Crux Mathematicorum

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MathemAttic

No. 34

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2022**.

MA166. Consider the set of all right triangles in the plane whose right angle vertex lies at the origin and whose other two vertices lie somewhere else on the graph of $y = \frac{x^2}{4}$. Prove that all such triangles must pass through $(0, 4)$.

MA167. A fair coin is a coin that will produce a result of either heads (H) or tails (T) when flipped with equal probability. Three separate trials are conducted. In each case, determine the expected number of flips required:

a) to get the first tail (T)?

b) to achieve the first occurrence of heads followed by tails (HT)?

c) to achieve the first occurrence of heads followed by heads (HH)?

MA168. What is the value of the positive integer n for which the least common multiple of 36 and n is 500 greater than the greatest common divisor of 36 and n?

MA169. Consider the following game. There are two players, player 1 and player 2. There is a pile of coins, each identical, on the table. Player 1 acts first, and must remove either 1, 2, or 3 coins. Player 2 acts next, and must remove either 1, 2, or 3 coins. The players continue taking turns in the manner described until there are no coins left on the table. The player who selects the last coin is the loser. It is known that player 1 has a strategy that will guarantee a win if the number of chips on the table is congruent to 0, 2, or 3 modulo 4. Explain what this strategy is, and prove that the strategy will guarantee the win for player 1.

MA170. A bicyclist rides 18 miles in exactly 72 minutes. Prove that there exists a contiguous 3-mile segment within these 18 miles that the rider completed in exactly 12 minutes.

. . . .

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2022**.

MA166. Prenons en considération tous les triangles rectangles dans le plan, dont l'angle rectangle se situe à l'origine et dont les deux autres sommets se trouvent en quelque part sur le graphique de $y = \frac{x^2}{4}$. Démontrer que tout tel triangle doit passer par $(0, 4)$.

MA167. Une pièce de monnaie est dite juste si elle donne pile (P) ou face (F) avec probabilités égales lorsqu'on joue à pile ou face.

a) Déterminer le nombre moyen de tirs requis afin d'obtenir la première occurrence de pile (P).

b) Déterminer le nombre moyen de tirs requis afin d'obtenir la première occurrence de face suivi de pile (FP).

c) Déterminer le nombre moyen de tirs requis afin d'obtenir la première occurrence de face suivi de face (FF).

MA168. Déterminer la valeur de l'entier positif n pour lequel le plus petit commun multiple de 36 et n est de 500 supérieur au plus grand commun diviseur de 36 et n.

MA169. Prenons en considération le jeu décrit ci-après. Il s'agit premièrement d'un jeu entre joueur 1 et joueur 2, où les deux joueurs retirent des pièces de monnaie d'une pile composée de pièces identiques. Joueur 1 commence et enlève 1, 2 ou 3 pièces de monnaie ; ensuite joueur 2 enlève 1, 2 ou 3 pièces de monnaie. Les deux joueurs alternent ainsi jusqu'à ce qu'il ne reste aucune pièce de monnaie. Le joueur à enlever la dernière pièce perd. Or, il semble bien connu que joueur 1 possède une stratégie assurant la victoire si le nombre de pièces au départ est équivalent à 0, 2 ou 3 modulo 4. Expliquer ce qu'est cette stratégie et expliquer pourquoi elle assure la victoire à joueur 1.

MA170. Un cycliste fait une promenade de 18 miles en exactement 72 minutes. Démontrer qu'il existe obligatoirement un segment de 3 miles contigus que le cycliste complète en exactement 12 minutes.

MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(9), p. 411–413.

MA141. *Proposed by Ed Barbeau.*

Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the cubes of the smallest $m+1$ integers is equal to the sum of the cubes of the largest m integers.

We received 5 submissions of which 4 were correct and complete. We present the solution by Michael Lin.

There are no such sets.

For $m \in \mathbb{N}$, let $M_m = \frac{1}{2}m(m+1)$. It is known that $M_m \in \mathbb{N}$, $\sum_{i=1}^{m} i = M_m$, and $\sum_{i=1}^{m} i^3 = M_m^2$.

Let $a \in \mathbb{Z}$ and $m \in \mathbb{N}$. If $\sum_{i=1}^{m}(a-i)^3 + a^3 = \sum_{i=1}^{m}(a+i)^3$, then

$$\begin{aligned} 0 &= \sum_{i=1}^{m}((a+i)^3 - (a-i)^3) - a^3 \\ &= \sum_{i=1}^{m}(2i^3 + 6a^2 i) - a^3 \\ &= 2M_m^2 + 6a^2 M_m - a^3. \end{aligned}$$

Viewed as a quadratic in M_m, we have

$$M_m = \frac{-6a^2 \pm \sqrt{36a^4 + 8a^3}}{4} = \frac{-6a^2 \pm 2|a|\sqrt{9a^2+2a}}{4}.$$

If $a \leq -1$, $(3a+1)^2 < (9a^2+6a+1) - (4a+1) = 9a^2 + 2a < (3a)^2$. Thus, $|3a+1| < \sqrt{9a^2+2a} < |3a|$. But then $M_m \notin \mathbb{Q}$, contradicting $M_m \in \mathbb{N}$.

If $a = 0$, then $M_m = 0$, contradicting $M_m \in \mathbb{N}$.

If $a \geq 1$, we have

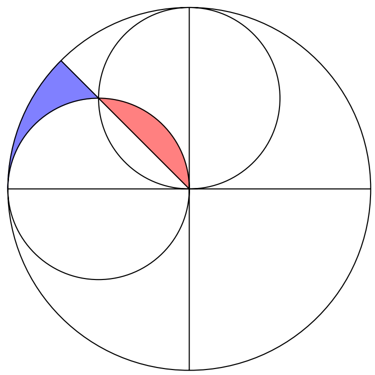
$$(3a)^2 < 9a^2 + 2a < 9a^2 + 6a + 1 = (3a+1)^2.$$

Therefore, we have $3a < \sqrt{9a^2+2a} < 3a+1$. But then $M_m \notin \mathbb{Q}$, contradicting $M_m \in \mathbb{N}$.

Thus, a contradiction always arises, so there is no $a \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that

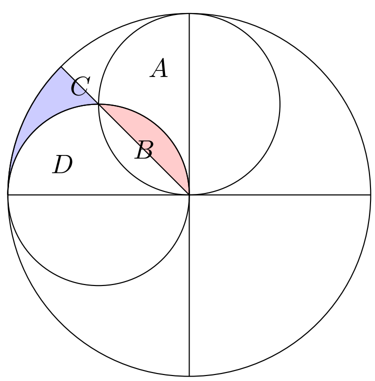
$$\sum_{i=1}^{m}(a-i)^3 + a^3 = \sum_{i=1}^{m}(a+i)^3.$$

MA142. The sketch shown is from the files of Leonardo da Vinci. Two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



Originally from 2013 Manitoba Mathematical Competition, question 6.

We received 9 submissions, all correct. We present the solution by Michael Lin.



Let regions A, B, C, D be labeled as indicated in the figure, and let $[X]$ denote the area of region X. Let the radius of the larger circle be $2r$, so the radius of the smaller circles is r. By symmetry, the red-shaded region has area $[B]/2$ and the blue-shaded region has area $[C]/2$. Note

$$[A]+[C]=\frac{1}{4}\pi(2r)^2-([B]+[D])=\pi r^2-\frac{1}{2}\pi r^2=\frac{1}{2}\pi r^2=[A]+[B],$$

so $[B]=[C]$. Thus,

$$[\text{red-shaded region}]=[B]/2=[C]/2=[\text{blue-shaded region}].$$

MA143. The integers from 1 to n are added to form the sum N and the integers from 1 to m are added to form the sum M, where $n > m+1$. If the difference between the two sums is $N - M = 2012$, then determine the value of $n+m$.

Originally from 2012 BC Secondary School Mathematics Contest, Senior Final, Part A, problem 5.

We received sixteen submissions, out of which fifteen were correct and complete. We present the solution by Akhil Sankar, lightly edited.

We have

$$N = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$
$$M = 1+2+3+\cdots+m = \frac{m(m+1)}{2}.$$

Therefore

$$2012 = N - M = \frac{n(n+1)-m(m+1)}{2}$$

or

$$4024 = (n-m)(n+m+1).$$

Since $n > m+1$ we know that both factors are positive. The prime factorization of 4024 is $2^3 \cdot 503$. Since $n-m$ and $n+m+1$ have opposite parity and since

$$1 < n-m < n+m+1,$$

the only possible solution is $n-m = 8$ and $n+m+1 = 503$ and so $n+m = 502$.

