Official Solutions for CJMO 2022

P1. Let ABC be an acute angled triangle with circumcircle Γ. The perpendicular from A to BC intersects Γ at D, and the perpendicular from B to AC intersects Γ at E. Prove that if $|AB| = |DE|$, then $\angle ACB = 60°$.

Solution. Since AB and ED are equal chords in the same circle, we either have $\angle ACB = \angle ECD$ or $\angle ACB + \angle ECD = 180°$. We compute
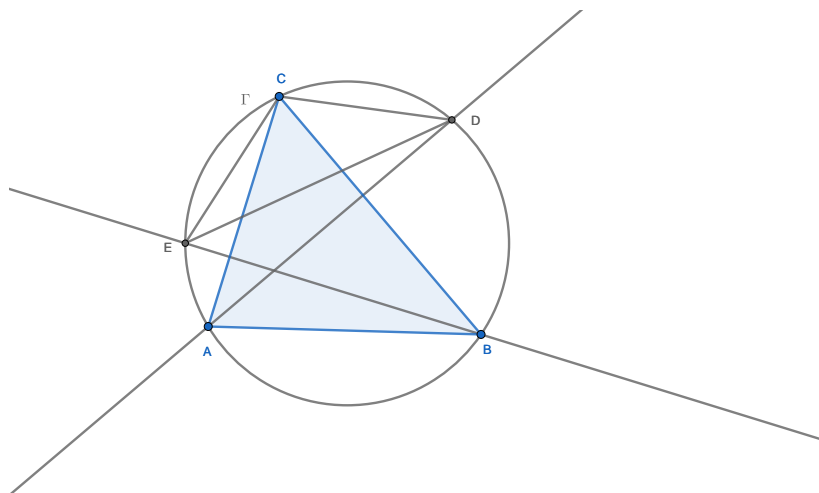

Figure 1: Illustration for Problem 1.

$$\begin{aligned}
\angle ECD &= \angle ECA + \angle ACB + \angle BCD \\
&= \angle EBA + \angle ACB + \angle BAD \\
&= 90° - \angle BAC + \angle ACB + 90° - \angle ABC \\
&= (180° - \angle BAC - \angle ABC) + \angle ACB \\
&= 2\angle ACB.
\end{aligned}$$

If $\angle ACB = \angle ECD$ then $\angle ACB = 0°$, contradiction. Thus $\angle ACB + \angle ECD = 180°$, whence $\angle ACB = 60°$. $\qquad\square$

P2. You have an infinite stack of T-shaped tetrominoes (composed of four squares of side length 1), and an $n \times n$ board. You are allowed to place some tetrominoes on the board, possibly rotated, as long as no two tetrominoes overlap and no tetrominoes extend off the board. For which values of n can you cover the entire board?
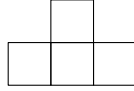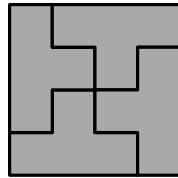

Figure 2: T-shaped tetromino.

Solution. Let us first note that one can cover the entire board if and only if $4 \mid n$. Indeed, one can cover a 4×4 board as follows:

Since for any $m \in \mathbb{N}$, $4m \times 4m$ board can be covered by a grid of m^2 4×4 squares, all multiples of four are possible.

Suppose then that $4 \nmid n$, and assume first that n is odd. Each tetromino covers 4 squares, hence if one can cover the entire board, then the final area covered must be a multiple of 4. Since n^2 is odd, this is impossible.

Finally, suppose that $n = 4k + 2$ for some $k \in \mathbb{N} \cup \{0\}$. For a contradiction, suppose that one can cover the entire board. Colour the squares of the board with white and black paint like a chessboard such that the bottom left corner square is white. Since n is even, there is the same number of white and black squares. Therefore, there will be $n^2/2 = 8k^2 + 8k + 2$ white squares overall, which is an even number. Note that, since white squares do not have a common border, each T-tetromino covers an odd number of white squares (exactly 1 or 3). Since we need to place $n^2/4 = 4k^2 + 4k + 1 \equiv 1 \pmod 2$ tetrominoes, which is an odd number, we will cover an odd number of white squares. This gives us the desired contradiction with the observation that we have an even number of white squares, and thus one cannot cover the board. $\square$

P3. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab + 1} + \sqrt{a^2 + b} \cdot \sqrt{b^2 + a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b}.$$

Solution. Let us rewrite the given equation as follows:

$$ab + \sqrt{a^2 + b}\sqrt{b^2 + a} = -\sqrt{ab + 1}.$$

Squaring this gives us

$$a^2 b^2 + 2ab\sqrt{a^2 + b}\sqrt{b^2 + a} + (a^2 + b)(b^2 + a) = ab + 1$$
$$(a^2 b^2 + a^3) + 2ab\sqrt{a^2 + b}\sqrt{b^2 + a} + (a^2 b^2 + b^3) = 1$$
$$\left(a\sqrt{b^2 + a} + b\sqrt{a^2 + b}\right)^2 = 1$$
$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b} = \pm 1.$$

