

Canadian Mathematical Olympiad Qualifying Repêchage 2022



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Official Solutions

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1 Let $n \geq 2$ be a positive integer. On a spaceship, there are n crewmates. At most one accusation of being an imposter can occur from one crewmate to another crewmate. Multiple accusations are thrown, with the following properties:

- Each crewmate made a different number of accusations.
- Each crewmate received a different number of accusations.
- A crewmate does not accuse themselves.

Prove that no two crewmates made accusations at each other.

Solution: We will prove this by induction on n .

If $n = 2$, there are two crewmates. Since each crewmate can only make 0 or 1 accusation, the only result is one crewmate is accusing the other crewmate, and not vice versa. Hence, the two crewmates are not accusing each other.

Now suppose the statement is true for $n = k$. Now we will consider the case for $k + 1$ crewmates.

The number of accusations that each crewmate can make is any number from $\{0, 1, 2, \dots, k\}$, since there are a total of $k + 1$ crewmates. Since each crewmate made a different number of accusations, one crewmate has made k accusations. In other words, this crewmate made an accusation against everyone else. Let this crewmate be denoted by C .

Note also that each crewmate received a different number of accusations, chosen from $\{0, 1, 2, \dots, k\}$.

Therefore, one crewmate is not accused by anyone. But since C has already accused everybody, everyone else has received at least one accusation. Therefore, the person who has not received any accusations is C . Consider the group without C . Since no one else accused C , the remaining k crewmates still has made a different number of accusations. Since C accused everyone, without C , the number of accusations received by each of the remaining crewmates decreased by 1. Hence the number of accusations each crewmate received remain all different. Therefore, the group without C is a group of size k that satisfy the condition given in the question. Therefore, by mathematical induction, in the group without C , no

two crewmates were accusing each other. Adding C back in, since nobody accused C , C could not have accused and be accused by another crewmate. Therefore, in both cases (including C or not including C), no two crewmates made accusations at each other. \square

2 Determine all pairs of integers (m, n) such that $m^2 + n$ and $n^2 + m$ are both perfect squares.

Solution: The answer is $(m, n) = (k^2, 0)$ or $(0, k^2)$ for any integer k , and $(m, n) = (-1, -1)$.

We will consider two cases for our analysis. The first case is when one of m, n is 0. The second case is when neither of m, n is 0.

If $m = 0$, then $m^2 + n = n$ and $n^2 + m = n^2$. The latter is already a perfect square. The former is a perfect square if and only if n is a perfect square. Therefore, $n = k^2$ for some integer k . Therefore, $(m, n) = (0, k^2)$. By symmetry, $(m, n) = (k^2, 0)$ is also a solution.

Now we will consider the case where neither of m, n are zeroes. Then $m^2 + n$ is not equal to m^2 and $n^2 + m$ is not equal to n^2 .

The closest perfect squares to m^2 are $(m-1)^2 = m^2 - (2m-1)$ and $(m+1)^2 = m^2 + (2m+1)$.

Therefore, $|n| \geq |2m-1|$, where $|\cdot|$ denotes absolute value. Similarly, $|m| \geq |2n-1|$.

Hence, we have the sequence of inequalities, and using the triangle inequality on absolute values, we have, $|n| \geq |2m-1| \geq |2m| - 1 \geq 2(|2n-1|) - 1 \geq 2(|2n| - 1) - 1 = 4|n| - 3$. Therefore, $|n| \leq 1$. Since $n \neq 0$, $n = -1, 1$. Similarly, $m = -1, 1$. By inspection checking $(m, n) = (-1, -1), (-1, 1), (1, -1), (1, 1)$, we can see that the only one that satisfies $m^2 + n$ and $n^2 + m$ being perfect squares is $(m, n) = (-1, -1)$.

Both cases have been accounted for. This completes the solution.

3 Consider n real numbers x_0, x_1, \dots, x_{n-1} for an integer $n \geq 2$. Moreover, suppose that for any integer i , $x_{i+n} = x_i$. Prove that

$$\sum_{i=0}^{n-1} x_i(3x_i - 4x_{i+1} + x_{i+2}) \geq 0.$$

Solution: Let us first note that

$$\sum_{i=1}^n x_i(3x_i - 4x_{i+1} + x_{i+2}) = \sum_{i=1}^n (2x_i^2 + x_i^2/2 + x_i^2/2 - 2x_i x_{i+1} - 2x_i x_{i+1} + x_i x_{i+2}).$$

Next, by the cyclic nature of this sum we may rewrite it as follows:

$$\begin{aligned} & \sum_{i=1}^n (2x_i^2 + x_{i-1}^2/2 + x_{i+1}^2/2 - 2x_i x_{i-1} - 2x_i x_{i+1} + x_{i-1} x_{i+1}) \\ &= \sum_{i=1}^n (2x_i^2 - 2x_i(x_{i-1} + x_{i+1}) + (x_{i-1} + x_{i+1})^2/2) \\ &= \sum_{i=1}^n \left(\sqrt{2}x_i - (x_{i-1} + x_{i+1})/\sqrt{2} \right)^2, \end{aligned}$$

which is clearly non-negative as it is the sum of n non-negative terms. This finishes the proof.

