

The 2022 Canadian Mathematical Olympiad



Official Solutions for CMO 2022

P1. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab+1} + \sqrt{a^2+b} \cdot \sqrt{b^2+a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2+a} + b\sqrt{a^2+b}.$$

Solution. Let us rewrite the given equation as follows:

$$ab + \sqrt{a^2+b}\sqrt{b^2+a} = -\sqrt{ab+1}.$$

Squaring this gives us

$$\begin{aligned} a^2b^2 + 2ab\sqrt{a^2+b}\sqrt{b^2+a} + (a^2+b)(b^2+a) &= ab+1 \\ (a^2b^2 + a^3) + 2ab\sqrt{a^2+b}\sqrt{b^2+a} + (a^2b^2 + b^3) &= 1 \\ \left(a\sqrt{b^2+a} + b\sqrt{a^2+b}\right)^2 &= 1 \\ a\sqrt{b^2+a} + b\sqrt{a^2+b} &= \pm 1. \end{aligned}$$

Next, we show that $a\sqrt{b^2+a} + b\sqrt{a^2+b} > 0$. Note that

$$ab = -\sqrt{ab+1} - \sqrt{a^2+b} \cdot \sqrt{b^2+a} < 0,$$

so a and b have opposite signs. Without loss of generality, we may assume $a > 0 > b$. Then rewrite

$$a\sqrt{b^2+a} + b\sqrt{a^2+b} = a(\sqrt{b^2+a} + b) - b(a - \sqrt{a^2+b})$$

and, since $\sqrt{b^2+a} + b$ and $a - \sqrt{a^2+b}$ are both positive, the expression above is positive. Therefore,

$$a\sqrt{b^2+a} + b\sqrt{a^2+b} = 1,$$

and the proof is finished. □

P2. Let $d(k)$ denote the number of positive integer divisors of k . For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n ,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).$$

Solution. For any integer k and set of integers S , let $f_S(k)$ be the number of multiples of k in S . We can count the number of pairs (k, s) with $k \in \mathbb{N}$ dividing $s \in S$ in two different ways, as follows:

- For each $s \in S$, there are $d(s)$ pairs that include s , one for each divisor of s .
- For each $k \in \mathbb{N}$, there are $f_k(S)$ pairs that include k , one for each multiple of k .

Therefore,

$$\sum_{s \in S} d(s) = \sum_{k \in \mathbb{N}} f_S(k).$$

Let

$$O = \{1, 3, 5, \dots, 2n-1\} \quad \text{and} \quad E = \{2, 4, 6, \dots, 2n\}$$

be the set of odd and, respectively, the set of even integers between 1 and $2n$. It suffices to show that

$$\sum_{k \in \mathbb{N}} f_O(k) \leq \sum_{k \in \mathbb{N}} f_E(k).$$

Since the elements of O only have odd divisors,

$$\sum_{k \in \mathbb{N}} f_O(k) = \sum_{k \text{ odd}} f_O(k).$$

For any odd k , consider the multiples of k between 1 and $2n$. They form a sequence

$$k, 2k, 3k, \dots, \left\lfloor \frac{2n}{k} \right\rfloor k$$

alternating between odd and even terms. There are either an equal number of odd and even terms, or there is one more odd term than even terms. Therefore, we have the inequality

$$f_O(k) \leq f_E(k) + 1$$

for all odd k . Combining this with the previous observations gives us the desired inequality:

$$\begin{aligned} \sum_{k \in \mathbb{N}} f_O(k) &= \sum_{k \text{ odd}} f_O(k) \\ &\leq \sum_{k \text{ odd}} (f_E(k) + 1) \\ &= \sum_{k \text{ odd}} f_E(k) + n \\ &= \sum_{k \text{ odd}} f_E(k) + f_E(2) \\ &\leq \sum_{k \in \mathbb{N}} f_E(k). \end{aligned}$$

□

P3: Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b , erases them, and writes either $a + b$ or $\min\{a^2, b^2\}$. After $n - 1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} < f(n) \leq 3^{n/3}.$$

Solution. Clearly $f(n)$ is a strictly increasing function, as we can form $f(n - 1)$ with $n - 1$ ones, and add the final one. However, we can do better; assume Vishal generates $f(n)$ on the board. After $n - 2$ minutes, there are two numbers left, say they were formed by x ones and y ones, where $x + y = n$. Clearly the numbers are at most $f(x), f(y)$ (and can be made to be equal to $f(x), f(y)$), and therefore we obtain

$$f(n) = \max_{x+y=n, 1 \leq x \leq y \leq n-1} \left(\max(f(x) + f(y), f(x)^2) \right) \quad (1)$$

where we used the fact that f is increasing to get that $\min(f(x)^2, f(y)^2) = f(x)^2$ when $x \leq y$. In particular, $f(n + 1) \geq f(n) + 1$, and $f(2n) \geq f(n)^2$ for all positive integers n .