Next, we show that $a\sqrt{b^2 + a} + b\sqrt{a^2 + b} > 0$. Note that

$$ab = -\sqrt{ab + 1} - \sqrt{a^2 + b} \cdot \sqrt{b^2 + a} < 0,$$

so a and b have opposite signs. Without loss of generality, we may assume $a > 0 > b$. Then rewrite

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b} = a(\sqrt{b^2 + a} + b) - b(a - \sqrt{a^2 + b})$$

and, since $\sqrt{b^2 + a} + b$ and $a - \sqrt{a^2 + b}$ are both positive, the expression above is positive. Therefore,

$$a\sqrt{b^2 + a} + b\sqrt{a^2 + b} = 1,$$

and the proof is finished. $\qquad\square$

P4. Let $d(k)$ denote the number of positive integer divisors of k. For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \le d(2) + d(4) + d(6) + \cdots + d(2n).$$

Solution. For any integer k and set of integers S, let $f_S(k)$ be the number of multiples of k in S. We can count the number of pairs (k, s) with $k \in \mathbb{N}$ dividing $s \in S$ in two different ways, as follows:

- For each $s \in S$, there are $d(s)$ pairs that include s, one for each divisor of s.

- For each $k \in \mathbb{N}$, there are $f_k(S)$ pairs that include k, one for each multiple of k.

Therefore,

$$\sum_{s \in S} d(s) = \sum_{k \in \mathbb{N}} f_S(k).$$

Let

$$O = \{1, 3, 5, \ldots, 2n-1\} \qquad \text{and} \qquad E = \{2, 4, 6, \ldots, 2n\}$$

be the set of odd and, respectively, the set of even integers between 1 and $2n$. It suffices to show that

$$\sum_{k \in \mathbb{N}} f_O(k) \le \sum_{k \in \mathbb{N}} f_E(k).$$

Since the elements of O only have odd divisors,

$$\sum_{k \in \mathbb{N}} f_O(k) = \sum_{k \text{ odd}} f_O(k).$$

For any odd k, consider the multiples of k between 1 and $2n$. They form a sequence

$$k, 2k, 3k, \ldots, \left\lfloor \frac{2n}{k} \right\rfloor k$$

alternating between odd and even terms. There are either an equal number of odd and even terms, or there is one more odd term than even terms. Therefore, we have the inequality

$$f_O(k) \le f_E(k) + 1$$

for all odd k. Combining this with the previous observations gives us the desired inequality:

$$
\begin{aligned}
\sum_{k \in \mathbb{N}} f_O(k) &= \sum_{k \text{ odd}} f_O(k) \\
&\le \sum_{k \text{ odd}} (f_E(k) + 1) \\
&= \sum_{k \text{ odd}} f_E(k) + n \\
&= \sum_{k \text{ odd}} f_E(k) + f_E(2) \\
&\le \sum_{k \in \mathbb{N}} f_E(k).
\end{aligned}
$$

$\square$

P5: Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b, erases them, and writes either $a + b$ or $\min\{a^2, b^2\}$. After $n - 1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} \; < \; f(n) \; \leq \; 3^{n/3}.$$

Solution. Clearly $f(n)$ is a strictly increasing function, as we can form $f(n-1)$ with $n-1$ ones, and add the final one. However, we can do better; assume Vishal generates $f(n)$ on the board. After $n - 2$ minutes, there are two numbers left, say they were formed by x ones and y ones, where $x + y = n$. Clearly the numbers are at most $f(x), f(y)$ (and can be made to be equal to $f(x), f(y)$), and therefore we obtain

$$f(n) = \max_{x+y=n,\, 1 \leq x \leq y \leq n-1} \left(\max\left(f(x) + f(y), f(x)^2 \right) \right) \tag{1}$$

where we used the fact that f is increasing to get that $\min(f(x)^2, f(y)^2) = f(x)^2$ when $x \leq y$. In particular, $f(n+1) \geq f(n) + 1$, and $f(2n) \geq f(n)^2$ for all positive integers n.

Upper bound:

First proof of upper bound. We use induction. We can check that $f(n) = n$ for $n \leq 4$, and these all satisfy the bound $f(n) = n \leq 3^{n/3}$. Assume it is true for all $m < n$ (some $n \geq 5$), and with x, y as in equation 1 we have

$$f(x)^2 \; \leq \; f\left(\left\lfloor \frac{n}{2} \right\rfloor \right)^2 \; \leq \; \left(3^{n/6} \right)^2 \; = \; 3^{n/3},$$

as desired. It thus remains to show that $f(x) + f(y) \leq 3^{n/3}$. By induction, it suffices to prove that

$$3^{x/3} + 3^{y/3} \; \leq \; 3^{(x+y)/3},$$

for $1 \leq x \leq y \leq n - 1$ and $x + y = n$. This is equivalent to

$$1 + 3^{(y-x)/3} \; \leq \; 3^{y/3}.$$

Let $w = 3^{(y-x)/3}$, and we require $3^{x/3} w \geq w + 1$. If $x \geq 2$, then this is true as $w \geq 1$, and if $x = 1$ then $w = 3^{(n-2)/3} \geq 3$ and the result is still true. Thus all terms in equation 1 are at most $3^{n/3}$, and so $f(n) \leq 3^{n/3}$, and the upper bound is proven.

