	+	4	+	3	=	1	5

  - One addition and one subtraction.

9	+	7	−	2	=	1	4

  - Two multiplications.

2	*	2	*	5	=	2	0

  - One multiplication and one division.

9	*	8	/	2	=	3	6

  - One multiplication and one subtraction.

6	*	8	−	3	=	4	5

  - One multiplication and one addition.

7	*	9	+	5	=	6	8

  - One division and one addition.

9	/	1	+	8	=	1	7

- Finally, if the "=" sign is on the seventh tile from the left, then we have eight possible scenarios involving the following on the left side of the "=" sign:
  - Two numbers (one three-digit and one two-digit) with:
    - One subtraction. Moreover, we know that the first tile will be "1" and the fifth tile will be "9".

1	0	4	−	9	9	=	5

    - One division.

7	9	2	/	9	9	=	8

- Three numbers (one two-digit and two one-digit) with:
  - ∗ Two subtractions.

1	8	−	8	−	9	=	1

  - ∗ One addition and one subtraction.

1	0	+	3	−	7	=	6

  - ∗ One division and one subtraction.

1	8	/	3	−	1	=	5

  - ∗ One division and one addition.

2	8	/	4	+	2	=	9

  - ∗ One multiplication and one subtraction.

3	∗	5	−	1	2	=	3

  - ∗ One division and one multiplication.

3	/	7	∗	2	1	=	9

Understanding various possible outcomes, as outlined above, as well as having reasonable starting guesses would likely help you to solve the puzzle in six attempts or less. Have fun and good luck!

The online game is accessible at https://nerdlegame.com/.

Doddy is a math enthusiast working as a nuclear engineer. The love of math and physics was the reason for him to choose this field. In his spare times, among other things he likes to solve math puzzles and problems. In addition to ***Crux***, the Project Euler has provided him with enough challenges and enjoyment in this area. Doddy and his family share their Oakville home with their four cats: Luke, Lorelai, Lincoln, and Lilian. Communications concerning the article can be shared with the author via email: `kastanya@yahoo.com`.

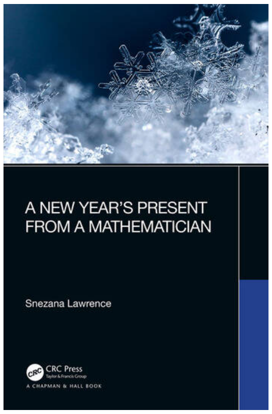
From the bookshelf of . . .

Veselin Jungic

*This new feature of **MathemAttic** brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to `MathemAttic@cms.math.ca`. Publishers are also welcome to send along books for possible review.*

A New Year's Present from a Mathematician
by Snezana Lawrence
ISBN 978-0367219369, 200 pages
Published by Chapman and Hall/CRC, Boca Raton, 2019.

Snezana Lawrence, the author of "A New Year's Present from a Mathematician," is a British historian of mathematics and a mathematics educator. An architect by undergraduate training, Lawrence earned her PhD in 2002 from the Open University, a British public research university. Her dissertation, "Geometry of Architecture and Freemasonry in 19th Century England," was supervised by Jeremy Gray, an English mathematician primarily interested in the history of mathematics. Currently, Lawrence is a senior lecturer in mathematics and design engineering at Middlesex University in London, England. Before her career as a university faculty, Lawrence worked as a high school mathematics teacher for several years. Among her numerous community engagements, Lawrence is Chair of the "History and Pedagogy of Mathematics" group, affiliated with the International Commission on Mathematical Instruction.



As this brief biographical sketch indicates, Lawrence's academic interests and experiences are wide and varied with mathematics, an engine and a mirror of human development over many centuries, at their centre. Having this in mind, "A New Year's Present from a Mathematician" reads as the author's intellectual memoir: Wherever we look in time and space, we find mathematics.

The author takes her readers on a journey through the wonderland of mathematics with the goal of visiting twelve landmarks depicted by the lives and work of 9 mathematicians (Isaac Newton, Christopher Wren, Emmy Noether, Maria Gaetana Agnesi, Luca Pacioli, John Dee, Paul Erdős, Jean-Baptiste le Rond d'Alambert, and Johannes Kepler), a famous group of mathematicians (under the collective pseudonym of Nicolas Bourbaki), an architectural masterpiece (Hagia Sophia), and an institution that marked the development of mathematics as we know it (The Royal Society of London for Improving Natural Knowledge). What connects these landmarks, spread out through Europe over two millennia, is their everlasting mathematical legacy ranging from the foundations of geometry to calculus to Ramsey theory.

Lawrence also demonstrates that it is impossible to look at the development of even a piece of mathematics in isolation: the landmarks she highlights are only a personal choice of a discrete set of points from a continuum that we call mathematics. This is underlined by author's choice not to follow chronological order in her narrative. Her approach of going back and forth in time, in this reader's view, further emphasizes the ever-presence of mathematics.

This insightful and warmly written book will be a pleasure to read for anyone who is learning, teaching, doing, or is just curious about mathematics and its history. It also may be used as a valuable teaching and learning resource to introduce, learn, or re-learn about the historical and mathematical topics that the book covers. When presenting mathematical topics, the author is a knowledgeable and objective narrator who makes sure that her readers are provided with all the information necessary to follow the narrative.

Still, to reduce this book to only its mathematical dimension would be an act of injustice not only towards the book but also towards its author.

Lawrence is a great storyteller and when she reflects about the personal lives of mathematicians and the circumstances in which they lived, she talks directly to her reader. In this way, the reader can see, feel, and understand these great individuals at a more intimate level. These moments are welcome reminders that learning and doing mathematics is both a personal and cultural experience that intertwines with an individual's everyday life and includes interactions with others who share similar mathematical and non-mathematical interests, knowledge, history, set of values, and so on.

The author's general kindness towards her protagonists and her genuine love for mathematics further distance this text from commonly impersonal mathematical books.

The wide range of topics covered in the book, the numerous beautiful illustrations, and the author's ability to capture the reader's attention by combining her personal reflections with the historical and mathematical facts, make this book a stimulating read.

For example, one may wonder who the "mathematician" from the book title is. Is this the book's author? Or perhaps this is a collective name for all of the mathematicians mentioned in the book? Or, perhaps more prosaically, a mathematician who does the last-minute holiday shopping? Or, as this reader would like to think, that unknown mathematics student, teacher, or researcher for whom learning, teaching, and creating mathematics is a life-long source of pride, happiness, and joy?

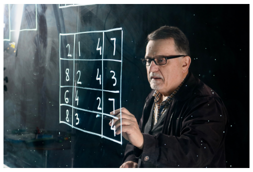
Or, as another example, one may ask to whom this book should be presented to as a gift? In other words, to which degree the knowledge presented in this text should be the outcome of the mathematical education of a student who is not aiming to become a professional mathematician? Would it be too much to ask an economist to list the five regular solids? Or to ask an engineer to know what the Ramsey number $R(3, 3)$ is?

More generally, what are the place and the role of mathematics in the knowledge of an educated person who is not a professional mathematician?

Instead of further elaborating on those questions, here are Lawrence's words that will hopefully encourage you to read her book:

> You may know a lot about mathematics already, but I hope that you will still be able to know more after you've read this book. Or you may know little or almost no mathematics, yet I am sure that you too will be able to learn from this book where to delve deeper into mathematics, and what to pick from the vast archive of abstract thought that mathematics has ways of neatly organizing.

. . . .



This book is a recommendation from the bookshelf of Veselin Jungic. Veselin Jungic is a 3M National Teaching Fellow and a Fellow of the Canadian Mathematical Society.

OLYMPIAD CORNER

No. 404

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **August 15, 2022**.

OC586. Prove that for any positive integers $a_1, a_2, \ldots, a_k$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} > 1,$$

the equation

$$\left\lfloor \frac{n}{a_1} \right\rfloor + \left\lfloor \frac{n}{a_2} \right\rfloor + \ldots + \left\lfloor \frac{n}{a_k} \right\rfloor = n$$

has at most $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ solutions in positive integers.

OC587. We call an arrangement of n ones and m zeros in a circle *good* if it is possible to swap an adjacent zero and one in such a way that we get an arrangement that differs from the original one by a rotation. For what natural numbers n and m does a good arrangement exist?

OC588. Let A be a finite ring and let $a, b \in A$ such that $(ab-1)b = 0$. Prove that $b(ab-1) = 0$.

OC589. Consider an acute triangle ABC where $AB < AC$. The bisector of the angle BAC intersects the side BC at point D. Point M is the midpoint of the side BC. Prove that the line passing through the centers of the triangles ABC and ADM is parallel to the line AD.

OC590. Find all real numbers c for which there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for each $x, y \in \mathbb{R}$ the following equality is satisfied:

$$f(f(x) + f(y)) + cxy = f(x + y).$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2022**.

OC586. Montrez que pour tout entiers positifs $a_1, a_2, \ldots, a_k$ vérifiant

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} > 1,$$

l'équation

$$\left\lfloor \frac{n}{a_1} \right\rfloor + \left\lfloor \frac{n}{a_2} \right\rfloor + \ldots + \left\lfloor \frac{n}{a_k} \right\rfloor = n$$

admet au plus $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ solutions par des entiers positifs.

OC587. On dit d'un arrangement circulaire de n uns et m zéros qu'il est *bon* s'il est possible d'interchanger zéro et un contigus de telle sorte qu'on obtienne un arrangement qui ne diffère de l'arrangement d'origine que par une rotation. Pour quels nombres naturels n et m existe-t-il un bon arrangement?

OC588. Soit A un anneau fini et soit $a, b \in A$ tels que $(ab-1)b = 0$. Montrez que $b(ab-1) = 0$.

OC589. Considérons un triangle acutangle ABC où $AB < AC$. La bissectrice de l'angle BAC rencontre le côté BC en D. Soit M le point milieu du côté BC. Montrez que la droite passant par les centres des cercles circonscrits respectifs des triangles ABC et ADM est parallèle à la droite AD.