Upper bound:

First proof of upper bound. We use induction. We can check that $f(n) = n$ for $n \leq 4$, and these all satisfy the bound $f(n) = n \leq 3^{n/3}$. Assume it is true for all $m < n$ (some $n \geq 5$), and with x, y as in equation ?? we have

$$f(x)^2 \leq f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^2 \leq (3^{n/6})^2 = 3^{n/3},$$

as desired. It thus remains to show that $f(x) + f(y) \leq 3^{n/3}$. By induction, it suffices to prove that

$$3^{x/3} + 3^{y/3} \leq 3^{(x+y)/3},$$

for $1 \leq x \leq y \leq n - 1$ and $x + y = n$. This is equivalent to

$$1 + 3^{(y-x)/3} \leq 3^{y/3}.$$

Let $w = 3^{(y-x)/3}$, and we require $3^{x/3}w \geq w + 1$. If $x \geq 2$, then this is true as $w \geq 1$, and if $x = 1$ then $w = 3^{(n-2)/3} \geq 3$ and the result is still true. Thus all terms in equation ?? are at most $3^{n/3}$, and so $f(n) \leq 3^{n/3}$, and the upper bound is proven.

Second proof of upper bound. Consider a second game with the same rules but in which Vishal can replace a and b by either $a + b$ or ab . Let $g(n)$ be the largest possible value for this new game. Then $f(n) \leq g(n)$ because $\min\{a^2, b^2\} \leq ab$.

We can check $g(n) = n$ for $n \leq 4$, so $g(n) \leq 3^{n/3}$ for these values. If x and y are both bigger than 1, then $g(x) + g(y) \leq g(x)g(y)$. Therefore, for $n > 4$, we have that

$$g(n) = \max \left\{ g(n-1) + 1, \max_{1 \leq x \leq n-1} g(x)g(n-x) \right\}$$

Now proceed similarly to the first proof. Assume $n > 4$ and $g(m) \leq 3^{m/3}$ for all $m < n$. If $1 \leq x \leq n-1$, then $g(x)g(n-x) \leq 3^{x/3} 3^{(n-x)/3} = 3^{n/3}$. And $g(n-1) + 1 \leq 3^{(n-1)/3} + 1$, which is shown to be less than $3^{n/3}$ in the first proof. It follows that $f(n) \leq g(n) \leq 3^{n/3}$.

Lower bound:

First proof of lower bound. We begin with a lemma.

Lemma 1. *Let m be a nonnegative integer. Then*

$$f(2^m) \geq 2^{2^{m-1}} \quad \text{and} \quad f(3 \cdot 2^m) \geq 3^{2^m}.$$

Proof. We prove the lemma by induction. One can check that $f(n) = n$ for $n \leq 3$, which proves the lemma for $m = 0$. For a general $m > 0$, we get

$$\begin{aligned} f(2^m) &\geq f(2^{m-1})^2 \geq \left(2^{2^{m-2}}\right)^2 = 2^{2^{m-1}} \\ f(3 \cdot 2^m) &\geq f(3 \cdot 2^{m-1})^2 \geq \left(3^{2^{m-1}}\right)^2 = 3^{2^m}, \end{aligned}$$

by induction, as required. □

(This lemma can also be proved more constructively. Briefly, if $n = 2^m$, then partition the 1's on the board into 2^{m-1} pairs, and then add each pair to get 2^{m-1} 2's ($2 = 2^0$); then multiply pairs of 2's to get 2^{m-2} 4's ($4 = 2^1$); then multiply pairs of 4's to get 2^{m-3} 16's ($16 = 2^2$); and so on, until there are 2 ($= 2^1$) copies of $2^{2^{m-2}}$, which then gets replaced with $2^{2^{m-1}}$. The process is similar for $n = 3 \cdot 2^m$, except that the first step is to partition the 1's into 2^m groups of 3, and then use addition within each group to get 2^m 3's on the board.)

Now assume $2^x \leq n < 3 \cdot 2^{x-1}$ for some integer x . Then we have

$$f(n) \geq f(2^x) \geq 2^{2^{x-1}} > 2^{n/3},$$

as required. If no such x exists, then there exists an integer x such that $3 \cdot 2^{x-1} \leq n < 2^{x+1}$. In this case, we have

$$f(n) \geq f(3 \cdot 2^{x-1}) \geq 3^{2^{x-1}} > 2^{2^{x+1}/3} > 2^{n/3}$$

where the second last inequality is equivalent to $2^{x-1} \log(3) \geq \frac{2^{x+1}}{3} \log(2)$, and by dividing out 2^x and clearing the denominator this is equivalent to $3 \log(3) \geq 4 \log(2)$, which is true as $3^3 = 27 > 16 = 2^4$.