MA144. A game is played on a 7×7 board, initially blank. Betty Brown and Greta Green make alternate moves, with Betty going first. In each of her moves, Betty chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Greta chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Betty. Then Greta paints the remaining blank squares green. Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?

Originally from Originally from 2009 Alberta High School Mathematics Competition, Part II, problem 3.

We received 4 submissions, of which 2 were correct and complete. We present the solution by Rynkeviit Dominyka.

There are nine squares at the intersections of even-numbered rows. Any block chosen by Betty will include one of these nine squares. Thus, Greta has to play in those nine squares in her first four moves. This way Betty will have at most five moves and will be able to paint at most twenty squares. Thus, Greta will win.

MA145. Determine all integers n for which $n^3 - 3n + 2$ is divisible by $2n + 1$.

Originally from 2010 Alberta High School Mathematics Competition, Part I, problem 16.

We received 14 submissions of which 9 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

Suppose n is an integer for which $n^3 - 3n + 2$ is divisible by $2n + 1$. By the division algorithm,

$$M = \frac{n^3 - 3n + 2}{2n + 1} = \frac{1}{2}n^2 - \frac{1}{4}n - \frac{11}{8} + \frac{27}{8(2n + 1)},$$

and M is an integer. Since

$$8M = 4n^2 - 2n - 11 + \frac{27}{2n + 1}$$

is also an integer, $2n + 1$ must be a divisor of 27. Solving $2n + 1 = d$ for each d in $\{\pm 1, \pm 3, \pm 9, \pm 27\}$, the set of divisors of 27, we get that n must be -14, -5, -2, -1, 0, 1, 4, or 13, and one readily verifies that these satisfy the condition.

TEACHING PROBLEMS

No. 16

Susan Milner, Shawn Godin and John McLoughlin

Rectangles

Puzzles surround us. Newspapers feature them. Puzzle apps are loaded onto many people's devices. Some of them entertain us as they are disguised as games. They are enjoyable because they pose a challenge by presenting us with a mystery to solve. Many engage us in mathematical problem-solving which is hidden beneath the veil of entertainment. This article is based upon material from an upcoming volume of the **ATOM** (*A Taste of Mathematics*) series published by the CMS.

Logic puzzles provide a way of engaging in mathematical thinking that is accessible to everyone, from people who don't think of themselves as mathematicians to those whom the world regards as outstanding mathematicians.

Playing brings engagement, which in turn leads to the development of mathematical habits of thought. This is observable even in the space of an hour in the classroom - at any age. The playful attitude itself in a math class can be entirely new for many students, even though it is characteristic of many professional mathematicians and educators.

Here are some of the desirable mathematical habits of thought we have observed in many classrooms from kindergarten to grade 12:

- students start to develop and to articulate longer chains of reasoning;
- students start making the distinction between what they know must be true as opposed to what might possibly work;
- students become comfortable with making mistakes and starting again;
- students learn to look for good places to start a puzzle, as opposed to picking somewhere at random; and
- students start to develop indirect forms of reasoning, such as spotting a move that will force a subsequent impossibility.

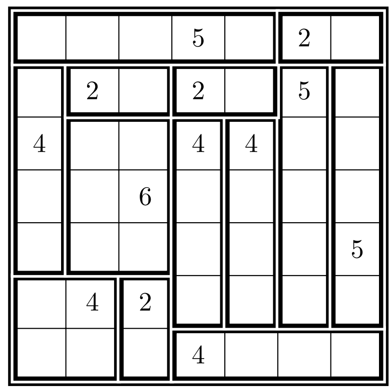
Incidental benefits are realized over time. These extend to students and teachers. For instance, playing with several different types of puzzles will generally facilitate growth in meeting novel puzzle situations. People start to look for the characteristics of puzzles. Teachers also become more adept at working with puzzles while gaining valuable insight into their students' thinking. One of the realizations may be the fact that the particular students who excel in such situations may not be those who typically excel in more formal mathematical assessments.

In most cases, when we encounter a puzzle for the first time, we are given the rules and then set the task of solving the puzzle. In some cases, strategies to aid in solving puzzles can be sought either from a book, article or online. However, with many puzzles, some or all of the rules could be deduced by examining some puzzles alongside their solutions.

Consider the following puzzle and its solution:

			5		2	
	2		2		5	
4			4	4		
		6				
						5
	4	2				
			4			

Puzzle 1



Puzzle 1 solved

Can you deduce the rules of the game from this example? Can you see your students deducing the rules of the game? This puzzle was likely fairly straight forward. Others will present more difficulties when it comes to figuring out the rules. Consider using this first step when introducing puzzles to students; the problem-solving need not be limited to finding a solution!

The rules for Rectangles:

- the background grid has to be covered in rectangles which do not overlap and which have no gaps between them;
- each rectangle contains exactly one number, which tells us how many of the background grid-squares must be in the rectangle.

4		2		4		
		2		3		
	2		5			
			3			
		3				4
2		4	3		8	

Puzzle 2

		4		2				
				2		2	4	2
			6	2				
			4		2			2
	2			9		3		2
6					8			
	2		2				2	
2			3	4		2	2	

Puzzle 3

Now that you know the rules, you are set to try some puzzles. However, solve the problem as a strategist. That is, as you are solving it, make some notes about things to help you solve the problem. Where is a good place to start and why? What configurations are easiest to deal with? Are there any configurations that we need to be careful about? See what you can discover as you try the puzzles above.

One observation is that since the numbers represent the areas of rectangles, we are interested in the possible ways to factor each number. Considering puzzle 2, we are dealing with

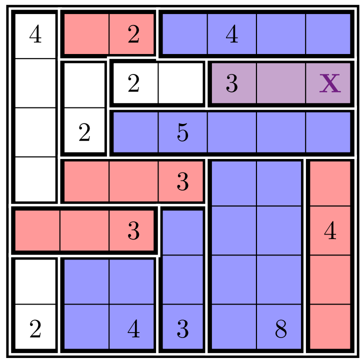
$$\begin{aligned}
&2 = 1 \times 2 = 2 \times 1,\\
&3 = 1 \times 3 = 3 \times 1,\\
&4 = 1 \times 4 = 2 \times 2 = 4 \times 1,\\
&5 = 1 \times 5 = 5 \times 1, \text{ and}\\
&8 = 1 \times 8 = 2 \times 4 = 4 \times 2 = 8 \times 1,
\end{aligned}$$

where 1×2 and 2×1 represent the same rectangle in different orientations. As such, prime values require only checking two possible dimensions of rectangles. However, since the number can appear in any of the interior squares of a rectangle, it is necessary to check various positions and spaces. Thus, for example, a 2 appearing on our grid, as shown, can only appear in one of the four rectangles: 2A, 2B, 2C, or 2D. However, a 3 can appear in six rectangles 3AA, 3BB, 3CC, 3DD, A3C, or D3B. Notice that the possibilities will be reduced for numbers along the edges of the entire grid, as compared to those in the interior.

						A		
	A					A		
D	2	B		D	D	3	B	B
	C					C		
						C		

Possible rectangles for entries 2 or 3

A good first strategy might be to look at our numbers to see if any of them only have one rectangle possible. For example, consider the diagram of puzzle 2 with the coloured markings shown here. The indicated rectangles (blue) around the 3, 4, 4, 5 and 8 are the only ones that do not include another number. Once those rectangles are firmly positioned, it follows that a few more (red) are defined. Repeated use of this idea finishes off the puzzle.

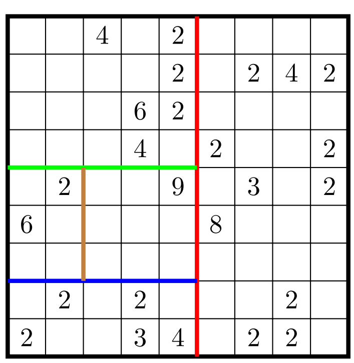


Strategies for puzzle 2

However, you may have noticed the square marked with the X (purple). After the first round, the *only* rectangle that could include this square comes from the 3. This forces the rectangle around the 3. In this puzzle, we didn't need to use that idea, but it does come in handy from time to time.

Another observation that you might have made is that the larger numbers often give us a good starting point. Even though they may yield more possible rectangle sizes, orientations and locations, it is common to find many of those choices eliminated by the fact that no other numbers can appear in them.