4 For a non-negative integer n , call a one-variable polynomial F with integer coefficients n -good if there exist exactly n values of c for which $F(c)$ is prime. Show that there exist infinitely many non-constant polynomials, F , each with the following properties:

1. $F(0) = 1$
2. For every positive integer c , $F(c) > 0$, and
3. F is not n -good for any n

Solution: We choose polynomials of the form $2^k x + 1$ for some k . Assume that this polynomial takes on finitely many primes. Consider a prime p dividing $c^{2^n} + 1$ for some even number c . Then p divides $c^{2^{n+1}} - 1$ and $c^{p-1} - 1$. If 2^{n+1} does not divide $p - 1$ then by the Euclidean algorithm, p must divide $c^{2^m} - 1$ for some $m \leq n$, and hence p divides $c^{2^n} - 1$, which is a contradiction. Thus, if p divides $c^{2^n} + 1$ for some c , then 2^{n+1} must divide $p - 1$. We finally choose $n = k - 1$ and c to be twice the product of all primes that $2^k x + 1$ takes. Take a prime factor of this number, and it must be an additional prime value the polynomials takes on.

5 Alice has four boxes, 327 blue balls, and 2022 red balls. The blue balls are labeled 1 to 327. Alice first puts each of the balls into a box, possibly leaving some boxes empty. Then, a random label between 1 and 327 (inclusive) is selected, Alice finds the box the ball with the label is in, and selects a random ball from that box. What is the maximum probability that she selects a red ball?

Solution: It is $\frac{2022}{327+2022} = \frac{2022}{2349} = \frac{674}{783}$. We show that Alice can do at least as well by merging the any two boxes, so it is optimal to merge all of the boxes. Let the first box to be merged have a blue balls and b red balls, and let the second box have c blue balls and d red balls. The original probability that Alice wins by selecting one of the blue balls in the two boxes is

$$\frac{1}{327} \left(\frac{ab}{a+b} + \frac{cd}{c+d} \right),$$

and the probability that Alice wins by selecting one of those balls after merging the boxes is,

$$\frac{1}{327} \frac{(a+c)(b+d)}{a+b+c+d}.$$

It remains to show the inequality

$$\frac{ab}{a+b} + \frac{cd}{c+d} \leq \frac{(a+c)(b+d)}{a+b+c+d}.$$

There are several approaches to finish from here. In particular, one may simply clear denominators and collect like terms and the resulting inequality is

$$(ad - bc)^2 \geq 0.$$

Alternatively, we have

$$\frac{ab}{a+b} + \frac{cd}{c+d} \leq \frac{(a+c)(b+d)}{a+b+c+d}$$

$$\begin{aligned} \Leftrightarrow \left(a - \frac{ab}{a+b}\right) + \left(c - \frac{cd}{c+d}\right) &\geq \left(a+c - \frac{(a+c)(b+d)}{a+b+c+d}\right) \\ \Leftrightarrow \frac{b^2}{a+b} + \frac{d^2}{c+d} &\geq \frac{(b+d)^2}{a+b+c+d} \\ \Leftrightarrow \left(\frac{b^2}{a+b} + \frac{d^2}{c+d}\right) ((a+b) + (c+d)) &\geq (b+d)^2, \end{aligned}$$

which is the Cauchy-Schwarz inequality.

6 Let a, b, c be real numbers, which are not all equal, such that

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3.$$

Prove that at least one of a, b, c is negative.

Solution 1: Suppose on the contrary that all of $a, b, c \geq 0$.

Rewriting the condition in the problem yields $a + b + c = 3$ and

$$\frac{ab + bc + ca}{abc} = 3.$$

Let $t = abc$. Then $ab + bc + ca = 3t$. We will be using these two statements in the following arguments.

Since $a, b, c \geq 0$, we can apply the AM-GM inequality on a, b, c .