OC590. Trouvez tous les nombres réels c pour lesquels il existe une fonction $f : \mathbb{R} \to \mathbb{R}$ telle que pour tout $x, y \in \mathbb{R}$ l'égalité suivante est vérifiée :

$$f(f(x) + f(y)) + cxy = f(x + y).$$

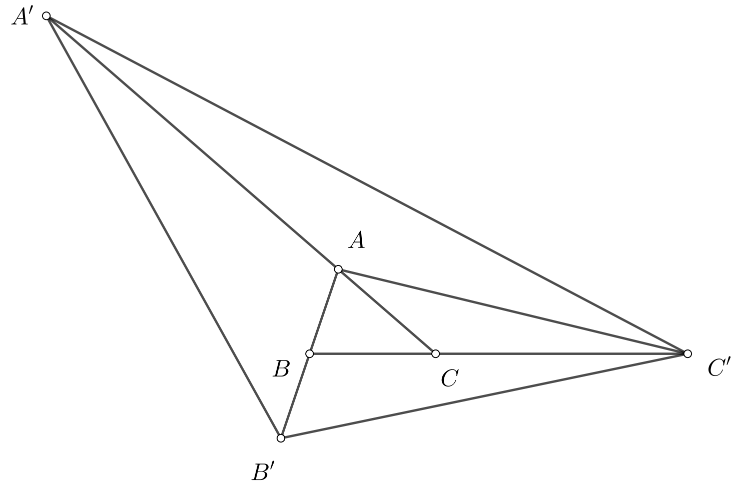
OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2022: 48(1), p. 19–20.

OC561. Let $\triangle ABC$ be an arbitrary triangle with area 1. The edge AB is extended past B to a point B' such that $|BB'| = |AB|$. Similarly, the edge BC is extended past C to a point C' such that $|CC'| = 2|BC|$; and CA is extended past A to a point A' such that $|AA'| = 3|CA|$. Find the area of $\triangle A'B'C'$.

Originally from the 2021 Science Atlantic undergraduate problem solving contest.

We received 14 submissions of which 13 were correct and complete. We present 2 solutions.



Solution 1, by Ivko Dimitrić.

It is clear that two triangles whose bases belong to the same line and have shared opposite vertex so that the altitudes from that vertex to the bases are equal have the ratio of their areas equal to the ratio of their bases. Let $\mathcal{A}$ denote the area of a triangle. Then

$$\mathcal{A}(BCB') = \mathcal{A}(ABC) = 1, \qquad \mathcal{A}(ACC') = 2\mathcal{A}(ABC) = 2,$$

$$\mathcal{A}(CB'C') = 2\mathcal{A}(BB'C) = 2, \qquad \mathcal{A}(AA'B) = 3\mathcal{A}(ABC) = 3,$$

$$\mathcal{A}(BA'B') = \mathcal{A}(AA'B) = 3, \qquad \mathcal{A}(AC'A') = 3\mathcal{A}(CC'A) = 6.$$

Adding up the areas of all seven triangles we get

$$\mathcal{A}(A'B'C') = 1 + 1 + 2 + 2 + 3 + 3 + 6 = 18.$$

Solution 2, by UCLan Cyprus Problem Solving Group.

We use barycentric coordinates. Let $A = (1, 0, 0), B = (0, 1, 0)$ and $C = (0, 0, 1)$. It is immediate to check that

$$B' = (-1, 2, 0), C' = (0, -2, 3), A' = (4, 0, -3).$$

For example, since $AA' = 3AC$, then the area of ABA' is triple the area of ABC. Furthermore, since the triples of points A, B, C and A, B, A' have opposite orientation, then the third coordinate of A' is -3. We now compute

$$\begin{vmatrix} 4 & 0 & -3 \\ -1 & 2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 0 & -3 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 2 \begin{vmatrix} 4 & -3 \\ -1 & 3 \end{vmatrix} = 18\,.$$

Since the triangle ABC has area 1, then the triangle $A'B'C'$ has area 18.

OC562. Ruby and Sapphire are celebrating Pi Day by sharing a circular pie. Ruby has two red birthday cake candles, and Sapphire has two blue candles. Ruby starting, they will alternately place one candle on the perimeter of the pie. (Of course, no two candles may be in the same place!) After all the candles are placed, each girl will get the portion of the pie that is closer to one of her candles than to any of the others. The goal is to get strictly more pie than one's opponent; an equal division is a draw.

Either find a winning strategy for one player and show that it is essentially unique, or show that the game, rationally played, is a draw.

Originally from the 2021 Science Atlantic undergraduate problem solving contest.

We received 3 solutions. We present the solution by UCLan Cyprus Problem Solving Group.

We will show that Sapphire has a winning strategy.

We will record the (anti-clockwise) angles from the first candle of Ruby (i.e. by which angle we need to rotate the first candle of Ruby about the center of the pie).

In her first move Sapphire places her candle diametrically opposite the candle of Ruby, i.e. at an angle of π. Assume that the next candle of Ruby is at an angle of ϑ. Without loss of generality $0 < \vartheta < \pi$. If Ruby places her second candle at an angle ϑ, then it is not difficult to see that Sapphire wins if and only if her next candle is at an angle

$$\varphi \in (\vartheta, \pi - \vartheta) \cup (\pi + \vartheta, 2\pi).$$

Note that the first of these two intervals might be empty (if $\vartheta \geq \pi/2$) but the second interval is never empty.

The above strategy of Sapphire is 'unique'. If she does not pick the angle π, then Ruby can do so and the game will end in a draw or a win for Ruby.

OC563. Find, with proof, $\int_0^{\pi/2} \cos^{31416}(x)dx$.

Originally from the 2021 Science Atlantic undergraduate problem solving contest.

We received 14 solutions. We present the solution by Oliver Geupel.

For $m \geq 0$, let $I_m = \int_0^{\pi/2} \cos^{2m} x \, \mathrm{d}x$. We prove that

$$I_m = \frac{\pi}{2^{2m+1}} \binom{2m}{m},$$

so that

$$\int_0^{\pi/2} \cos^{31416} x \, \mathrm{d}x = \frac{\pi}{2^{31417}} \binom{31416}{15708} = 0.00707 \ldots.$$

We have $I_0 = \int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2}$. For $m > 0$, integration by parts yields

$$\begin{aligned} I_m &= \int_0^{\pi/2} \cos^{2m-1} x \, \mathrm{d} \sin x \\ &= \left[\cos^{2m-1} x \sin x\right]_0^{\pi/2} + \int_0^{\pi/2} (2m-1) \cos^{2m-2} x \sin^2 x \, \mathrm{d}x \\ &= (2m-1) \int_0^{\pi/2} \cos^{2m-2} x \left(1 - \cos^2 x\right) \mathrm{d}x \\ &= (2m-1) \left(I_{m-1} - I_m\right), \end{aligned}$$

from which we obtain the recursion

$$I_m = \frac{2m-1}{2m} I_{m-1}.$$

It follows

$$\begin{aligned} I_m &= \frac{(2m-1)!!}{(2m)!!} \cdot I_0 \\ &= \frac{(2m)!}{2^{2m}(m!)^2} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{2^{2m+1}} \binom{2m}{m}. \end{aligned}$$

The proof is complete.

Remark. A related identity with similar proof is

$$\int_0^{\pi/2} \cos^{2m+1} x \, \mathrm{d}x = \frac{4^m}{(2m+1)\binom{2m}{m}}.$$

The formulas are well-known, cf. Gradshteyn, I.S., Ryzhik, I.M., *Tables of Integrals, Series, and Products*, Seventh Edition, Elsevier, 2007, formulas 3.621.3–4.

OC564. Define a "Fibonacci-like" sequence as follows: $A_1 = A_2 = 1$, and $A_n = 2A_{n-2} + A_{n-1}$ for $n \geq 3$; so $A_3 = 2 \times 1 + 1 = 3$, $A_4 = 2 \times 1 + 3 = 5$, and so on. Prove that for odd n,

$$\sum_{i=1}^{n-1} A_i = A_n - 1$$

Originally from the 2021 Science Atlantic undergraduate problem solving contest.

We received 19 correct solutions. We present 2 solutions.

Solution 1, by Henry Ricardo.

For $n = 3$, we see that $\sum_{i=1}^{2} A_i = 1 + 1 = 2 = A_3 - 1$.

Now assume that the result holds for some odd integer $N > 3$. Then the next higher odd number is $N + 2$ and we have

$$\begin{aligned}\sum_{i=1}^{(N+2)-1} A_i = \sum_{i=1}^{N+1} A_i = \sum_{i=1}^{N-1} A_i + A_N + A_{N+1} &= (A_N - 1) + A_N + A_{N+1} \\ &= 2A_N + A_{N+1} - 1 \\ &= A_{N+2} - 1,\end{aligned}$$

and our inductive proof is complete.

Solution 2, by Ivko Dimitrić.

To solve the linear recurrence relation

$$A_n - A_{n-1} - 2A_{n-2}, \quad A_1 = A_2 = 1,$$

we find the roots of the characteristic equation $r^2 - r - 2 = 0$ to be both real, $r_1 = 2$, $r_2 = -1$. Then $A_n = b\,2^n + c\,(-1)^n$ where constants b and c are found from the initial values for $n = 1, 2$ to be $b = 1/3$ and $c = -1/3$, so that

$$A_n = \frac{1}{3}\Big[2^n + (-1)^{n+1}\Big]. \tag{1}$$

Hence when n is odd, we have $(-1)^{n+1} = 1$ and $A_n - 1 = \frac{1}{3}(2^n - 2)$. On the other hand, using (1) and summing geometric sequences we have

$$\begin{aligned}\sum_{i=1}^{n-1} A_i &= \frac{1}{3}\left(\sum_{i=1}^{n-1} 2^i + \sum_{i=1}^{n-1} (-1)^{i+1}\right) \\ &= \frac{1}{3}\left(2 \cdot \frac{2^{n-1} - 1}{2 - 1} + \frac{1 - (-1)^{n+1}}{1 - (-1)}\right),\end{aligned}$$

which reduces to $\frac{1}{3}(2^n - 2)$ when n is odd. This proves the claim.

OC565. Given that $\sin(xy) = 1$, find the least upper bound of $\sin(x)\sin(y)$, and show that this is never achieved.

Originally from the 2021 Science Atlantic undergraduate problem solving contest.