For example, if we consider puzzle 3 the largest number is 9. A quick glance tells us that no 1×9 rectangle will be devoid of numbers, and hence, the only option is a 3×3 rectangle. The 2 and the 8 on the right of the 9 let us see that any rectangle that contains the 9 cannot extend to the right past the red line. Similarly the 4 helps define the upper limits (green), the two 2s the lower limit (blue) and the other 2 the leftmost limit (brown). These restrictions give us only one viable placement of the rectangle of area 9. Once that rectangle is in place, similar logic yields the one around the 8 which means we have accounted for over 20% of the total area. Much of the rest of the problem falls away using only our original observations.



Looking at the 9 in puzzle 3

It is worthwhile having students discuss their solutions and strategies with each other. Over time they will refine their strategies as well as gaining appreciation for the importance of varied approaches. Suggest that students number the rectangles in the order they draw them to plot their progress through the puzzle. This will give them some discussion points with their peers.

Below are some general hints for solving these types of puzzles:

- A good starting place is a number for which there is only one option.
- Another good starting point is a grid-square that can be reached by only one rectangle.
- It can help to look for possibilities that would block another number's rectangle, leaving no further choice for that second rectangle. Then you can reject those possibilities.
- Look also for possible rectangles that would leave a grid-square unreachable by any other rectangle. Then those possible rectangles can be rejected.
- It can help to mark in grid-squares that you know must form part of a given rectangle.
- Guessing or making assumptions is likely to cause frustration! It can be difficult to catch yourself making assumptions, so sometimes all you can do is erase the entire puzzle and start again, being careful to identify your reasoning for each step you make.
- If you think you've been absolutely logical but still can't solve the puzzle, double-check your counting in the longer rectangles.

Now, if we replace "puzzle(s)" with "problem(s)", we are in a world familiar to mathematics educators. A puzzle offers a different sort of problem situation. Puzzles offer a fun and engaging way to have students develop their problem-solving and communication skills. Working them into your classroom routine can add a little life to the environment.

This article is based on an upcoming volume of the CMS's **ATOM** (*A Taste Of Mathematics*) series. The volume will be made up of various puzzles, including Rectangles. Each chapter will delve into a particular puzzle in depth. It will be set up to allow the reader to think about the rules of the game as well as strategies for dealing with the puzzles. Readers interested in receiving advanced copies of material to use in their classrooms can contact the editor at `GodinMathStuff@gmail.com`. We would love to have your feedback on the presentation of the material.

To find a variety of classroom-tested puzzles, visit `susansmathgames.ca`, where there is also a discussion of effective ways of presenting puzzles to classes, as well as templates for puzzles that use manipulatives. The website is hosted by the Pacific Institute for the Mathematical Sciences.

You'll find many more Rectangles puzzles to download or play on-line at Simon Tatham's Portable Puzzles.
`http://www.chiark.greenend.org.uk/ sgtatham/puzzles/js/rect.html`

We leave you with a more challenging puzzle for your enjoyment.

						2	2			
						3			6	
	15						3	2		
2					10	10				
			8			12				5
				8						
		4								
5			2			4				
	15								6	
									12	
							3			
	2				5					
2						7				
								8		2

Puzzle 4

Susan Milner taught post-secondary mathematics in British Columbia for 29 years, retiring in 2015 with Professor Emerita status at the University of the Fraser Valley. In 2014, she was awarded the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize. Now living in Nelson, she loves sharing math and logic games with people of all ages. Before Covid, she travelled all over BC for Science World, visiting K-12 classrooms. Lately, she's been sharing puzzles via video link, both with school children and with third-age learners in Nelson's Learning in Retirement community.

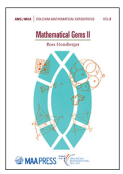
From the bookshelf of . . .

Ed Barbeau

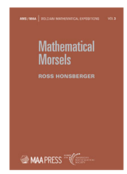
This new feature of MathemAttic *brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`*. Publishers are also welcome to send along books for possible review.*

The Honsberger Corpus

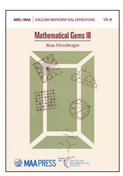
Mathematical Gems II
by Ross Honsberger
ISBN 0-88385-319-1,
182 pages
Published by MAA,
1976.



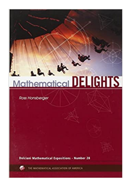
Mathematical Morsels
by Ross Honsberger
ISBN 088385-303-5,
249 pages
Published by MAA,
1979.



Mathematical Gems III
by Ross Honsberger
ISBN 0-88385-318-3,
250 pages
Published by MAA,
1985.



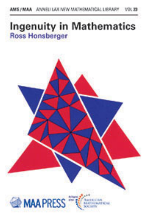
Mathematical Delights
by Ross Honsberger
ISBN 0-88385-334-5,
252 pages
Published by MAA,
2004.



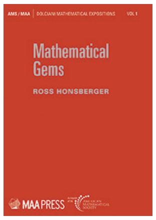
For many years beginning in the late 1960s, attendees at the annual meeting of the Ontario Association for Mathematics Education flocked to a mathematical concert performed by Ross Honsberger. A superlative witty expositor, he regaled his auditors with about ten selections from his repertoire of mathematics problems, novelties and oddities.

Fortunately, the fruits of his erudition are available. Somehow, he came to the attention of the Mathematical Association of America, which, in 1970, published his first book *Ingenuity in Mathematics*. Some time later the Association decided to embark on a new series of books, the *Dolciani Mathematical Exposition Series*, to provide polished essays for the general mathematical reader. The inaugural four volumes along with four later ones in the series were written or edited by Honsberger. In all, he published 13 books with the MAA, many with titles such as *Gems*, *Morsels*, *Chestnuts*, *Diamonds* and *Delights*, all preceded by the word "Mathematical". The Wikipedia entry for Honsberger lists his works. That is a lot of beautiful mathematics!

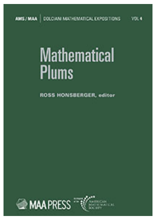
Ingenuity in Mathematics
by Ross Honsberger
ISBN 0-88385-633-9,
206 pages
Published by MAA, 1970.



Mathematical Gems
by Ross Honsberger
ISBN 0-88385-301-9,
176 pages
Published by MAA, 1974.



Mathematical Plums
Edited by Ross Honsberger
ISBN 0-88385-304-3,
182 pages
Published by MAA, 1979.

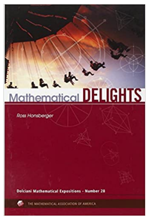


Most of the essays are on combinatorics, algebra, number theory and geometry. It is hard to give a fair sample of such rich material. Mathematicians will find along with familiar material, much that is new and fascinating. With an eye to the bizarre and unexpected, Honsberger included this checker-jumping problem in *Mathematical Gems II*:

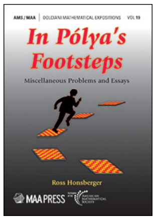
> One starts by arranging a number of men in the starting zone, which consists of the half-plane of lattice points on or below the x-axis. The object is to get a man as far as possible above the axis by jumps over a single man (which is subsequently removed) in the direction of the lattice lines onto a vacant point. Determine the least number of men in the starting zone in order to get a man to the prescribed height above the x-axis.

You can get to the first level above the x-axis starting with two men. With three men at adjacent points along the x-axis and one below one of the end men, it is possible to bring a man to the second level. Eight men are needed for one to survive to the third level, and twenty to the fourth level. As John Conway (who else?) discovered, no number of men will allow one to get a man to the fifth level above the x-axis!

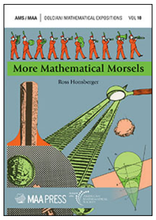
Mathematical Diamonds
by Ross Honsberger
ISBN 0-88385-332-9,
241 pages
Published by MAA, 2003.



In Pólya's Footsteps
by Ross Honsberger
ISBN 0-88385-326-4,
315 pages
Published by MAA, 1997.



More Mathematical Morsels
by Ross Honsberger
ISBN 0-88385-314-0,
312 pages
Published by MAA, 1991.



Some problems, while interesting, are more accessible. Suppose that a semicircle is drawn outwardly on a chord AB of a unit circle with centre O. Determine the position of the chord for which a point on the semicircle is maximum distance from O (*Mathematical Morsels, Problem 5*). If you have a (non-self-intersecting) $n-$sided polygon that is not necessarily convex, then it has at least $n-3$ diagonals that are entirely inside it, and exactly $n-3$ if and only if no two inner diagonals intersect. (*ibid, Problem 51*).