Second proof of lower bound. We shall prove the stronger result $f(n) \geq 2^{(n+1)/3}$ for $n \geq 2$ by induction. One can check that $f(n) = n$ for $n = 2, 3, 4$, which proves the result for these values. Assume that $n \geq 5$ and that $f(k) \geq 2^{(k+1)/3}$ for all $k = 2, 3, \dots, n-1$. Then

$$\begin{aligned} f(n) &\geq f(\lfloor n/2 \rfloor)^2 \\ &\geq \left(2^{(\lfloor n/2 \rfloor + 1)/3}\right)^2 && \text{since } \lfloor \frac{n}{2} \rfloor \geq 2 \\ &= 2^{(2\lfloor n/2 \rfloor + 2)/3} \\ &\geq 2^{(n+1)/3} && \text{since } \lfloor \frac{n}{2} \rfloor \geq \frac{n-1}{2}. \end{aligned}$$

The result follows by induction. □

Remark 1. One can show that f satisfies the recurrence $f(n) = n$ for $n = 1, 2$, $f(2n) = f(n)^2$ for $n \geq 2$, and $f(2n+1) = f(2n) + 1$ for $n \geq 1$. The upper bound in the problem is tight (equality holds for $n = 3 \cdot 2^x$), but the lower bound is not.

P4. Let n be a positive integer. A set of n distinct lines divides the plane into various (possibly unbounded) regions. The set of lines is called “nice” if no three lines intersect at a single point. A “colouring” is an assignment of two colours to each region such that the first colour is from the set $\{A_1, A_2\}$, and the second colour is from the set $\{B_1, B_2, B_3\}$. Given a nice set of lines, we call it “colourable” if there exists a colouring such that

1. no colour is assigned to two regions that share an edge;
2. for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ there is at least one region that is assigned with both A_i and B_j .

Determine all n such that every nice configuration of n lines is colourable.

Solution. The answer is $n \geq 5$. If $n \leq 4$, consider n parallel lines. There are 6 total colour combinations required, and only $n + 1 \leq 5$ total regions, hence the colouring is not possible.

Now, assume $n \geq 5$. Rotate the picture so that no line is horizontal, and orient each line so that the “forward” direction increases the y -value. In this way, each line divides the plane into a right and left hand side (with respect to this forward direction). Every region of the plane is on the right hand side of k lines and on the left hand side of $n - k$ lines for some $0 \leq k \leq n$. Furthermore, there is a region for every k : let w be large enough so that w is greater than the y -value of any intersection point of two lines. Consider the horizontal line $y = w$: a point very far on the left of this line is left of every single line, and as we cross over all lines in the problem, we hit all values of k .

Finally, take a region that is on the right hand side of k lines. Colour it A_1 if k is odd, and A_2 if it is even. Similarly, colour it B_i if $k \equiv i \pmod{3}$. By the previous paragraph, there are regions for at least $k = 0, 1, \dots, 5$, whence there is a region coloured A_i and B_j for all (i, j) . Furthermore, two regions that share an edge will be on the right hand side of k and $k + 1$ lines for some k . By construction, the A_i and B_i colours of the regions must differ, hence we have proven that the set of lines is colourable. \square

P5. Let $ABCDE$ be a convex pentagon such that the five vertices lie on a circle and the five sides are tangent to another circle inside the pentagon. There are $\binom{5}{3} = 10$ triangles which can be formed by choosing 3 of the 5 vertices. For each of these 10 triangles, mark its incenter. Prove that these 10 incenters lie on two concentric circles.

Solution. Let I be the incenter of pentagon $ABCDE$. Let I_A denote the incenter of triangle EAB and I_a the incenter DAC . Define $I_B, I_b, I_C, I_c, I_D, I_d, I_E, I_e$ similarly.

We will first show that $I_AI_BI_CI_DI_E$ are concyclic. Let ω_A be the circle with center at the midpoint of arc DE and passing through D and E . Define $\omega_B, \omega_C, \omega_D, \omega_E$ similarly. It is well-known that the incenter of a triangle lies on such circles, in particular, I_A lies on ω_C and ω_D . So the radical axis of ω_C, ω_D is the line AI_A . But this is just the angle bisector of $\angle EAB$, which I also lies on. So I is in fact the radical center of $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$! Inverting about I swaps I_A and A and since $ABCDE$ are concyclic, $I_AI_BI_CI_DI_E$ are concyclic as well.

Let O be the center of the circle $I_AI_BI_CI_DI_E$. We will now show that $OI_a = OI_d$ which finishes the problem as we can consider the cyclic versions of this equation to find that $OI_a = OI_d = OI_b = OI_e = OI_c$. Recall a well-known lemma: For any cyclic quadrilateral $WXYZ$, the incenters of XYZ, YZW, ZWX, WXY form a rectangle. Applying this lemma on $ABCD$, we see that I_B, I_C, I_a, I_d form a rectangle in that order. Then the perpendicular bisector of I_BI_C is exactly the perpendicular bisector of I_aI_d . Thus, O is equidistant to I_a and I_d and we are done. \square