We received 4 correct solutions. We present the solution by UCLan Cyprus Problem Solving Group.

By continuity of $\cos x$, given $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\cos(\frac{1}{n}) > 1 - \varepsilon$. Given this n, we find the largest odd natural number smaller than $(4n+1)\pi/2$, which leaves remainder 1 when divided by 4. Say $4k+1 < (4n+1)\pi/2 \leqslant 4k+5$. Then

$$0 \leqslant \frac{4k+5}{4n+1} - \frac{\pi}{2} < \frac{4}{4n+1} < \frac{1}{n}\,,$$

thus

$$\sin\left(\frac{4k+5}{4n+1}\right) = \cos\left(\frac{4k+5}{4n+1} - \frac{\pi}{2}\right) > 1 - \varepsilon\,.$$

Taking $x = \dfrac{(4n+1)\pi}{2}$ and $y = \dfrac{4k+5}{4n+1}$ we have $\sin(xy) = \sin\left(\left(2k+2+\frac{1}{2}\right)\pi\right) = 1$ and

$$\sin(x)\sin(y) = \sin\left(\left(2n+\frac{1}{2}\right)\pi\right)\sin\left(\frac{4k+5}{4n+1}\right) = \sin\left(\frac{4k+5}{4n+1}\right) > 1 - \varepsilon\,.$$

So the upper bound is indeed 1.

If the upper bound can be achieved, then there are odd integers r, s, t such that $x = r\pi/2, y = s\pi/2$ and $xy = t\pi/2$. But then $\pi = \frac{2t}{rs}$, a contradiction as π is irrational.

Two More Proofs of the Inequality $-1 < \cos A \cos B \cos C \leq \frac{1}{8}$

Abbas Galehpour Aghdam

Problem 1. *Let A, B, and C denote the angles of a triangle ABC. Show that the following inequality holds:*

$$-1 < \cos A \cos B \cos C \leq \frac{1}{8}. \tag{1}$$

In [2], Murty discussed this problem and presented four proofs for inequality. We will also solve it by examining two different approaches.

The left-hand side of (1) follows immediately since the values of $\cos A, \cos B$ and $\cos C$ are always more than -1, then the product of cosines is also more than -1.

For the right-hand side of (1), there are three cases:

Case 1 ABC is a right triangle.

In this case, the product of the cosines of the three angles is zero since $\cos 90^\circ = 0$. Clearly, zero is less than $\frac{1}{8}$, so the right-hand side of (1) is valid.

Case 2 ABC is an obtuse triangle.

In any obtuse triangle, the product of the cosines is negative since one of the angles is an obtuse angle. Clearly, every negative number is less than $\frac{1}{8}$.

Case 3 ABC is an acute triangle.

For this case, we present two solutions to the right-hand side of (1).

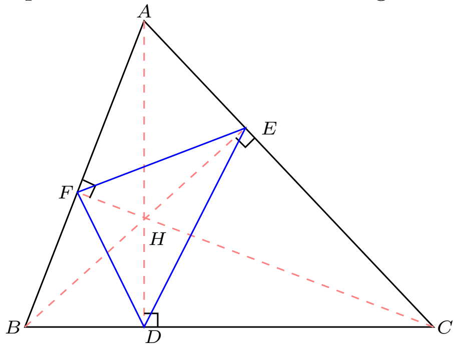


Figure 1: Altitudes of ΔABC are concurrent at orthocenter H of ΔDEF

Method 1

Let AD, BE, CF be the altitudes of ΔABC, concurrent at orthocenter $H \cdot \Delta DEF$ is the orthic triangle of ΔABC and we denote its side lengths by $EF = a'$, $DF = b'$, and $DE = c'$ (Figure 1).

As illustrated in Figure 2, circumscribe ΔABC, let O be the center of the circumcircle of ΔABC and its radius be R. We extend HD, HE, HF to meet the circumcircle of ΔABC at A', B', and C' respectively.

$\Delta A'B'C'$ is named the expanded orthic of ΔABC; both triangles have the same circumcircle with center O and radius R.

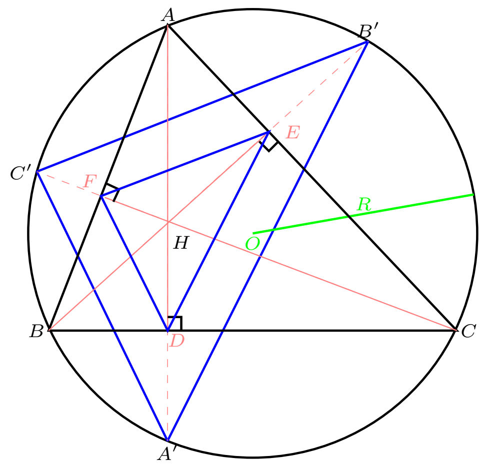


Figure 2: circumscribe ΔABC

To prove $\cos A \cos B \cos C \leq \dfrac{1}{8}$, we first show that:

$$\frac{area[\Delta A'B'C']}{area[\Delta ABC]} = 8 \cos A \cos B \cos C \qquad (2)$$

To show this, we need the following three theorems that have been proven by Bill Pang in [1], that any one can refer to.

Theorem 1. *The ltitudes of an acute triangle bisect the angles of its orthic triangle. Thus (as in Figure 1), AD, BE and CF are the angle bisectors of ΔDEF.*

Theorem 2. *The expanded orthic and the orthic triangle are homothetic with center of dilation H and factor of dilation = 2, thus:*

$$\frac{area[\Delta A'B'C']}{area[\Delta DEF]} = 4$$

Theorem 3. *The areas of $\Delta A'B'C'$ and ΔABC can be determined in terms of a', b', c' that are the side lengths of ΔDEF:*

$$area[\Delta A'B'C'] = \sqrt{(a'+b'+c')(a'+b'-c')(a'+c'-b')(b'+c'-a')}$$
$$area[\Delta ABC] = \frac{a'b'c'(a'+b'+c')}{\sqrt{(a'+b'+c')(a'+b'-c')(a'+c'-b')(b'+c'-a')}}$$

Note. Theorems 2 and 3 are valid in any triangle, but Theorem 1 is only valid in acute triangles.

We also need the following four results that are given here as exercises for the readers.

Exercise 1. *In any acute triangle, the angles of the orthic triangle are the supplements of twice the angles of the original triangle. Thus (as in Figure 1), we have:*

$$\angle D = 180^\circ - 2\angle A, \quad \angle E = 180^\circ - 2\angle B \quad \text{and} \quad \angle F = 180^\circ - 2\angle C.$$

Exercise 2. *Let R and R' denote the circumradii of the original triangle and its orthic triangle respectively. Prove that $R' = \frac{R}{2}$.*

Exercise 3. *In Figure 2, prove that:*

$$area[\Delta DEF] = \frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C.$$

Exercise 4. *In any triangle with side lengths a, b and c, the following inequality is valid:*

$$abc \geq (a+b-c)(a+c-b)(b+c-a)$$

Now, we are ready to show equality (2). With the help of Theorem 2 and Exercise 3, we can write:

$$area[\Delta A'B'C'] = 4\ area[\Delta DEF] = 2R^2 \sin 2A \sin 2B \sin 2C \tag{3}$$

Also, we have the following well-known relation:

$$area[\Delta ABC] = 2R^2 \sin A \sin B \sin C \tag{4}$$

By dividing (3) by (4), we get:

$$\begin{aligned}\frac{area[\Delta A'B'C']}{area[\Delta ABC]} &= \frac{2R^2 \sin 2A \sin 2B \sin 2C}{2R^2 \sin A \sin B \sin C} \\ &= \frac{(2\sin A \cos A)(2\sin B \cos B)(2\sin C \cos C)}{\sin A \sin B \sin C} \\ &= 8\cos A \cos B \cos C.\end{aligned}$$

Now, we need only establish that: $area[\Delta A'B'C'] \leq area[\Delta ABC]$. For this, we use Exercise 4. For a', b' and c' (the side lengths of ΔDEF), we can say:

$$a'b'c' \geq (a' + b' - c')(a' + c' - b')(b' + c' - a').$$

Multiplying by $(a' + b' + c')$ yields:

$$a'b'c'(a' + b' + c') \geq (a' + b' + c')(a' + b' - c')(a' + c' - b')(b' + c' - a').$$

Clearly, $(a' + b' + c')$ is positive. Moreover, based on the well-known triangle inequality, the other three terms on the right-hand side are also more than zero, hence we can write:

$$a'b'c'(a' + b' + c') \geq \left(\sqrt{(a' + b' + c')(a' + b' - c')(a' + c' - b')(b' + c' - a')}\right)^2.$$

Dividing both sides by $\sqrt{(a' + b' + c')(a' + b' - c')(a' + c' - b')(b' + c' - a')}$ and using Theorem (3), gives us the fact that $area[\Delta A'B'C'] \leq area[\Delta ABC]$, so $8 \cos A \cos B \cos C \leq 1$, or

$$\cos A \cos B \cos C \leq \frac{1}{8}.$$

Method 2

We first present an inequality and leave its proof as an exercise to the readers.