Of course, the work of Erdós and his collaborators (in this case, Benkoski) is evident in some of the material: given a set of distinct positive integers all of whose subsets have different sums, the sum of their reciprocals is less than 2 (*Mathematical Gems III, Chapter 17*).

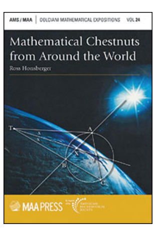
The topics are attractive for different reasons. Sometimes the result itself surprises (as Honsberger often asked in his lectures, "how does someone think of such things?"), while the solution is often standard and sometimes even tedious. At other times, there is an unusual strategy leading to a straightforward *dénouement*. But the most satisfying solutions are clever, unexpected and brief. Honsberger celebrates human ingenuity that generated the problems and solutions.

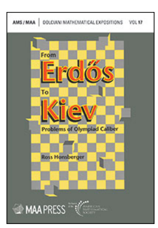
Let me close with a notoriously difficult competition problem: the 1980 IMO problem to show that $(a^2+b^2)/(ab+1)$ is square whenever a and b are integers for which $ab+1$ divides a^2+b^2, solved by a Bulgarian student during the competition (*Mathematical Delights*).

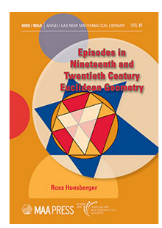
Mathematical Chestnuts from Around the World
by Ross Honsberger
ISBN 0-88385-330-2,
310 pages
Published by MAA, 2001.

From Erdos to Kiev - Problems of Olympiad Caliber
by Ross Honsberger
ISBN 0-88385-324-8,
257 pages
Published by MAA, 1996.

Episodes in Nineteenth and Twentieth Century Euclidean Geometry
by Ross Honsberger
ISBN 0-88385-639-5,
174 pages
Published by MAA, 1995.







These books are a recommendation from the bookshelf of Edward Barbeau. Ed is professor emeritus of mathematics at the University of Toronto. Besides being one of the editors of ***Crux***, Ed has been involved in mathematical enrichment for decades. He has worked on numerous mathematics competitions, has visited classrooms, presented to teachers, as well as writing books and articles for the expert and the general public alike.

An Introduction to Knot Theory

Rishi Nair

This is one of the three winning articles of the 2021 MathemAttic Article Contest. The other winning pieces will appear in later issues of this Volume.

We provide a brief introduction to some fundamentals of knot theory and its applications. The topics covered include knot notations and an overview of proving knot equivalence by Reidemeister moves and knot invariants. We conclude with a discussion on some practical applications of knot theory and review some of the recent advances in this field.

1 What is Knot Theory?

Knot theory is a branch of topology concerned with the properties of mathematical knots. When we think of knots, shoelaces, sailing knots, and neckties often come to our minds. Mathematical knots are very similar to the listed examples but have their two loose ends fused together. Formally, a mathematical knot is a simple closed polygonal curve in three-dimensional Euclidean space $\mathbb{R}^3$. You can also think of a mathematical knot as a twisted circle; if you trace your fingers along the knot from a starting point on the knot, you should be able to find your way back to the starting point.

The most basic example of a knot is a circle. It meets the criteria of a knot we have set above since it is essentially a line with no tangles whose ends have been fused. In knot theory, we classify the circle as an unknot or a trivial knot – this is because certain knots can be untangled or unknotted into a circle. The trefoil knot is the simplest non-trivial knot, which can be created by forming an overhand knot and joining its two loose ends as shown in Figure 1b. There are a plethora of knots, of which many are equivalent, meaning they can be manipulated by a sequence of carefully designed operations to have the same appearance.

The first steps in the study of mathematical knots were taken by Carl Friedrich Gauss in the 1830s; however, the origins of knot theory as we know it today can be traced back to theoretical research on the atomic structure of matter later in the same century. During that period, some scientists theorized that vacuum was made up of a special matter called the "aether." In an attempt to classify and organize atoms, mathematician/physicist William Thomson proposed that an atom can be modeled as a knotted vortex of aether [1]. This idea increasingly grew in popularity among the scientific community. Mathematician/physicist Peter Guthrie Tait was inspired by Thomson's idea and attempted to create a system to classify knots,

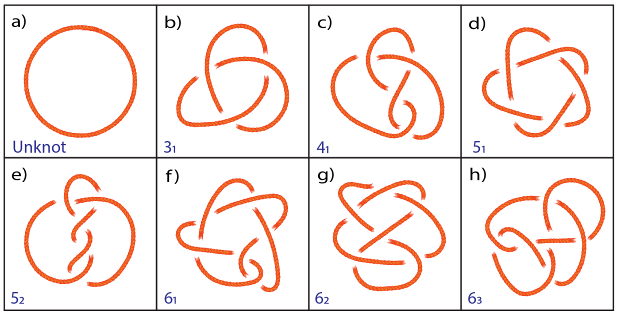


Figure 1: A visual representation of some elementary knots along with their Alexander-Briggs notation.

which could then be used to find order amongst atoms [1]. Eventually new theories came along that disproved the "aether" hypothesis. Nevertheless, these innovative ideas developed in the context of atomic theory laid the foundations of modern knot theory.

A graphical representation of eight basic knots that are distinct from one another is shown in Figure 1. Although physical knots exist in a three-dimensional space, we can represent them in two dimensions by splitting the line strand that goes underneath another line strand at each intersection point (notice the small gaps between the strands at the intersection points in Figure 1). Such graphical visualizations of knots are known as knot diagrams.

Mathematicians often use the so-called Alexander-Briggs notation to numerically represent and classify knots. In the Alexander-Briggs notation, any knot is represented as X_Y, where X denotes the crossing number and Y denotes the order of the knot. The crossing number equals the minimum number of intersection points in any projection of the knot, while the order of a knot is a number assigned to each knot such that no two distinct knots have the same Alexander-Briggs notation. Figure 1 shows the Alexander-Briggs notation for some basic knots. It can be noted that knots d) and e) have the same crossing number; however, since these two knots are distinct, we assign them different orders so that they can be differentiated by their Alexander-Briggs notation.

Mathematicians have devised different ways of representing and analyzing knots which enable us to discover unique patterns and properties. We will cover some of these ideas in the next section.

2 Some Fundamental Concepts in Knot Theory

A fundamental question in knot theory is whether two knots are equivalent. In other words, it is possible that two knots can look different but fundamentally be the same. If two knots have a different appearance but are fundamentally the

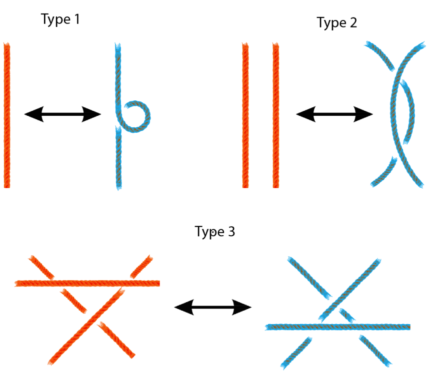


Figure 2: The three Reidemeister moves.

same, we call these two knots equivalent. In the 1920s, Kurt Reidemeister [7] proved that two knots are equivalent if one knot can be rearranged into the other knot using a sequence of certain moves that does not involve cutting or tearing the knot. There are three such types of moves which are now known as Reidemeister moves.

Figure 2 shows a graphical representation of the three Reidemeister moves. The type 1 Reidemeister move can be described as inserting or removing a twist in a knot. The type 2 move can be described as creating or eliminating two crossings within a knot. The type 3 move can be described as sliding a line strand from one side of a crossing to the other side.

Consider the seemingly complicated knot shown in Figure 3a. We will apply a sequence of Reidemeister moves to show that this knot is equivalent to the unknot (circle). We first apply a type 2 move to arrive at Figure 3b. Following this, we apply another type 2 move followed by a type 2 and a type 1 move to arrive at the unknot.