By the AM-GM inequality, we have

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}.$$

Since $a + b + c = 3$, this is equivalent to $1 \geq \sqrt[3]{t}$. Therefore, $t \leq 1$. (*)

Also by the AM-GM inequality,

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2}.$$

This can be rewritten as $3t \geq 3\sqrt[3]{t^2} \Rightarrow t \geq t^{2/3} \Rightarrow t^{1/3} \geq 1 \Rightarrow t \geq 1$.

Combining this with (*) yields $t = 1$. Therefore, $abc = 1$. Hence, $ab + bc + ca = 3t = 3$. Since $a + b + c = 3$, a, b, c are roots of the polynomial $x^3 - 3x^2 + 3x - 1$. This factors as $(x - 1)^3$. Therefore, $a = b = c = 1$, all of which are equal. This contradicts the statement that a, b, c are not all equal.

Therefore, one of a, b, c must be negative.

Solution 2: Suppose on the contrary that all of $a, b, c \geq 0$. Then by Cauchy-Schwarz inequality, we have

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \left(a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c} \right)^2 = 9,$$

with equality if and only if $a/(1/a) = b/(1/b) = c/(1/c)$, i.e. $a = b = c = 1$. This equality holds with the conditions given in the question. since $a + b + c = 3$ and $1/a + 1/b + 1/c = 3$. Since not all of a, b, c are equal, this inequality could not have held. Therefore, one of a, b, c is negative. \square

Comment: $(a, b, c) = (-1, 2 + \sqrt{3}, 2 - \sqrt{3})$ is a solution to the equation with negative values.

- 7 Let ABC be a triangle with $|AB| < |AC|$, where $|\cdot|$ denotes length. Suppose D, E, F are points on side BC such that D is the foot of the perpendicular on BC from A , AE is the angle bisector of $\angle BAC$, and F is the midpoint of BC . Further suppose that $\angle BAD = \angle DAE = \angle EAF = \angle FAC$. Determine all possible values of $\angle ABC$.

Solution: The only possible value is $\angle ABC = 90^\circ$.

Let $\theta = \angle BAD = \angle DAE = \angle EAF = \angle FAC$.

Let O be the circumcentre of $\triangle ABC$. Then note that

$\angle BAD = 90 - \angle ABD = 90 - \angle ABC = 90 - \frac{1}{2}\angle AOC = \angle OAC$. Therefore, $\theta = \angle OAC = \angle FAC$.

Therefore, A, F, O are collinear.

Since $FB = FC$ and $OB = OC$, F, O lie on the perpendicular bisector of BC . If F, O are distinct points, then A is also on the perpendicular bisector of BC . But $|AB| < |AC|$, thus this is impossible. Therefore, $F = O$. Thus, the midpoint of BC is the circumcentre of $\triangle ABC$. We conclude that $\angle BAC = 90^\circ$.

- 8 Let m, n, k be positive integers. k coins are placed in the squares of an $m \times n$ grid. A square may contain any number of coins, including zero. Label the k coins C_1, C_2, \dots, C_k . Let r_i be the number of coins in the same row as C_i , including C_i itself. Let s_i be the number of coins in the same column as C_i , including C_i itself. Prove that

$$\sum_{i=1}^k \frac{1}{r_i + s_i} \leq \frac{m+n}{4}.$$

Solution: Let R be any row, and suppose it has x coins $C_{n_1}, C_{n_2}, \dots, C_{n_x}$. Note that $r_{n_1}, r_{n_2}, \dots, r_{n_x}$ each equal to x . Therefore,

$$\sum_{i=1}^x \frac{1}{r_{n_i}} = 1.$$

Since each coin belongs to exactly one row,

$$\sum_{i=1}^k \frac{1}{r_i} \leq m,$$

since this left-hand expression is equal to the number of rows in the grid containing coins, which is at most m . Analogously,

$$\sum_{i=1}^k \frac{1}{s_i} \leq n.$$

Note that by $AM - HM$ inequality, using the arithmetic and harmonic mean of $\frac{1}{x}$ and $\frac{1}{y}$, we have

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$$

You can also prove this easily using first principles. Therefore,

$$\frac{1}{r_i} + \frac{1}{s_i} \geq \frac{4}{r_i + s_i},$$

for each $i \in \{1, 2, \dots, k\}$.

We conclude that,

$$m + n \geq \sum_{i=1}^k \left(\frac{1}{r_i} + \frac{1}{s_i} \right) \geq \sum_{i=1}^k \frac{4}{r_i + s_i},$$

which is equivalent to the equality in the problem statement. This completes the proof.