Exercise 5. *In any triangle ABC with side lengths a, b and c, the following inequality holds:*

$$\left(\frac{b}{c} + \frac{c}{b}\right) \cos A + \left(\frac{a}{c} + \frac{c}{a}\right) \cos B + \left(\frac{a}{b} + \frac{b}{a}\right) \cos C = 3 \tag{5}$$

For any positive number n, we have $n + \dfrac{1}{n} \geq 2$, so we can write:

$$\frac{b}{c} + \frac{c}{b} \geq 2; \tag{6}$$

$$\frac{a}{c} + \frac{c}{a} \geq 2; \tag{7}$$

$$\frac{a}{b} + \frac{b}{a} \geq 2; \tag{8}$$

(6) $\times \cos A +$ (7) $\times \cos B +$ (8) $\times \cos C$ yields:

$$\left(\frac{b}{c} + \frac{c}{b}\right) \cos A + \left(\frac{a}{c} + \frac{c}{a}\right) \cos B + \left(\frac{a}{b} + \frac{b}{a}\right) \cos C \geq 2(\cos A + \cos B + \cos C). \tag{9}$$

(5) and (9) give us:

$$(\cos A + \cos B + \cos C) \leq \frac{3}{2}. \tag{10}$$

Since ABC is an acute triangle, thus $\cos A, \cos B, \cos C$ are positive. The AM-GM inequality gives us:

$$\frac{\cos A + \cos B + \cos C}{3} \geq \sqrt[3]{\cos A \cos B \cos C},$$

or:

$$(\cos A + \cos B + \cos C)^3 \geq 27 \cos A \cos B \cos C. \tag{11}$$

(10) and (11) yield:

$$27 \cos A \cos B \cos C \leq \left(\frac{3}{2}\right)^3 = \frac{27}{8},$$

or:

$$\cos A \cos B \cos C \leq \frac{1}{8}.$$

References

[1] B. Pang, From the orthic triangle, *Pi in the Sky*, Issue 12, Fall 2008, p. 12–15, https://media.pims.math.ca/pi_in_sky/pi12.pdf

[2] V. N. Murty, Four proofs of the inequality $-1 < \cos A \cos B \cos C \leq \frac{1}{8}$, *Crux Mathematicorum with Mathematical Mayhem*, Volume 29 (2), February 2003, p. 82–83, https://cms.math.ca/publications/crux/issue/?volume=29&issue=2

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2022**.

4751. *Proposed by Michel Bataille.*

Let m, n be integers such that $0 \leq m \leq n$. Evaluate in closed form

$$\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{2n-m-k}{k}.$$

4752. *Proposed by George Stoica.*

Let $\displaystyle\sum_{n=1}^{\infty} x_n = \infty$ for $x_n > 0$, $n = 1, 2, \ldots$ Prove that $\displaystyle\sum_{n=1}^{\infty} x_n^{1+\frac{\varepsilon}{n}} = \infty$ for any $\varepsilon > 0$.

4753. *Proposed by Luu Dong.*

Let $ABCD$ be a quadrilateral for which AD is not parallel to BC. Fix a point E on AB different from A, B, and let F be a variable point on the line CD. Denote the projections of C and D on the line EF by M and N, respectively. If P is the intersection of the lines through M perpendicular to AD and through N perpendicular to BC, prove that the circumcenter of triangle MNP lies on a fixed circle as F moves along CD.

4754. *Proposed by Mihaela Berindeanu.*

If triangle ABC has circumradius $R = \sqrt{2}$ and its angles satisfy

$$2\sin A + 3\cos B\cos C = 4,$$

determine its area.

4755. *Proposed by Nguyen Tien Lam.*

Let x, y, z be positive integers such that $gcd(x, y, z) = 1$ and

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$$

is an integer. Prove that xyz is a perfect square.

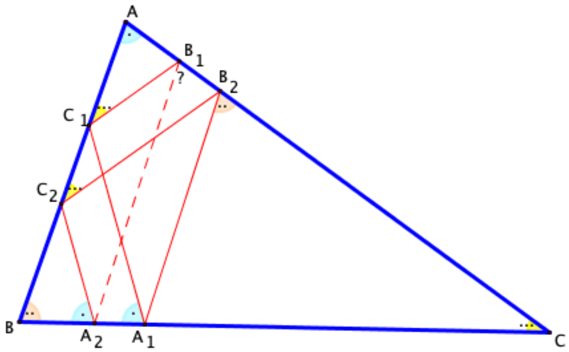
4756. *Proposed by Daniel Sitaru, modified by the Editorial Board.*

Evaluate

$$\lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} \log_n k.$$

4757. *Proposed by J. Chris Fisher.*

Recall that the line PQ is *antiparallel to BC with respect to $\angle BAC$* if $P \in AC$, $Q \in AB$, and $\angle PQA = \angle BCA$. (Equivalently, if ΔAPQ is oppositely similar to ΔABC; equivalently, if the points B, C, P, Q are concyclic.) Given a triangle ABC with points $A_i \in BC$, $B_i \in CA$, and $C_i \in AB$, $i = 1, 2$, arranged so that



$$\begin{array}{lll} B_1C_1, & B_2C_2 & \text{are both antiparallel to } BC \text{ with respect to } \angle BAC; \\ C_1A_1, & C_2A_2 & \text{are both antiparallel to } CA \text{ with respect to } \angle CBA; \text{ and} \\ & A_1B_2 & \text{is antiparallel to } AB \text{ with respect to } \angle ACB, \end{array}$$

prove that A_2B_1 is antiparallel to AB (with respect to $\angle ACB$).

4758. *Proposed by Florentin Visescu.*

Show that

$$\arctan\frac{a}{b+c} + \arctan\frac{b}{c+a} + \arctan\frac{c}{a+b} \geq \arctan\frac{11}{2}.$$

for all positive a, b, c.

4759. *Proposed by Boris Čolaković.*

For all positive real numbers a, b, c, x, y, z, prove that

$$\frac{y+z}{x}\cdot\frac{b+c}{a} + \frac{z+x}{y}\cdot\frac{c+a}{b} + \frac{x+y}{z}\cdot\frac{a+b}{c} \geq 8\left(\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y}\right).$$

4760. *Proposed by Goran Conar.*

Let $a_i \in \left\langle 0, \frac{1}{2} \right\rangle, i \in \{1, 2, \ldots, n\}$ be real numbers such that $\sum_{i=1}^{n} a_i = 1$. Prove that

$$n\sqrt{\frac{n-1}{n+1}} \leq \sum_{i=1}^{n} \sqrt{\frac{1-a_i}{1+a_i}} < (n+1)\sqrt{\frac{n-1}{n+1}}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2022**.

4751. *Soumis par Michel Bataille.*

Soit m et n des entiers vérifiant $0 \leq m \leq n$. Évaluez, sous forme close, l'expression suivante :

$$\sum_{k=0}^{n} (-1)^k \binom{2n-2k}{n-k} \binom{2n-m-k}{k}.$$

4752. *Soumis par George Stoica.*

Soit $\sum_{n=1}^{\infty} x_n = \infty$ où $x_n > 0$ pour $n = 1, 2, \ldots$. Montrez que $\sum_{n=1}^{\infty} x_n^{1+\frac{\varepsilon}{n}} = \infty$ pour tout $\varepsilon > 0$.

4753. *Soumis par Luu Dong.*

Soit $ABCD$ un quadrilatère pour lequel AD n'est pas parallèle à BC. Fixons un point E sur AB distinct de A et B. Soit F un point variable sur la droite CD. Désignons les projections de C et D sur la droit EF par M et N respectivement. Si P désigne l'intersection de la droite perpendiculaire à AD passant par M avec la droite perpendiculaire à BC passant par N, montrez que – lorsque F parcours CD – le centre du cercle circonscrit au triangle MNP est situé sur un cercle fixe.

4754. *Soumis par Mihaela Berindeanu.*

Soit ABC un triangle dont le cercle circonscrit a un rayon $R = \sqrt{2}$ et dont les angles satisfont

$$2\sin A + 3\cos B\cos C = 4.$$

Trouvez l'aire de ce triangle.

4755. *Soumis par Nguyen Tien Lam.*

Soit x, y et z des entiers positifs vérifiant $\mathrm{PGCD}(x, y, z) = 1$ et pour lesquels

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$$

est un entier. Montrez que xyz est un carré parfait.

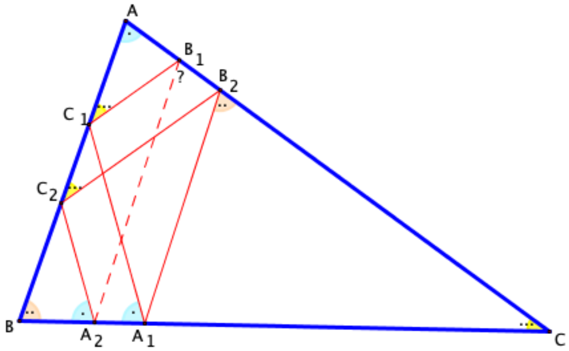
4756. *Soumis par Daniel Sitaru, modifié par le comité de rédaction.*

Évaluez

$$\lim_{n\to\infty} \frac{1}{n^{n+1}} \sum_{k=2}^{n^n} \log_n k.$$

4757. *Soumis par J. Chris Fisher.*

Rappelons que la droite PQ est dite *anti-parallèle à BC par rapport à $\angle BAC$* si $P \in AC$, $Q \in AB$ et $\angle PQA = \angle BCA$. (De façon équivalente, si ΔAPQ est inversement similaire à ΔABC; ou encore, si les points B, C, P, Q sont cocycliques.) Étant donné un triangle ABC et des points $A_i \in BC$, $B_i \in CA$ et $C_i \in AB$, $i = 1, 2$, disposés de sorte que



B_1C_1, B_2C_2	sont toutes deux anti-parallèles à BC par rapport à $\angle BAC$;	
C_1A_1, C_2A_2	sont toutes deux anti-parallèles à CA par rapport à $\angle CBA$; et	
A_1B_2	est anti-parallèle à AB par rapport à $\angle ACB$,	

montrez que A_2B_1 est anti-parallèle à AB (par rapport à $\angle ACB$).

4758. *Soumis par Florentin Visescu.*

Montrez que

$$\arctan\frac{a}{b+c}+\arctan\frac{b}{c+a}+\arctan\frac{c}{a+b}\geq\arctan\frac{11}{2}.$$

pour tout nombres réels positifs a, b et c.

4759. *Soumis par Boris Čolaković.*

Montrez que

$$\frac{y+z}{x}\cdot\frac{b+c}{a}+\frac{z+x}{y}\cdot\frac{c+a}{b}+\frac{x+y}{z}\cdot\frac{a+b}{c}\geq 8\left(\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}\right).$$

pour tout nombres réels positifs a, b, c, x, y et z.

4760. *Soumis par Goran Conar.*

Soit $a_i \in \left\langle 0, \frac{1}{2}\right\rangle$, $i \in \{1, 2, \ldots, n\}$ des nombres réels vérifiant $\sum_{i=1}^n a_i = 1$. Montrez que

$$n\sqrt{\frac{n-1}{n+1}}\leq\sum_{i=1}^{n}\sqrt{\frac{1-a_i}{1+a_i}}<(n+1)\sqrt{\frac{n-1}{n+1}}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2022: 48(6), p. 41–44.