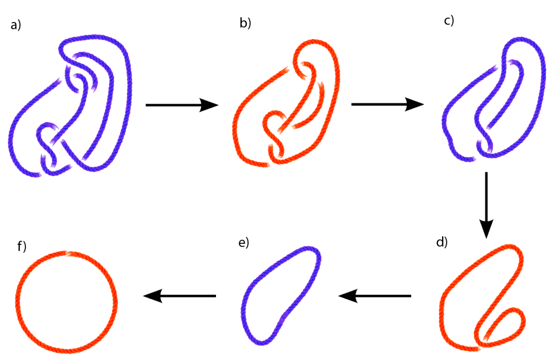


Figure 3: Reidemeister moves being used to simplify a knot into an unknot.

Mathematicians have studied if an upper bound can be established on the number of Reidemeister moves required to prove that two knots are equivalent. Haas and Lagarias proved an interesting result that the upper bound on the number of moves required to transform a complex knot with n crossings into an unknot is given by 2^{nc}, where $c = 10^{11}$ [8]. In other words, if we cannot transform a given complex knot into an unknot by 2^{nc} moves, then the given knot is not equivalent to an unknot. Note that the upper bound is significantly larger than the number of atoms in the known universe! Although this upper bound is incredibly large, the upper bound is computable. Having an upper bound therefore enables us to determine, in principle, if a knot is equivalent to an unknot in a finite amount of time.

A more elegant approach for finding if two knots are equivalent is to use the notion of knot invariants. A knot invariant can be thought of as a function that assigns a quantity for any given knot such that equivalent knots are assigned the same quantity. While these quantities are the same for equivalent knots, knot invariants by themselves aren't necessarily enough to tell if two knots are equivalent. If the invariants computed for two knots are not equal, then the knots are distinguishable from each other, and if the quantities are the same, they could potentially be equivalent. Two things to consider how effective an invariant is are how easy it is to compute that invariant for a given knot and how selective it is at differentiating nonequivalent knots.

One of the most basic knot invariants is tricolourability. A knot is considered to be tricolourable if each of its line strands can be assigned a colour such that: (1) all line strands are coloured, (2) a minimum of two colours are used and a maximum of three colours are used, and (3) at each intersection, the colours of the three line strands must all be the same colour or all be different colours. It can be shown that tricolourability is preserved by the three Reidemeister moves, and therefore is an invariant.

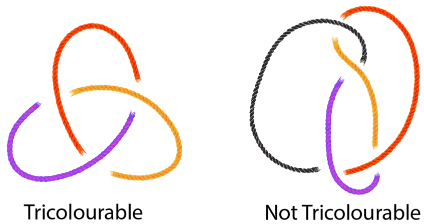


Figure 4: The tricolourability invariant idea applied to two knots.

The unknot is not tricolourable since it consists of only a single strand and therefore we cannot use at least 2 colours. Figure 4 illustrates how the tricolourability invariant can be computed for two knots. The knot on the left (trefoil) is tricolourable since we can colour each strand (following rules 1 and 2) such that at each intersection, the colours of the three line strands are all different. Therefore the trefoil knot is not equivalent to the unknot. On the other hand, the knot on the right (4_1) of the Figure is not tricolourable. We have left one strand uncoloured because no matter which colour we assign to this strand, we cannot meet the third rule of tricolourability. This means that we cannot use the tricolourability invariant to prove that the 4_1 knot is not equivalent to the unknot. However, the tricolourability invariant can be used to tell that the two knots of Figure 4 are distinct from each other.

Knot polynomials (typically in one or two variables) are another type of knot invariants. The Alexander polynomial introduced in 1923 is considered to be the first knot polynomial and was the only polynomial invariant for nearly 50 years. This invariant was later refined by John Conway and is now known as the Alexander-Conway polynomial. In 1984, Vaughan Jones discovered a new polynomial invariant which inspired further work on this topic. A notable development was the Kauffman polynomial which can be considered to be a generalization of the Jones polynomial. Knot polynomials do not provide a definitive way of determining whether two knots are equivalent. In practice, certain knots that are not equivalent can have the same knot polynomials. Nevertheless, knot polynomials have proved to be powerful tools to compare complex knots using computational methods.

Interested readers can refer to the introductory text by Adams [10] for a detailed exposition on the fundamentals of knot theory. It is worth noting that even though knot theory is abstract in nature, many of the fundamental concepts are accessible to anyone with a basic understanding of mathematics.

3 Practical Applications and Recent Advances

While knot theory is generally thought of as a field of pure mathematics lying at the intersection between topology, algebra and combinatorics, it plays an important role in several other fields of study in the theoretical and applied sciences. Knot theory is used in biochemistry and chemistry to study and analyze knotted structures such as DNA, proteins, knotted polymer structures and certain enzymes. Due to the long structure of DNA strands, they typically form complex knots [2]. Certain biological processes such as mitosis (cell division) can alter and change these knots by tangling and untangling them. Knot theory can provide valuable insights into the structural changes that happen during these processes.

Knot theory has a diverse range of applications and can also be used in optics, cryptography and even in history, anthropology and archaeology. [3]. Broadly speaking, in a field of study where objects that resemble knots are present, we can apply knot theory to classify and find properties of the knots. This information can then provide us with useful insights. Let's take archaeology for example. We can examine knotted structures present in embroidery and in sculptures. If we take two different sculptures with knots in them and find that the knots are equivalent, we can hypothesize that these structures came from the same geographical location or are from regions with similar cultural influences.

Recently, Lisa Piccirillo, a former graduate student at the University of Texas (who is now an Assistant Professor at MIT) solved a famous knot theory problem related to the Conway knot. The Conway knot, as shown in Figure 5, is a fairly simple knot whose crossing number is only 11. For nearly 50 years, ever since John Horton Conway first proposed this knot, there has been a mystery about this knot that has baffled many mathematicians. This mystery was whether the Conway knot was a slice knot. Slice knots are knots that can be found when a knotted sphere in a four-dimensional space is sliced open; an example of a slice knot is 6_1 shown in figure 1f. In 2020, Lisa Piccirillo answered this long-standing question and proved that the Conway knot was not a slice knot [4].

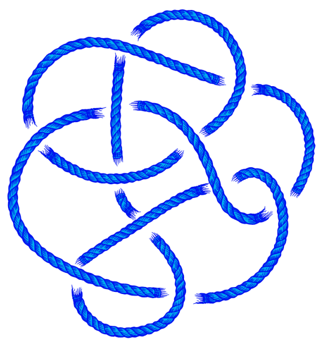


Figure 5: The Conway knot.

In 2017, researchers at the University of Illinois Chicago wrote a research paper on analyzing RNA molecules and proteins using knot theory. Due to the long structure of RNA molecules, they knot themselves in order to fit into human cells. These researchers created a new system of knot polynomials designed specifically for the purpose of studying RNA. Using this new system of knot polynomials, they classified the topology of different RNA molecules and modeled certain effects and bonds in various knotted protein molecules [5].

Not only can knot theory be used to analyze knotted biological structures but it can also be used to design bullet proof vests. In 2017, a team of scientists led by Professor David Leigh at the University of Manchester created the world's tightest knot and were awarded a Guinness World Record for this breakthrough. They created a 20 nanometer long molecular knot that has 8 crossings and is composed of 192 atoms [6]. A computer generated molecular model of the 192-atom knot is shown in Figure 6. This knot was created using a "self-assembly" technique which involves knitting molecular strands around metal ions. These tight molecular knots can be used to create ultra-strong materials.

Bullet-proof vests are currently made up of a material called kevlar, a plastic made up of tightly packed rigid molecular rods. In the near future, kevlar may be replaced by tightly knotted polymer strands. Polymer strands tend to be much lighter and stronger when compared to materials such as kevlar. By tightly knot-

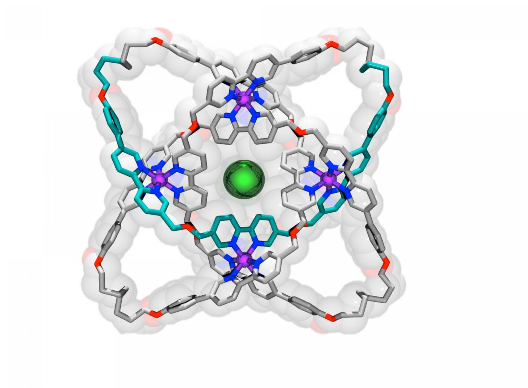


Figure 6: A model of the "worlds tightest" 192-atom knot [9].

ting these polymer strands, material scientists are able to create strong and durable construction materials [6]. The phenomena of polymer strands being stronger when knotted can be observed in spider webs. Spider web strands are very fragile and can be easily broken apart. However, once woven together, they are very strong and durable.

