P1. Assume that real numbers \( a \) and \( b \) satisfy

\[
ab + \sqrt{ab} + 1 + \sqrt{a^2 + b} + \sqrt{b^2 + a} = 0.
\]

Find, with proof, the value of

\[
a\sqrt{b^2 + a} + b\sqrt{a^2 + b}.
\]

P2. Let \( d(k) \) denote the number of positive integer divisors of \( k \). For example, \( d(6) = 4 \) since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers \( n \),

\[
d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).
\]

P3. Let \( n \geq 2 \) be an integer. Initially, the number 1 is written \( n \) times on a board. Every minute, Vishal picks two numbers written on the board, say \( a \) and \( b \), erases them, and writes either \( a + b \) or \( \min\{a^2, b^2\} \). After \( n - 1 \) minutes there is one number left on the board. Let the largest possible value for this final number be \( f(n) \). Prove that

\[
2^{n/3} < f(n) \leq 3^{n/3}.
\]

P4. Let \( n \) be a positive integer. A set of \( n \) distinct lines divides the plane into various (possibly unbounded) regions. The set of lines is called “nice” if no three lines intersect at a single point. A “colouring” is an assignment of two colours to each region such that the first colour is from the set \( \{A_1, A_2\} \), and the second colour is from the set \( \{B_1, B_2, B_3\} \). Given a nice set of lines, we call it “colourable” if there exists a colouring such that

(a) no colour is assigned to two regions that share an edge;
(b) for each \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \) there is at least one region that is assigned with both \( A_i \) and \( B_j \).

Determine all \( n \) such that every nice configuration of \( n \) lines is colourable.
P5. Let $ABCDE$ be a convex pentagon such that the five vertices lie on a circle and the five sides are tangent to another circle inside the pentagon. There are \( \binom{5}{3} = 10 \) triangles which can be formed by choosing 3 of the 5 vertices. For each of these 10 triangles, mark its incenter. Prove that these 10 incenters lie on two concentric circles.

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**Important!**

*Please do not discuss this problem set online for at least 24 hours!*