4701. *Proposed by Michel Bataille.*

Let AD, BE, CF be the internal angle bisectors of ΔABC (with D on BC, E on CA, F on AB). Let the perpendicular to BC through D intersect the perpendicular bisector of AD at A' and let B', C' be similarly constructed. Prove that the lines AA', BB', CC' are concurrent and that

$$AA' \cdot BB' \cdot CC' \leq \left(\frac{3R}{4}\right)^3,$$

where R is the circumradius of ΔABC.

There were 5 solutions and 1 incomplete submission. We present two solutions.

Solution 1, by the proposer.

Let O and R be the centre and radius of the circumcircle of ABC. Suppose that AD produced meets the circumcircle again at U. Since BU and CU subtend equal angles at A, $BU = CU$ and OU right bisects BC. Hence $DA' \| UO$. Since triangles ADA' and AUO are isosceles and AU is a transversal of DA' and UO,

$$\angle DAA' = \angle ADA' = \angle AUO = \angle UAO = \angle DAO.$$

Thus O lies on AA'. Similarly, O lies on BB' and CC'. (The triangles ADA' and AUO are related by a homothey with centre A.)

Since $\angle AUB = \angle ACB = \angle ACD$ and $\angle BAU = \angle DAC$, then triangles AUB and ACD are similar. Hence

$$AU : b = AU : AC = AB : AD = c : AD$$

and $AU \cdot AD = bc$.

Triangles DAA' and UAO are similar, so $AA' : R = AA' : AO = AD : AU$. Therefore $AA' = R(AD^2/bc)$. Likewise $BB' = R(BE^2/ac)$ and $CC' = R(CF^2/ab)$.

Applying the Law of Cosines to triangles ABD and ACD, using $BD = ac(b+c)^{-1}$, $CD = ab(b+c)^{-1}$ and eliminating the cosines of the angles at D, we find that

$$AD^2 = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} = \frac{4bcs(s-a)}{(b+c)^2} \leq s(s-a).$$

Also $BE^2 \le s(s-b)$ and $CF^2 \le s(s-c)$. Since $abc = 4R\sqrt{s(s-a)(s-b)(s-c)}$,

$$AA' \cdot BB' \cdot CC' = R^3 \left(\frac{AD \cdot BE \cdot CF}{abc}\right)^2 \le \frac{Rs^2}{16}.$$

From the concavity of the sine function,

$$2s = a + b + c = 2R(\sin A + \sin B + \sin C) \le 6R\sin(\tfrac{A+B+C}{3}) = 3\sqrt{3}R.$$

Therefore $s^2 \le 27R^2/4$ and $AA' \cdot BB' \cdot CC' \le 27R^3/64$, as desired. Equality occurs if and only if the triangle is equilateral.

Solution 2, by UCLan Cyprus Problem Solving Group.

Let α, β, γ be the angles at A, B, C. Then $\angle ADC = (\alpha/2) + \beta$ and

$$\begin{aligned}\angle BAA' &= \angle BAD + \angle DAA' = (\alpha/2) + \angle ADA' \\ &= (\alpha/2) + \angle ADC - 90^\circ = \alpha + \beta - 90^\circ = 90^\circ - \gamma.\end{aligned}$$

On the other hand, $\angle AOB = 2\gamma$ and $\angle BAO = \angle ABO = 90^\circ - \gamma$. Therefore $\angle BAA' = \angle BAO$ and so O lies on the line AA'. Similarly, O lies on BB' and CC'.

We have that

$$\begin{aligned}AA' &= \frac{AD}{2\cos\angle ADA'} = \frac{AD}{2\sin\angle ADB} = \frac{AD^2}{2c\sin\beta} \\ &= \frac{R \cdot AD^2}{bc} = R\left(1 - \frac{a^2}{(b+c)^2}\right)^2,\end{aligned}$$

where we have used the Law of Sines on triangle ADB to replace $\sin\angle ADB$ and the fact that $b = 2R\sin\beta$. There are similar expressions for BB' and CC'.

Let $x = a/(b+c)$, $y = b/(c+a)$, $z = c/(a+b)$. Since

$$\begin{aligned}6 + 2(x+y+z) &= 2(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \\ &= ((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \ge 9,\end{aligned}$$

$x + y + z \ge 3/2$ and

$$x^2 + y^2 + z^2 \ge \tfrac{1}{3}(x+y+z)^2 \ge \tfrac{3}{4}.$$

Finally, using the arithmetic-geometric means inequality, we obtain that

$$\begin{aligned}AA' \cdot BB' \cdot CC' = R^3(1-x^2)(1-y^2)(1-z^2) &\le \frac{R^3}{3^3}[3 - (x^2+y^2+z^2)]^3 \\ &\le \frac{R^3}{3^3}\left(\frac{9}{4}\right)^3 = \left(\frac{3R}{4}\right)^3.\end{aligned}$$

Equality holds if and only if the triangle is equilateral.

Comments by the editor. If AA' is produced to meet BC at A'', BB' to meet AC at B'' and CC' to meet AB at C'', Walther Janous used the trigonometric form of the converse of Ceva's Theorem to show that AA'', BB'' and CC'' met at a common point. Marie Nicole Gras and he, independently, obtained the equation

$$AA' \cdot BB' \cdot CC' = R^3 \left(\frac{4sr}{s^2 + 2Rr + r^2} \right)^2 .$$

Some solutions involved a significant amount of trigonometric manipulation. For example, C.R. Pranesachar started with $AD = (2bc \cos \frac{\alpha}{2})/(b+c)$, whereupon

$$AA' = \frac{AD}{2\cos(\beta - \gamma)/2} = \frac{4Rbc \cos^2 \alpha/2}{(b+c)^2}.$$

Along with similar expressions for BB' and CC', we are led to

$$AA' \cdot BB' \cdot CC' = \frac{64R^3 a^2 b^2 c^2 (\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2})^2}{(b+c)^2 (c+a)^2 (a+b)^2}.$$

The cosine product is dominated by $3\sqrt{3}/8$ and we have $4ab \leq (a+b)^2$ and the like to obtain the desired upper bound.

4702. *Proposed by S. Chandrasekhar.*

Let p be a prime which is congruent to 3 (mod 4). Let S denote the set of square elements in the field of integers modulo p. Then show that

$$\prod_{\substack{a \,<\, b \\ a,\, b \,\in\, S}} (a+b) = \pm 1 \pmod{p}.$$

We received 8 submissions and 6 of them were complete and correct. We present two solutions. The first solution is based on elementary number theory, and the second solution requires some basic knowledge of finite fields.

First, observe that it suffices to show that

$$P := \prod_{\substack{a \neq b \\ a,b \in S}} (a+b) \equiv 1 \pmod{p}.$$

Solution 1, by the majority of solvers, slightly modified by the editor.

Let $S^* = S \setminus \{0\}$. Since $p \equiv 3 \pmod 4$ and $(-1)^{(p-1)/2} \equiv -1 \pmod p$, we have $-1 \notin S$. It follows that for $a \neq 0$, we have $a \in S$ if and only if $-a \notin S$.

For $a \in S$, we define

$$P_a := \prod_{b \in S \setminus \{a\}} (a+b).$$

Note that if $a \in S^*$, then as b runs over $S \setminus \{a\}$, ba^{-1} also runs over $S \setminus \{1\}$. Thus if $a \in S^*$, then

$$P_a = \prod_{b \in S \setminus \{a\}} (a + b) \equiv a^{(p-1)/2} \prod_{b \in S \setminus \{a\}} (1 + ba^{-1}) \equiv \prod_{c \in S \setminus \{1\}} (1 + c) \pmod{p}.$$

This shows that $P_a \equiv P_1 \pmod{p}$ for each $a \in S^*$.

To compute P_0, we apply Wilson's theorem to obtain

$$-1 \equiv (p-1)! \equiv \prod_{a \in S^*} a \prod_{a \in S^*} (-a) \equiv (-1)^{(p-1)/2} P_0^2 \equiv -P_0^2.$$

It follows that $P_0 \equiv \pm 1 \pmod{p}$. However, since $P_0 = \prod_{a \in S^*} a \in S$, we must have $P_0 \equiv 1 \pmod{p}$.

To compute P_1, recall that we have the polynomial identity

$$\prod_{a \notin S} (x - a) \equiv x^{(p-1)/2} + 1 \pmod{p}.$$

It follows that

$$\prod_{a \in S^*} (x + a) \equiv \prod_{a \notin S} (x - a) \equiv x^{(p-1)/2} + 1 \pmod{p}.$$

In particular,

$$2P_1 \equiv \prod_{a \in S} (1 + a) \equiv 1^{(p-1)/2} + 1 \equiv 2 \pmod{p},$$

and thus $P_1 \equiv 1 \pmod{p}$. We conclude that

$$P \equiv P_0 P_1^{(p-1)/2} \equiv 1 \pmod{p}.$$

Solution 2, by Marie-Nicole Gras, slightly modified by the editor.

Let $\mathbb{F}_p$ be the field with p elements, and $\mathbb{F}_{p^2}$ be the field with p^2 elements. Since $p \equiv 3 \pmod{4}$, -1 is not a square in $\mathbb{F}_p^*$. Thus, $x^2 + 1$ is an irreducible polynomial over $\mathbb{F}_p$ and $\mathbb{F}_p[x]/(x^2+1) \cong \mathbb{F}_{p^2}$. Let $i \in \mathbb{F}_{p^2}$ such that $i^2 = -1$; then we can identify $\mathbb{F}_p \times \mathbb{F}_p$ with $\mathbb{F}_{p^2}$ by identifying (u, v) with $u + iv$, where $u, v \in \mathbb{F}_p$. Consider the norm map $N : \mathbb{F}_{p^2}^* \to \mathbb{F}_p^*$ with respect to this quadratic extension $\mathbb{F}_{p^2}/\mathbb{F}_p$:

$$N(u + iv) = u^2 + v^2, \quad \forall (u, v) \neq (0, 0).$$

Note that N is a homomorphism of multiplicative groups and N is surjective. It follows that $\ker(N) = \frac{p^2-1}{p-1} = p + 1$. Thus, for each $w \in \mathbb{F}_p^*$, there are exactly $p + 1$ solutions to the equation $u^2 + v^2 = N(u + iv) = w$, where $u, v \in \mathbb{F}_p$ such that $(u, v) \neq (0, 0)$.