Although knot theory started off as the core ingredient in a now discredited theory of atomic matter based on aether, things have almost come back a full circle. Mathematicians and physicists have found interesting connections between knot theory and quantum field theory. When a classical particle moves from one point to another, it follows a smooth path that obeys Newton's laws of motion. In contrast, a quantum particle follows an irregular path when travelling from one point to another. Due to the Heisenberg uncertainty principle, the exact location of a quantum particle at some point in time and its trajectory of motion are both unknown. This means that there are multiple possible paths that a quantum particle can take. When analyzing the trajectory of a quantum particle, we usually have to compute averages over all possible paths. For each possible path, there is a "probability amplitude" for the quantum particle to reach its destination. This amplitude is given by the Wilson operator. If we regard the space-time trajectory of a quantum particle as a knot, the value of its Jones polynomial (a polynomial invariant) is equal to the average value of the Wilson operator [11]. This means that the Jones polynomial of the trajectory of a quantum particle can be used to estimate its expected path. Edward Witten's seminal work in this area led to him being awarded the Fields medal in 1990. He is the first physicist to win this prestigious award.

Researchers have investigated how the relationship between knot theory and quantum field theory can be leveraged to solve problems related to quantum computers. When building quantum computers, engineers face the problem of decoherence, which is when a quantum system loses its quantum properties due to interactions with the external environment. In principle, a topological quantum computer that effectively drags particles around each other in a knotted space-time orbit may be less vulnerable to decoherence and therefore fault-tolerant [12]. The quest to build a robust topological quantum computer that could potentially revolutionize fields from drug discovery to cryptography continues [13].

Acknowledgement

I would like to thank Malors Emilio Espinosa Lara from the Department of Mathematics at the University of Toronto for introducing me to knot theory and providing valuable feedback on a draft version of this article.

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Rishi Nair is a grade 11 student at the University of Toronto Schools. He is interested in various aspects of mathematics with a particular emphasis on the applications of mathematics to physics and computer science. His first exposure to topology was at the 2021 Math Academy summer program on knot theory organized by the Department of Mathematics at the University of Toronto. This experience inspired him to write an expository article on knot theory to introduce others to this fascinating topic. Rishi Nair is also an avid programmer, guitar player and music enthusiast.

From the lecture notes of . . .

Henry Finlayson, as told by Bill Sands

Bill Sands was the Editor of *Crux* from 1986 to 1995. He retired in 2013 after teaching at the University of Calgary for over 30 years, and now lives in Qualicum Beach, BC, where he keeps his hand in mathematics by serving as editor-at-large of *Crux*, contributing to a couple of math contests, and trying to do some research.

Professor Henry Finlayson, known to everyone as "Hank", died on February 25, 2022 at the age of 92. Hank was a Professor of Mathematics at the University of Manitoba for many years, and was well-liked by students (and everyone). He was my professor in one undergraduate course in the late 1960's. I may also been a teaching assistant for him in an introductory calculus course back then, because I remember him talking about a particular problem in the course text (the name of which has fled my brain in the meantime!)

Here is the **problem**:

A hemisphere 1 metre high sits flat on the ground. The sun is setting. You are standing at a short distance from the hemisphere, holding a small ball at a height of 1 metre above the ground. The shadow of the ball cast by the setting sun just touches the top of the hemisphere. At time $t = 0$ you drop the ball, and its shadow moves down the surface of the hemisphere, both ball and shadow reaching the ground at the same time.

(a) Find the velocity $v(t)$ of the shadow on the hemisphere at time t.

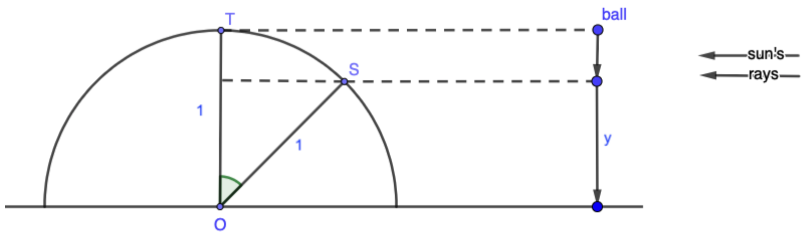
(b) What is $\lim_{t\to 0} v(t)$?

A solution is given below. We are most interested in the answer to part (b), which turns out to be $\sqrt{9.8}$ m/s (taking 9.8 m/s^2 as the acceleration due to gravity). Hank found this answer counterintuitive: as he put it, the ball is motionless, then you let it go, and *instantaneously the shadow of the ball is moving at over 3 metres per second!*

Some may object that in its initial position, the ball is not casting a shadow on the hemisphere at all. Fair enough, but there is a shadow at all times after the ball is released. Besides, suppose you hold the ball at a height of just less than 1 metre above the ground before letting it fall; then there certainly is a shadow before the ball is released, so shouldn't "continuity" say that this shadow should behave very much like the shadow behaves when the ball is exactly 1 metre from the ground? I leave such matters for you readers to think about for yourselves!

In any case, this is an example I still recall, after what must be over 50 years. Here's to you, Hank!

Solution.



As in the diagram, let T be the top and O the centre of the hemisphere, and S the shadow of the ball on the surface of the hemisphere at time t. Letting y be the distance (in metres) of the ball from the ground at time t, we have that $y = 1 - 4.9t^2$ (valid for $0 \le t \le 1/\sqrt{4.9}$) and $dy/dt = -9.8t$, and also $y = \cos\theta$, where $\theta = \angle TOS$ at time t. Thus

$$-9.8t = \frac{dy}{dt} = \frac{d}{dt}(\cos\theta) = -\sin\theta\frac{d\theta}{dt}\ .$$

The distance TS along the hemisphere at time t is also θ, and so

$$v(t) = \frac{d\theta}{dt} = \frac{9.8t}{\sin\theta} = \frac{9.8t}{\sqrt{1-y^2}} = \frac{9.8t}{\sqrt{9.8t^2-(4.9t^2)^2}} = \frac{9.8}{\sqrt{9.8-(4.9t)^2}}\ .$$

Thus

$$\lim_{t\to 0}(v(t)) = \sqrt{9.8}$$

as claimed.

OLYMPIAD CORNER

No. 402

The problems in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **June 15, 2022**.

OC576. Let x, y, z be real numbers such that the numbers

$$\frac{1}{|x^2+2yz|}, \qquad \frac{1}{|y^2+2zx|}, \qquad \frac{1}{|z^2+2xy|}$$

are side-lengths of a (non-degenerate) triangle. Find all possible values of the expression $xy + yz + zx$.

OC577. Let $n \geq 2$ be an integer and let $A \in \mathcal{M}_n(\mathbb{C})$ such that A and A^2 have different ranks. Prove that there exists a nonzero matrix $B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA = B^2 = O_n$.

OC578. Let $\mathcal{F}$ be the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition $e^{f(x)} + f(x) \geq x + 1$ for all $x \in \mathbb{R}$. Find the minimum value of

$$I(f) = \int_0^e f(x)\ dx$$

as f runs through $\mathcal{F}$.

OC579. Consider an n-element subset S of the plane consisting of points with both integer coordinates, where n is an odd number. The injective function $f : S \to S$ satisfies the following condition: for each pair of points $A, B \in S$, the distance between points $f(A)$ and $f(B)$ is not greater than the distance between the points A and B. Prove that there exists a point $X \in S$ that $f(X) = X$.

OC580. Consider an acute triangle ABC where $AB < AC$. Points E and F are the feet of its altitudes drawn from vertices B and C, respectively. The tangent line at point A to the circumcircle ABC intersects the line BC at point P. The line parallel to line BC passing through point A intersects the line EF at point Q. Prove that the line PQ is perpendicular to the median of triangle ABC drawn from vertex A.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2022**.

OC576. Soient x, y, z des nombres réels tels que les nombres

$$\frac{1}{|x^2+2yz|}, \qquad \frac{1}{|y^2+2zx|}, \qquad \frac{1}{|z^2+2xy|}$$

sont les longueurs des côtés d'un triangle non dégénéré. Déterminer toutes les valeurs possibles de l'expression $xy + yz + zx$.

OC577. Soit $n \geq 2$ un entier; soient aussi $A \in \mathcal{M}_n(\mathbb{C})$ telle que A et A^2 ont des rangs différents. Démontrer qu'il existe une matrice non nulle $B \in \mathcal{M}_n(\mathbb{C})$ telle que $AB = BA = B^2 = O_n$.