Second proof of upper bound. Consider a second game with the same rules but in which Vishal can replace a and b by either $a + b$ or ab. Let $g(n)$ be the largest possible value for this new game. Then $f(n) \leq g(n)$ because $\min\{a^2, b^2\} \leq ab$.

We can check $g(n) = n$ for $n \leq 4$, so $g(n) \leq 3^{n/3}$ for these values. If x and y are both bigger than 1, then $g(x) + g(y) \leq g(x)g(y)$. Therefore, for $n > 4$, we have that

$$g(n) = \max\left\{ g(n-1) + 1, \max_{1 \leq x \leq n-1} g(x)g(n-x) \right\}$$

Now proceed similarly to the first proof. Assume $n > 4$ and $g(m) \leq 3^{m/3}$ for all $m < n$. If $1 \leq x \leq n - 1$, then $g(x)g(n-x) \leq 3^{x/3} 3^{(n-x)/3} = 3^{n/3}$. And $g(n-1) + 1 \leq 3^{(n-1)/3} + 1$, which is shown to be less than $3^{n/3}$ in the first proof. It follows that $f(n) \leq g(n) \leq 3^{n/3}$.

<u>Lower bound:</u>
First proof of lower bound. We begin with a lemma.

Lemma 1. *Let m be a nonnegative integer. Then*

$$f(2^m) \geq 2^{2^{m-1}} \quad and \quad f(3 \cdot 2^m) \geq 3^{2^m}.$$

Proof. We prove the lemma by induction. One can check that $f(n) = n$ for $n \leq 3$, which proves the lemma for $m = 0$. For a general $m > 0$, we get

$$f(2^m) \geq f(2^{m-1})^2 \geq \left(2^{2^{m-2}} \right)^2 = 2^{2^{m-1}}$$
$$f(3 \cdot 2^m) \geq f(3 \cdot 2^{m-1})^2 \geq \left(3^{2^{m-1}} \right)^2 = 3^{2^m},$$

by induction, as required. $\square$

(This lemma can also be proved more constructively. Briefly, if $n = 2^m$, then partition the 1's on the board into 2^{m-1} pairs, and then add each pair to get 2^{m-1} 2's $(2 = 2^{2^0})$; then multiply pairs of 2's to get 2^{m-2} 4's $(4 = 2^{2^1})$; then multiply pairs of 4's to get 2^{m-3} 16's $(16 = 2^{2^2})$; and so on, until there are 2 $(= 2^1)$ copies of $2^{2^{m-2}}$, which then gets replaced with a$2^{2^{m-1}}$). The process is similar for $n = 3 \cdot 2^m$, except that the first step is to partition the 1's into 2^m groups of 3, and then use addition within each group to get 2^m 3's on the board.)

Now assume $2^x \leq n < 3 \cdot 2^{x-1}$ for some integer x. Then we have

$$f(n) \geq f(2^x) \geq 2^{2^{x-1}} > 2^{n/3},$$

as required. If no such x exists, then there exists an integer x such that $3 \cdot 2^{x-1} \leq n < 2^{x+1}$. In this case, we have

$$f(n) \geq f(3 \cdot 2^{x-1}) \geq 3^{2^{x-1}} > 2^{2^{x+1}/3} > 2^{n/3}$$

where the second last inequality is equivalent to $2^{x-1} \log(3) \geq \frac{2^{x+1}}{3} \log(2)$, and by dividing out 2^x and clearing the denominator this is equivalent to $3 \log(3) \geq 4 \log 2$, which is true as $3^3 = 27 > 16 = 2^4$.

Second proof of lower bound. We shall prove the stronger result $f(n) \geq 2^{(n+1)/3}$ for $n \geq 2$ by induction. One can check that $f(n) = n$ for $n = 2, 3, 4$, which proves the result for these values. Assume that $n \geq 5$ and that $f(k) \geq 2^{(k+1)/3}$ for all $k = 2, 3, \ldots, n-1$. Then

$$
\begin{aligned}
f(n) \ &\geq \ f\left(\lfloor n/2 \rfloor\right)^2 \\
&\geq \ \left(2^{(\lfloor n/2 \rfloor + 1)/3}\right)^2 \qquad && \text{since } \left\lfloor \tfrac{n}{2} \right\rfloor \geq 2 \\
&= \ 2^{(2\lfloor n/2 \rfloor + 2)/3} \\
&\geq \ 2^{(n+1)/3} \qquad && \text{since } \left\lfloor \tfrac{n}{2} \right\rfloor \geq \tfrac{n-1}{2}.
\end{aligned}
$$

The result follows by induction. □

Remark 1. *One can show that f satisfies the recurrence $f(n) = n$ for $n = 1, 2$, $f(2n) = f(n)^2$ for $n \geq 2$, and $f(2n+1) = f(2n) + 1$ for $n \geq 1$. The upper bound in the problem is tight (equality holds for $n = 3 \cdot 2^x$), but the lower bound is not.*