If $p \equiv 7 \pmod 8$, then 2 is a square in $\mathbb{F}_p^*$. If w is a square in $\mathbb{F}_p^*$, then $w = 2u^2$ for some $u \in \mathbb{F}_p$, so the number of solutions to $a+b=w$, where $a, b \in S$ such that $a \neq b$ is

$$\frac{p+1-8}{4} + 2 = \frac{p+1}{4}.$$

(Each solution $w = a + b$ corresponds to 4 solutions to $w = u^2 + v^2$ in the generic case, but we need to consider the special case that $a = b$ or $ab = 0$.) If w is a non-square in $\mathbb{F}_p^*$, then the number of solutions to $a + b = w$, where $a, b \in S$ such that $a \neq b$ is also $\frac{p+1}{4}$. Consequently, Wilson's theorem implies that

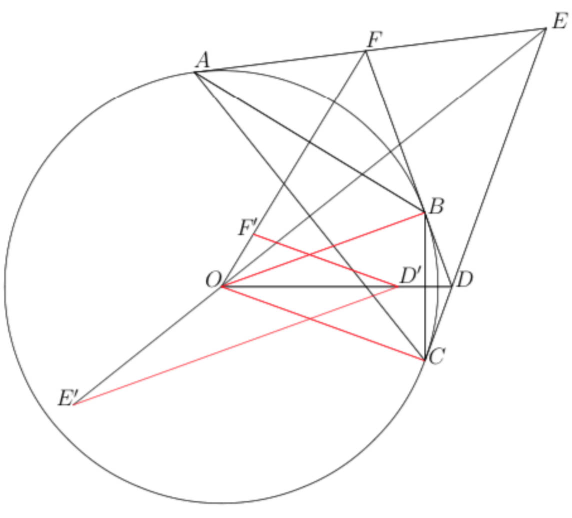
$$P = \prod_{\substack{a,b \in S \\ a \neq b}} (a+b) = \prod_{w \in \mathbb{F}_p^*} w^{(p+1)/4} = [(p-1)!]^{(p+1)/4} = (-1)^{(p+1)/4} = 1.$$

If $p \equiv 3 \pmod 8$, then 2 is not a square in $\mathbb{F}_p^*$. We can perform a similar computation to show that for any $w \in \mathbb{F}_p^*$, the number of solutions to $a + b = w$, where $a, b \in S$ such that $a \neq b$ is $\frac{p+1}{4} + 1$ if w is a square, and is $\frac{p+1}{4} - 1$ if w is a non-square. We can then use a similar argument to get the desired conclusion.

4703. *Proposed by Jiahao Chen.*

Given a triangle ABC with circumcenter O, denote by DEF the triangle formed by the tangents to the circumcircle at A, B, C with A on EF, B on FD, and C on DE. If D', E', F' are the reflections of D, E, F in the lines BC, CA, AB, respectively, prove that $D'E' || OB$ if and only if $D'F' || OC$.

All 9 submissions that we received were correct; we feature a composite of the solutions from Ivko Dimitrić, who used complex numbers, and Marie-Nicole Gras, who used Cartesian coordinates.



We assume that ΔABC is inscribed in the unit circle centered at the origin of the complex plane; points are denoted by capital letters while the complex numbers that represent them are named by the corresponding lower-case letters. We further assume that the real axis is the perpendicular bisector of the chord BC, so that $\bar{b} = c = \frac{1}{b}$ and, of course, $\bar{a} = \frac{1}{a}$. Line DF through B consists of points Z for which $BZ \perp OB$, i.e.

$$\overline{\left(\frac{z-b}{b}\right)} + \frac{z-b}{b} = 0 \quad \Longleftrightarrow \quad \frac{\bar{z}}{\bar{b}} - 1 + \frac{z}{b} - 1 = 0.$$

Then an equation of line DF and similarly obtained equations of ED and FE (recalling that $c = \bar{b}$) are

$$b\bar{z} + \bar{b}\,z = 2, \qquad \bar{b}\,\bar{z} + b\,z = 2, \qquad a\,\bar{z} + \bar{a}\,z = 2 \tag{1}$$

Solving the systems of three pairs of equations from this set of three, we get the intersection points D, E, F of the tangents as follows:

$$d = \frac{2b\bar{b}}{b+\bar{b}}, \qquad e = \frac{2a\bar{b}}{\bar{b}+a}, \qquad f = \frac{2ab}{a+b}. \tag{2}$$

Additionally, we compute

$$\bar{d} = d = \frac{2}{b+\bar{b}}, \qquad \bar{e} = \frac{2}{\bar{b}+a}, \qquad \bar{f} = \frac{2}{a+b}. \tag{3}$$

Since triangles DCB, FBA, EAC are isosceles, the segments such as DD' and BC intersect at the common midpoints, so $d + d' = b + \bar{b}$, and similarly $e + e' = \bar{b} + a$, $f + f' = a + b$; thus,

$$d' = b + \bar{b} - d \qquad e' = \bar{b} + a - e, \qquad f' = a + b - f. \tag{4}$$

The relation $D'E' \parallel OB$ means (by virtue of (4)) that

$$\frac{e' - d'}{b} = \frac{a + d - e}{b} - 1 \quad \text{is real.} \tag{5}$$

This is equivalent to $\overline{\left(\frac{a+d-e}{b}\right)} = \frac{a+d-e}{b}$,

$$\begin{aligned}
\Longleftrightarrow \quad & \frac{b}{a} + b\left(\frac{2}{b+\bar{b}} - \frac{2}{\bar{b}+a}\right) = \frac{a}{b} + \frac{1}{b}\left(\frac{2b\bar{b}}{b+\bar{b}} - \frac{2a\bar{b}}{\bar{b}+a}\right) \\
\Longleftrightarrow \quad & \frac{2(b^2+\bar{b}^2)(a-b)}{b(b+\bar{b})(\bar{b}+a)} = \frac{(a+b)(a-b)}{ab} \\
\Longleftrightarrow \quad & (a+b)(b+\bar{b})(\bar{b}+a) = 2a(b^2+\bar{b}^2) \\
\Longleftrightarrow \quad & a^2(b+\bar{b}) \; - a(b-\bar{b})^2 + b\bar{b}(b+\bar{b}) = 0.
\end{aligned} \tag{6}$$

To conclude, we note that the relation $D'F' \parallel OC$ means that

$$\frac{f' - d'}{\bar{b}} = \frac{a + d - f}{\bar{b}} - 1 \quad \text{is real,}$$

and that criterion can be obtained from (5) by interchanging the roles of b and $\bar{b}$. Because (5) is equivalent to an equation, namely (6), that is symmetric in b and $\bar{b}$, we conclude that $D'E' \parallel OB$ if and only if $D'F' \parallel OC$, as desired.

Further Remarks. It is interesting to note that equation (6) provides information concerning the existence of triangles for which $D'E' \parallel OB$. Denoting the argument of b by β (which we have assumed to equal $\frac{1}{2}\angle COB$), we have $2\cos\beta = b + \bar{b}$ and $2i\sin\beta = b - \bar{b}$, whence equation (6) implies that a is a zero of the equation

$$x^2 + \frac{2\sin^2\beta}{\cos\beta}x + 1 = 0.$$

Because this is a monic polynomial with real coefficients, a is a zero if and only if $\bar{a}$ is also, and (denoting the argument of a by α so that $\cos\alpha = \frac{a+\bar{a}}{2}$) we conclude that $D'E' \parallel OB$ if and only if

$$\cos\alpha = -\frac{\sin^2\beta}{\cos\beta}.$$

If follows that our triangle exists for any b for which $-1 \leq \frac{\sin^2\beta}{\cos\beta} \leq 1$.

Editor's comments. As part of his solution, Oliver Geupel provided an explicit example of our triangle: With the notation of the featured solution, take ABC to be the isosceles triangle with $a = 1$ whose altitude is the golden section, namely $\frac{1+\sqrt{5}}{2}$; that is, b satisfies $2\cos\beta = b + \bar{b} = 1 - \sqrt{5}$. C.R. Pranesachar submitted the only coordinate free solution; he observed that the parallelism of $D'E'$ and OB in a triangle with angles A, B, and C is equivalent to three further equivalent statements:

- $\cos A \sin B \sin C = \frac{1}{2}$;
- $\cos 2A - \cos 2B - \cos 2C = 1$;
- $\tan A = \frac{\Delta}{R^2}$, where Δ and R are the area and circumradius of ΔABC.

4704. *Proposed by Daniel Sitaru.*

For $a, b, c, d \in [0, 1)$, prove that

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^2} \geq \frac{2}{1-(abc)^2} + \frac{2}{1-abcd}.$$

We received 13 submissions, all of which were correct. We present two solutions.

Solution 1, by Mohamed Amine Ben Ajiba, enhanced slightly by the editor.

By the AM-GM inequality, we have for all $x, y \in [0, 1)$ that

$$\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{\sqrt{(1-x^2)(1+y^2)}} = \frac{2}{\sqrt{(1-xy)^2-(x-y)^2}} \geq \frac{2}{1-xy}$$

with equality when $x = y$. Using this inequality, we have

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} \geq \frac{2}{1-(ab)^3} \qquad (1)$$

$$\frac{1}{1-c^6} + \frac{1}{1-(abc)^2} \geq \frac{2}{1-abc^4} \qquad (2)$$

$$\frac{1}{1-d^2} + \frac{1}{1-(abc)^2} \geq \frac{2}{1-abcd} \qquad (3)$$

$$2\left(\frac{1}{1-(ab)^3} + \frac{1}{1-abc^4}\right) \geq \frac{4}{1-(abc)^2} \qquad (4)$$

Adding (1)-(4), we get:

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^6} + \frac{2}{1-(abc)^2} + \frac{2}{1-(ab)^3} + \frac{2}{1-abc^4}$$
$$\geq \frac{2}{1-(ab)^3} + \frac{2}{1-abc^4} + \frac{2}{1-abcd} + \frac{4}{1-(abc)^2},$$

which simplifies to

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^6} \geq \frac{2}{1-(abc)^2} + \frac{2}{1-abcd},$$

completing the proof. Equality holds if and only if $a = b = c = \lambda$, $d = \lambda^3$ for some $\lambda \in [0, 1)$.