OC578. Soit $\mathcal{F}$ l'ensemble des fonctions continues $f : \mathbb{R} \to \mathbb{R}$ telles que $e^{f(x)} + f(x) \geq x + 1$ pour tout $x \in \mathbb{R}$. Déterminer la valeur minimale de

$$I(f) = \int_0^e f(x)\, dx,$$

où $f \in \mathcal{F}$.

OC579. Considérer un ensemble S formé de n points à coordonnées entières dans le plan, où n est un nombre impair. Une fonction injective $f : S \to S$ est telle que pour toute paire de points $A, B \in S$, la distance entre les points $f(A)$ et $f(B)$ ne dépasse pas la distance entre les points A et B. Démontrer qu'il existe un point $X \in S$ tel que $f(X) = X$.

OC580. Soit ABC un triangle acutangle tel que $AB < AC$. Les points E et F sont les pieds des altitudes tracées à partir des sommets B et C respectivement. La ligne tangente en A au cercle circonscrit de ABC intersecte la ligne BC au point P. De plus, la ligne passant par A et parallèle à la ligne BC intersecte la ligne EF au point Q. Démontrer que la ligne PQ est perpendiculaire à la médiane du triangle ABC, émanant de A.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(9), p. 425–426.

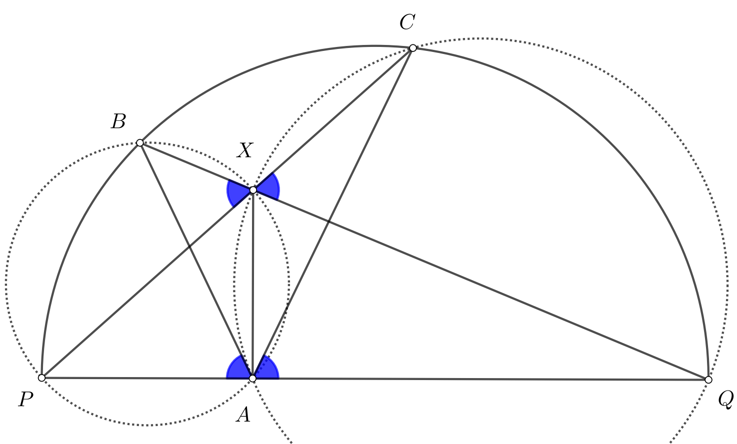
OC551. A semicircle k with diameter PQ is given. A chord BC of fixed length d is constructed on it, the endpoints of which are different from the points P and Q. From point B draw a ray so that this ray cuts the diameter PQ at a point A such that $\angle PAB = \angle QAC$. Prove that the magnitude of $\angle BAC$ does not depend on the position of the chord BC on the semicircle k.

Originally from 2018 Czech-Slovakia Math Olympiad, 2nd Problem, Category A, Regional Round.

We received 7 correct solutions. We present two solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

We may assume that P, B, C, Q appear on the semicircle in this order.



Let X be the point of intersection of PC and BQ. Let ω_1 be the circumcircle of triangle PBX and ω_2 the circumcircle of triangle QCX. Let A_1 be the other point of intersection of PQ with ω_1 and A_2 the other point of intersection of ω_2 with PQ.

Since $\angle XBP = \angle QBP = 90°$, then PX is a diameter of ω_1 and $\angle XA_1P = 90°$. Similarly $\angle XA_2Q = 90°$. Since $XA_1 \perp PQ$ and $XA_2 \perp PQ$, then $A_1 = A_2$.

We have $\angle BA_1P = \angle BXP = \angle CXQ = \angle CA_2Q$. So we must have $A_1 = A_2 = A$ as A is uniquely determined. (If A' is another such point and it is strictly inside the segment PA, then

$$\angle BA'P > \angle BAP = \angle CAQ > \angle CA'Q\,.$$

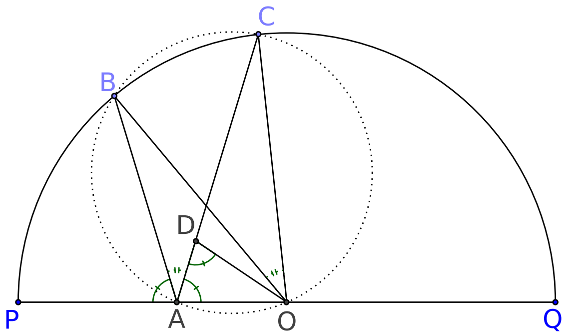
So $\angle BA'P \neq \angle CA'Q$, a contradiction. A similar contradiction is obtained in the case that A' is strictly inside the segment AQ.)

We now have

$$\angle BAC = \angle BAX + \angle XAC = \angle BPC + \angle BQC\,.$$

But $BC = 2R\sin(\angle BPC) = 2R\sin(\angle BQC)$ where R is the radius of the semicircle. So $\angle BPC = \angle BQC$ and therefore also $\angle BAC$ depends only on the length of BC and not on its position.

Solution 2, by Oliver Geupel.



Let O be the midpoint of PQ. It is enough to prove that

$$\angle CAB = \angle COB, \tag{1}$$

because $\angle COB$, the central angle subtending BC, does not depend on the position of the chord BC.

In the case $A = O$ the result (1) is immediate. So assume $A \neq O$. Without loss of generality suppose that A is an interior point of segment PO. Since one of the angles $\angle OAB$ and $\angle OAC$ is acute, we may assume that $\angle OAC$ is acute. Let point D be the reflection of A in the perpendicular to AC through O. Then,

$$\angle OAB = 180^\circ - \angle BAP = 180^\circ - \angle OAD = 180^\circ - \angle ADO = \angle ODC.$$

Moreover, $OB = OC > OA = OD$. That is, triangles AOB and DOC have two pairs of sides equal in length and the pair of angles opposite to the greater side equal in measurement. Hence, the triangles are congruent, so that

$$\angle ABO = \angle DCO = \angle ACO.$$

Thus, the points A, B, C, and O are concyclic. The result (1) follows.

OC552. Let a and b be two distinct positive real numbers. Consider the equation

$$\lfloor ax+b \rfloor = \lfloor bx+a \rfloor,$$

where $\lfloor y \rfloor$ denotes the integer part of the real number y. Prove that there exists an interval of length at least

$$\frac{1}{\max\{a,b\}}$$

all of whose points are solutions of the given equation.

Originally from 2018 Czech-Slovakia Math Olympiad, 3rd Problem, Category A, Regional Round.

We received 4 correct solutions. We present the solution by UCLan Cyprus Problem Solving Group.

Without loss of generality $a \geq b$. Assume also that $a+b=n+r$ where n is an integer and $0 \leq r < 1$.

We claim that every $x \in (\frac{a-r}{a}, \frac{1+a-r}{a})$ is a solution of the equation. Indeed for every such x we have

$$n = (a-r)+b < ax+b < (1+a-r)+b = n+1$$

and

$$n = -r+(n+r) \leq -\frac{rb}{a}+b+a < bx+a < \frac{b}{a}(1-r)+b+a \leq (1-r)+(n+r) = n+1\,.$$

Thus $\lfloor ax+b \rfloor = \lfloor bx+a \rfloor = n$ for every such x. This interval has length $1/a$ as required.

OC553. Determine all the pairs of integers (a,b) such that a^2+2b^2+2a+1 is a divisor of $2ab$.

Originally from 2018 Romania Math Olympiad, 2nd Problem, Grade 7, District Round.

We received 13 submissions, of which 12 were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.

If $a=0$ or $b=0$ then a^2+2b^2+2a+1 is a divisor of $2ab$ unless $a^2+2b^2+2a+1=0$. This happens if and only if $(a+1)^2+2b^2=0$ or equivalently if and only if $a=-1, b=0$.

From now on we assume that $ab \neq 0$. So if a^2+2b^2+2a+1 is a divisor of $2ab$ then $2|ab| \geq a^2+2b^2+2a+1$. By Cauchy-Schwarz we have

$$2|ab| \geq (a+1)^2+2b^2 \geq 2\sqrt{2}|(a+1)b|$$

giving $\left|1+\frac{1}{a}\right| \leq \frac{\sqrt{2}}{2}$. So a must be negative. Since a must be integer we must have $a=-1,-2,-3$ since otherwise $\left|1+\frac{1}{a}\right| \geq \frac{3}{4} > \frac{\sqrt{2}}{2}$.