Solution 2, by Marian Dincă.

Using the AM-GM inequality repeatedly, we obtain

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6} \geq \sqrt[3]{\frac{1}{1-a^6}\cdot\frac{1}{1-b^6}\cdot\frac{1}{1-c^6}} = \frac{3}{\sqrt[3]{(1-a^6)(1-b^6)(1-c^6)}} \qquad (5)$$

and

$$\sqrt[3]{(1-a^6)(1-b^6)(1-c^6)} \leq \frac{(1-a^6)+(1-b^6)+(1-c^6)}{3} = 1 - \frac{a^6+b^6+c^6}{3}$$
$$\leq 1 - \sqrt[3]{a^6b^6c^6} = 1-(abc)^2. \qquad (6)$$

From (5) and (6), we have

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} \geq \frac{3}{1-(abc)^2} = \frac{2}{1-(abc)^2} + \frac{1}{1-(abc)^2}. \qquad (7)$$

Next,

$$\begin{aligned}
&\frac{1}{1-(abc)^2}+\frac{1}{1-d^2}\ge 2\sqrt{\frac{1}{1-(abc)^2}\cdot\frac{1}{1-d^2}}\\
&=\frac{2}{\sqrt{1-(abc)^2(1-d^2)}}\ge\frac{2}{\frac{1-(abc)^2+1-d^2}{2}}\\
&=\frac{2}{1-\frac{(abc)^2+d^2}{2}}\ge\frac{2}{1-\sqrt{(abc)^2d^2}}=\frac{2}{1-abcd}. \qquad (8)
\end{aligned}$$

Finally, from (7) and (8), the conclusion follows.

4705. *Proposed by Nguyen Viet Hung.*

Find the following limit

$$\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}.$$

We received 33 submissions, of which 30 were correct and complete. We present two solutions.

Solution 1, by the UCLan Cyprus Problem Solving Group.

Since $f(x)=\frac{1}{\sqrt[3]{x}}$ is strictly decreasing in $(0,1]$, then for each $k=1,2,\ldots,n-1$ we have

$$\frac{1}{n}f\left(\frac{k+1}{n}\right)<\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(x)\,\mathrm{d}x<\frac{1}{n}f\left(\frac{k}{n}\right).$$

Thus

$$\frac{1}{\sqrt[3]{n^2}}\frac{1}{\sqrt[3]{k+1}}<\frac{3}{2}\left(\left(\frac{k+1}{n}\right)^{2/3}-\left(\frac{k}{n}\right)^{2/3}\right)<\frac{1}{\sqrt[3]{n^2}}\frac{1}{\sqrt[3]{k}}.$$

Summing from $k=1$ to $n-1$ and noting that the middle sum is telescopic, we have:

$$\frac{1}{\sqrt[3]{n^2}}\sum_{k=2}^{n}\frac{1}{\sqrt[3]{k}}<\frac{3}{2}\left(1-\left(\frac{1}{n}\right)^{2/3}\right)<\frac{1}{\sqrt[3]{n^2}}\sum_{k=1}^{n-1}\frac{1}{\sqrt[3]{k}}.$$

Hence,

$$\frac{3}{2}\left(1-\left(\frac{1}{n}\right)^{2/3}\right)+\frac{1}{n}<\frac{1}{\sqrt[3]{n^2}}\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}<\frac{3}{2}\left(1-\left(\frac{1}{n}\right)^{2/3}\right)+\frac{1}{\sqrt[3]{n^2}}.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}=\frac{3}{2}.$$

Solution 2, by Marie-Nicole Gras.

By the integral test, if f is a function defined on $[1,\infty)$, continuous, monotone decreasing in this interval, then for all integers $n \geq 1$,

$$\int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(x)dx.$$

We apply this test with $f(x) = x^{\frac{-1}{3}}$; since

$$\int x^{\frac{-1}{3}}\,dx = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2}x^{\frac{2}{3}},$$

we obtain

$$\frac{3}{2}\Big((n+1)^{\frac{2}{3}} - 1\Big) \leq \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} \leq 1 + \frac{3}{2}\Big(n^{\frac{2}{3}} - 1\Big).$$

Since

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\frac{2}{3}} = 1,$$

we deduce

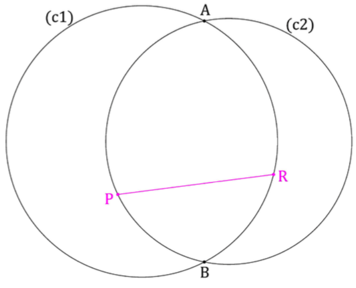
$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} = \frac{3}{2}.$$

Editor's comment. Using Bernoulli's inequality, Vivek Mehra generalized the result by proving that for $-1 < t < 0$,

$$\lim_{n\to\infty} \frac{1}{n^{1+t}} \sum_{k=1}^{n} k^t = \frac{1}{1+t}.$$

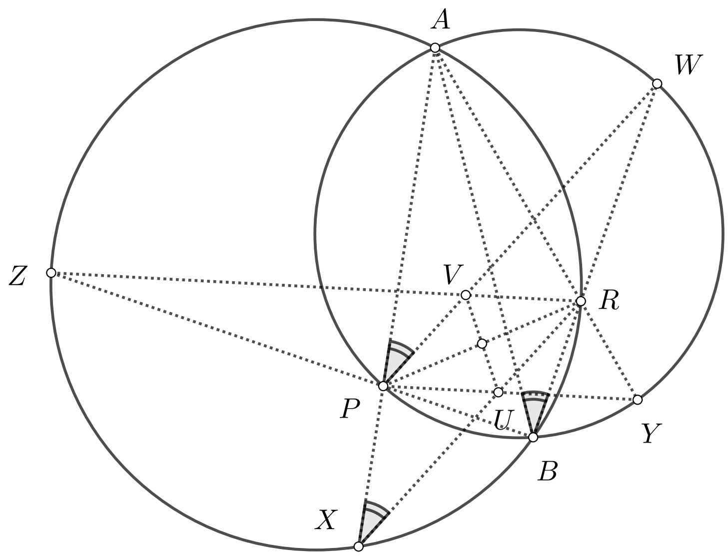
4706. *Proposed by Thanos Kalogerakis.*

In the figure below, find the midpoint of segment PR using the straightedge alone and prove that your construction works.



We received 6 correct submissions. We present the solutions provided by UCLan Cyprus Problem Solving Group.

Let X be the other point of intersection of AP with (c_1), Y the other point of intersection of AR with (c_2), Z the other point of intersection of BP with (c_1), and W the other point of intersection of BR with (c_2).



Let U be the point of intersection of PY and RX, and V be the point of intersection of PW and RZ. We claim that UV intersects PR at its midpoint.

It is enough to show that $PURV$ is a parallelogram. We have

$$\angle VPA = \angle WPA = \angle WBA = \angle RBA = \angle RXA.$$

So VP is parallel to RU. Similarly VR is parallel to PU so the result follows.

Note: On the triangle BZW, the points P and R are internal points of the segments BZ and BW. Then PW and RZ must have a point of intersection. I.e. V is well-defined. Similarly U is also well-defined.

4707. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$. Prove that

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = \frac{n^2-1}{3} \quad \text{and} \quad \sum_{k=1}^{n-1} \csc^4\left(\frac{k\pi}{n}\right) = \frac{n^4+10n^2-11}{45}.$$

There were 14 correct solutions submitted by 13 solvers. Half of the solutions used the method of the solution below. The remainder relied on the use of analysis or recourse to more obscure results from the trigonometric literature.

The sums are readily checked for $n = 2, 3, 4$. Let $n \geq 5$. Then $\cot(k\pi/n)$ $(1 \leq k \leq n-1)$ are the roots of the polynomial

$$f(z) = \binom{n}{1} z^{n-1} - \binom{n}{3} z^{n-3} + \binom{n}{5} z^{n-5} + \cdots .$$

One way to see this is to note that

$$\left(\frac{\cot\theta + i}{\cot\theta - i}\right)^n = \left(\frac{e^{i\theta}}{e^{-i\theta}}\right)^n = e^{2ni\theta},$$

where the right side takes the value 1 for $0 < \theta < \pi$ when $\theta = k\pi/n$ $(1 \leq k \leq n-1)$. Thus the cotangents are the roots of the polynomial $(z+i)^n - (z-i)^n = 2if(z)$.

Alternatively, by de Moivre's theorem,

$$\sin n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta,$$

whence

$$\frac{\sin n\theta}{\sin^n\theta} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k+1} \cot^{n-2k-1}\theta.$$

For $0 < \theta < \pi$, the left side vanishes if and only if $\theta = k\pi/n$ and we again obtain the same polynomial with the cotangent roots.

When $1 \leq k \leq 4$, the symmetric functions of degree k of the roots are given by

$$\sigma_1 = \sigma_3 = 0, \qquad \sigma_2 = \frac{-(n-1)(n-2)}{6}, \qquad \sigma_4 = \frac{(n-1)(n-2)(n-3)(n-4)}{120}.$$

The sum p_k of the kth powers of the roots are given by

$$p_1 = 0, \qquad p_2 = \sigma_1^2 - 2\sigma_2 = \frac{(n-1)(n-2)}{3}, \qquad p_3 = \sigma_1 p_2 - \sigma_2 p_1 + 3\sigma_3 = 0,$$

$$\begin{aligned} p_4 &= \sigma_1 p_3 - \sigma_2 p_2 + \sigma_3 p_1 - 4\sigma_4 = \frac{(n-1)(n-2)(n^2+3n-13)}{45} \\ &= \frac{n^4 - 20n^2 + 45n - 26}{45}. \end{aligned}$$

Since $\csc^2\theta = 1 + \cot^2\theta$ and $\csc^4\theta = 1 + 2\cot^2\theta + \cot^4\theta$,

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = (n-1) + \frac{(n-1)(n-2)}{3} = \frac{n^2-1}{3},$$

and

$$\sum_{k=1}^{n-1} \csc^4\left(\frac{k\pi}{n}\right) = (n-1) + \frac{2(n-1)(n-2)}{3} + \frac{(n-1)(n-2)(n^2+3n-13)}{45}$$
$$= \frac{n^4+10n^2-11}{45}.$$

Comment from the editor. Paolo Perfetti pursued a strategy that was straightforward conceptually but forbidding computationally. With $z_k = \exp(2k\pi i/n)$, he derived $\sum_{k=1}^{n-1} \csc^2(k\pi/n) = 4A - 4B$, where

$$A = \sum_{k=1}^{n-1}(1-z_k)^{-1}, \quad B = \sum_{k=1}^{n-1}(1-z_k)^{-2}.$$

Since

$$\sum_{k=1}^{n-1} \frac{1}{z-z_k} = \frac{P'(z)}{P(z)} - \frac{1}{z-z_0},$$

where $P(z) = z^n - 1 = \prod_{k=0}^{n-1}(z-z_k)$, the sum A can be found, with the aid of l'Hôpital's Rule, to be $\frac{1}{2}(n-1)$. Since

$$\sum_{k=1}^{n-1}(z-z_k)^{-2} = -\left(\sum_{k=1}^{n-1}(z-z_k)^{-1}\right)',$$

a similar process leads to $B = -(n^2/12) + (n/2) - (5/12)$.

Note that the sum $\sum_{k=1}^{n-1} \csc^4(k\pi/n)$ can be expressed as a linear combination of $\sum(1-z_k)^{-2}$, $\sum(1-z_k)^{-3}$ and $\sum(1-z_k)^{-4}$ and can be determined in a similar way invoking higher order derivatives of $\sum(z-z_k)^{-1}$.

4708. *Proposed by Conar Goran.*

Let α, β, γ be angles of an arbitrary triangle. Prove that the following inequality holds

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{3} \le \cot\left(\frac{3}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}\right).$$

When does the equality occur?

We received 6 solutions, all of which were correct. We present the solution by Mohamed Amine Ben Ajiba.

Let $s = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, and define $f(x) = x\cot x$ for $x \in (0,\pi)$. Using $\cos x \le 1$ and $\sin x \le x$, we have

$$f'(x) = (\cos x \sin x - x)\csc^2 x \le (\sin x - x)\csc^2 x \le 0$$

for all $x \in (0, \pi)$. Thus f is decreasing on $(0, \pi)$ and $f(x) \leq \lim_{x \to 0^+} f(x) = 1$ for all $x \in (0, \pi)$. Also

$$f''(x) = -2\,(1 - f(x)) \csc^2 x \leq 0$$

for all $x \in (0, \pi)$, so f is concave on $(0, \pi)$, and by Jensen's inequality, since $\frac{1}{s\alpha} + \frac{1}{s\beta} + \frac{1}{s\gamma} = 1$, we have

$$\frac{1}{s\alpha} \cdot f(\alpha) + \frac{1}{s\beta} \cdot f(\beta) + \frac{1}{s\gamma} \cdot f(\gamma) \leq f\left(\frac{3}{s}\right).$$

This is equivalent to

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{3} \leq \cot\left(\frac{3}{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}\right),$$

as desired. Equality holds if and only if $\alpha = \beta = \gamma$, i.e, the triangle is equilateral.

4709. *Proposed by Ion Patrascu.*

Let ABC be an acute triangle and O the center of its circumcircle. We denote by D, E and F the intersections of the lines AO and BC, BO and CA, CO and AB, respectively. If $BD\cos A = CE\cos B = AF\cos C$, prove that ABC is an equilateral triangle.

All but one of the fourteen submissions were correct; we feature a typical solution and a generalization.

Solution 1, by Mohamed Amine Ben Ajiba.

We denote by A, B, C the angles $\angle BAC, \angle CBA, \angle ACB$, respectively. The Law of Sines applied to triangle ABD gives us $BD = AB\frac{\sin\angle BAD}{\sin\angle ADB}$, where

$$\angle BAD = \frac{\pi - \angle AOB}{2} = \frac{\pi}{2} - C \quad \text{and} \quad \angle ADB = \pi - B - \angle BAD = \frac{\pi}{2} - (B - C).$$

Thus,

$$BD = AB\frac{\cos C}{\cos(B - C)}.$$

Similarly, we have

$$CE = BC\frac{\cos A}{\cos(C - A)}.$$

It follows that

$$\frac{BD}{CE} = \frac{AB}{BC} \cdot \frac{\cos C\cos(C - A)}{\cos A\cos(B - C)} = \frac{\sin C}{\sin A} \cdot \frac{\cos C\cos(C - A)}{\cos A\cos(B - C)}.$$

But we are given that $\frac{BD}{CE} = \frac{\cos B}{\cos A}$; consequently,

$$\begin{aligned}\frac{\cos B}{\cos A} = \frac{\sin 2C\cos(C-A)}{2\sin A\cos A\cos(B-C)} \quad &\Leftrightarrow \quad \frac{\sin 2C\cos(C-A)}{2\sin A(\cos B)\cos(B-C)} = 1\\ &\Leftrightarrow \quad \frac{\sin 2C[(2\sin B)\cos(C-A)]}{(\sin 2B)[2\sin A\cos(B-C)]} = 1\\ &\Leftrightarrow \quad \frac{\sin 2C[\sin(B+C-A)+\sin(B-C+A)]}{\sin 2B[\sin(A+B-C)+\sin(A-B+C)]} = 1\\ &\Leftrightarrow \quad \frac{\sin 2C(\sin 2A+\sin 2C)}{\sin 2B(\sin 2C+\sin 2B)} = 1 \qquad (1)\end{aligned}$$

Without loss of generality we assume that the triangle has been labeled so that B is less than or equal to the other two angles; let

$$x = \sin 2A, \quad y = \sin 2B, \quad z = \sin 2C.$$

Note that $x, z \geq y > 0$. The necessary and sufficient condition in (1) now reads $\frac{BD}{CE} = \frac{\cos B}{\cos A}$ if and only if $\frac{z(x+z)}{y(z+y)} = 1$, or

$$z^2 - y^2 = z(y - x).$$

But because $y = \min\{x, y, z\}$, we must have $z^2 - y^2 \geq z(y - x)$. We conclude, therefore, that $x = y = z$, which means that $\sin 2A = \sin 2B = \sin 2C$. Of course, $\sin 2A = \sin 2B$ implies only that $A = B$ or $A = \frac{\pi}{2} - B$; however, the latter is ruled out because that would imply that C be a right angle, contrary to our assumption that all angles are acute. Thus, we are left with $A = B$; similarly, we have $A = C$, and we conclude that ΔABC is equilateral, as desired.

Solution 2, by Vivek Mehra.

If R is the circumradius of the given triangle, the lengths of the perpendiculars from O to its sides are $R\cos A, R\cos B$, and $R\cos C$. Consequently, from $BD\cos A = CE\cos B = AF\cos C$ follows $[OBD] = [OCE] = [OAF]$ (where square brackets denote areas). This suggests the following generalization:

Theorem. For an arbitrary point P in the interior of ΔABC we denote by D, E, and F the intersections of the lines AP and BC, BP and CA, CP and AB, respectively. If $[PBD] = [PCE] = [PAF]$ then P is the centroid of ΔABC.

Note that the desired result follows immediately from the theorem: the centroid of a triangle coincides with the circumcenter if and only if the triangle is equilateral.

Proof of the theorem. We use Barycentric coordinates. Set

$$A = (1,0,0),\ B = (0,1,0),\ C = (0,0,1), \ \text{ and } \ P = (p,q,r)$$

with $p, q, r > 0$ and $p + q + r = 1$. We thus have

$$D = \left(0, \frac{q}{q+r}, \frac{r}{q+r}\right), \quad E = \left(\frac{p}{p+r}, 0, \frac{r}{p+r}\right), \quad F = \left(\frac{p}{p+q}, \frac{q}{p+q}, 0\right).$$

Our goal is to prove that $p = q = r$, which makes $P = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ the centroid. To that end we compute

$$\frac{[PBD]}{[ABC]} = \begin{vmatrix} p & 0 & 0 \\ q & 1 & \frac{q}{q+r} \\ r & 0 & \frac{r}{q+r} \end{vmatrix} = \frac{pr}{q+r};$$

similarly,

$$\frac{[PCE]}{[ABC]} = \frac{pq}{p+r} \quad \text{and} \quad \frac{[PAF]}{[ABC]} = \frac{qr}{p+q}.$$

Therefore, from $[PBD] = [PCE] = [PAF]$ and $p + q + r = 1$ we have

$$q(1-p) = r(1-q) = p(1-r).$$

Suppose that P had been chosen so that $p \geq q, r$; then either $p > q$ or $1-r > 1-p$ would contradict the equation $p(1-r) = q(1-p)$, so we would necessarily conclude that $p = q = r$. We likewise find a contradiction with either q or r assumed greatest. We conclude that $p = q = r$, as claimed.

4710★. *Proposed by Omar Sonebi, modified by the Editorial Board.*

Show that there exist 2021 consecutive natural numbers none of which is the sum of a perfect square and a perfect cube.

We received 8 solutions, out of which we present the one by Oliver Geupel.

We prove the following generalisation: For every natural number n, there exist n consecutive natural numbers in the set $A = \left\{1, 2, 3, \ldots, 64n^6\right\}$ none of which is the sum of a perfect square and a perfect cube. In fact, the perfect squares in A are the $8n^3$ numbers

$$1^2, 2^2, 3^2, \ldots, \left(8n^3\right)^2,$$

whereas the perfect cubes in A are the $4n^2$ numbers

$$1^3, 2^3, 3^3, \ldots, \left(4n^2\right)^3.$$

Hence, the subset B of members of A that can be written as the sum of a perfect square and a perfect cube has not more than $8n^3 \cdot 4n^2 = 32n^5$ elements.

Decompose the set A into $64n^5$ disjoint intervals of length n. By the Pigeonhole Principle, one of the intervals is disjoint to B. Hence the result.