Again by Cauchy-Schwarz we must have

$$a^2 + b^2 \geq 2|ab| \geq a^2 + 2b^2 + 2a + 1$$

giving $b^2 \leq -2a - 1$.

If $a = -1$, then $b^2 \leq 1$, so $b = \pm 1$ and both choices are accepted. If $a = -2$, then $b^2 \leq 3$, so $b = \pm 1$ but both choices are rejected (as $a^2 + 2b^2 + 2a + 1 = 3$ but $2ab = \pm 4$). If $a = -3$, then $b^2 \leq 5$, so $b = \pm 1, \pm 2$ and all choices are accepted.

So the valid pairs are $(-1, 1), (-1, -1), (-3, 1), (-3, -1), (-3, 2), (-3, -2)$ as well as $(0, b)$ for any $b \in \mathbb{Z}$ and $(a, 0)$ for any $a \in \mathbb{Z} \setminus \{-1\}$.

OC554. Let $ABCD$ be a rectangle and let $E \in CD$ and $F \in AD$. The perpendicular line through point E to line FB intersects line BC at point P and the perpendicular line through point F to line EB intersects line AB at point Q. Prove that points P, D, Q are collinear.

Originally from 2018 Romania Math Olympiad, 3rd Problem, Grade 7, District Round.

We received 10 correct solutions. We present 2 solutions.

Solution 1, by Miguel Amengual Covas.

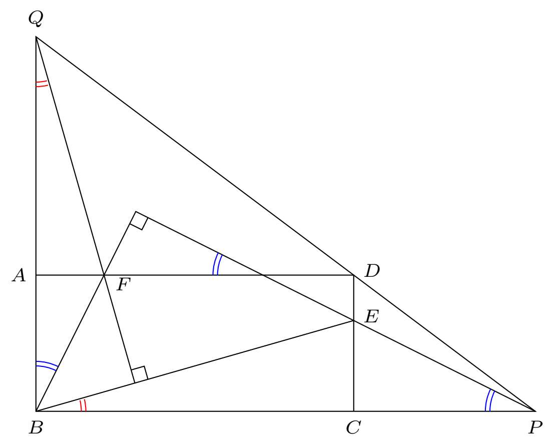
See the figure on the next page.

We have

$$\begin{aligned}
\tan(\angle CDP) = \frac{CP}{CD} &= \frac{CP}{AB} \\
&= \frac{CE}{AF} \quad \text{(from similar right-angled triangles } ABF \text{ and } CPE\text{)} \\
&= \frac{BC}{AQ} \quad \text{(from similar right-angled triangles } AQF \text{ and } CBE\text{)} \\
&= \frac{AD}{AQ} \\
&= \cot(\angle QDA),
\end{aligned}$$

making $\angle CDP$ and $\angle QDA$ complementary angles, that is,

$$\angle CDP + \angle QDA = 90^{\circ}.$$



Hence

$$\angle QDA + \angle ADC + \angle CDP = \angle QDA + 90^\circ + \angle CDP = 180^\circ,$$

which implies that points P, D, and Q are collinear, as claimed.

Solution 2, by Ivko Dimitrić.

Introduce rectangular coordinates with the origin at B, x-axis along $\overrightarrow{BA}$ and y-axis along $\overrightarrow{BC}$. Let $|BA| = a$ and $|BC| = c$. We can assume, without loss of generality, that $E \in \overline{CD}$ and $F \in \overline{AB}$ belong to the corresponding sides of the rectangle $ABCD$, so that $E = (u, c)$ and $F = (a, v)$ for some $0 \le u \le a$ and $0 \le v \le c$. It suffices to show that the triangles DCP and QAD are similar. Line $\overleftrightarrow{BF}$ has the slope v/a, so that the line $\overleftrightarrow{PE}$ through E perpendicular to it has an equation

$$y - c = -\frac{a}{v}(x - u)$$

and intersects $BC\,(x = 0)$ at $P\left(0, \frac{au+cv}{v}\right)$. Likewise, line $\overleftrightarrow{BE}$ has slope c/u so that the line $\overleftrightarrow{QF}$ through F perpendicular to it has an equation

$$y - v = -\frac{u}{c}(x - a)$$

and intersects $\overleftrightarrow{AB}\,(y = 0)$ at $Q\left(\frac{au+cv}{u}, 0\right)$. Then $PC = PB - BC = au/v$ and $AQ = BQ - AB = cv/u$. Consequently,

$$\frac{PC}{CD} = \frac{au/v}{a} = \frac{u}{v} \qquad \text{and} \qquad \frac{AD}{AQ} = \frac{c}{cv/u} = \frac{u}{v}.$$

This shows that the right triangles DCP and QAD are similar and hence $\angle CDP = \angle AQD$. Then

$$\angle QDP = \angle QDA + \angle ADC + \angle CDP = (90^\circ - \angle AQD) + 90^\circ + \angle CDP = 180^\circ,$$

that is, points P, D, Q are collinear.

The same analysis applies to the cases when one or both points E, F belong to the extensions of the corresponding sides, with necessary adjustments for evaluating PC and AQ.

OC555. Let $p > 3$ be a prime. Let K be the number of permutations $(a_1, a_2, \ldots, a_p)$ of $\{1, 2, \ldots, p\}$ such that

$$a_1a_2 + a_2a_3 + \cdots + a_{p-1}a_p + a_pa_1$$

is divisible by p. Prove $K + p$ is divisible by p^2.

Originally from 2018 Poland Math Olympiad, 6th Problem, Final Round.

We received 4 correct solutions. We present the solution by Oliver Geupel.

Let us start with some ad hoc definitions. We define the functions ('difference') $d : (\mathbb{F}_p)^p \to (\mathbb{F}_p)^p$ and ('value') $v : (\mathbb{F}_p)^p \to \mathbb{F}_p$ by

$$\begin{aligned} d\left(x_1, x_2, \ldots, x_p\right) &= \left(x_2 - x_1, x_3 - x_2, \ldots, x_p - x_{p-1}, x_1 - x_p\right), \\ v\left(x_1, x_2 \ldots, x_p\right) &= x_1x_2 + x_2x_3 + \cdots + x_{p-1}x_p + x_px_1. \end{aligned}$$

Let $h \in \mathbb{F}_p$. We define functions ('lift by increment h') $\ell_h : (\mathbb{F}_p)^p \to (\mathbb{F}_p)^p$ and ('shift by increment h') $s_h : (\mathbb{F}_p)^p \to (\mathbb{F}_p)^p$ as follows:

$$\begin{aligned} \ell_h\left(x_1, x_2, \ldots, x_p\right) &= \left(x_1 + h, x_2 + h, \ldots, x_p + h\right), \\ s_h\left(x_1, x_2, \ldots, x_p\right) &= \left(x_{1+h}, x_{2+h}, \ldots, x_{p+h}\right), \end{aligned}$$

where indices and arithmetic are modulo p.

For every $X \in (\mathbb{F}_p)^p$, we have that $\ell_1\left(s_1(X)\right) = s_1\left(\ell_1(X)\right)$. Therefore,

$$\begin{aligned} \ell_{h_1} \circ s_{k_1} \circ \ell_{h_2} \circ s_{k_2} \cdots \ell_{h_n} \circ s_{k_n}(X) &= \ell_{h_1+h_2+\cdots+h_n}\left(s_{k_1+k_2+\cdots+k_n}(X)\right) \\ &= s_{k_1+k_2+\cdots+k_n}\left(\ell_{h_1+h_2+\cdots+h_n}(X)\right). \end{aligned}$$

For $X, Y \in (\mathbb{F}_p)^p$, we write $X \sim Y$ if $Y = s_h(X)$ holds for some appropriate increment h. The relation $\sim$ is an equivalence relation on $(\mathbb{F}_p)^p$. Let π and ϱ be permutations of $\mathbb{F}_p$. We write $\pi \approx \varrho$ if $d(\pi) \sim d(\rho)$. The relation $\approx$ is an equivalence relation on the set $\mathcal{S}_p$ of permutations of $\mathbb{F}_p$. For all $h, k \in \mathbb{F}_p$, it holds $\ell_h(\pi) \approx \pi \approx s_k(\pi)$. Hence, $\pi \approx s_k\left(\ell_h(\pi)\right)$, i.e., the equivalence class $[\pi]_\approx$ is the set of all permutations that can be obtained from π by successive lifting and shifting. For $\pi = (a_1, \ldots, a_p)$ it holds

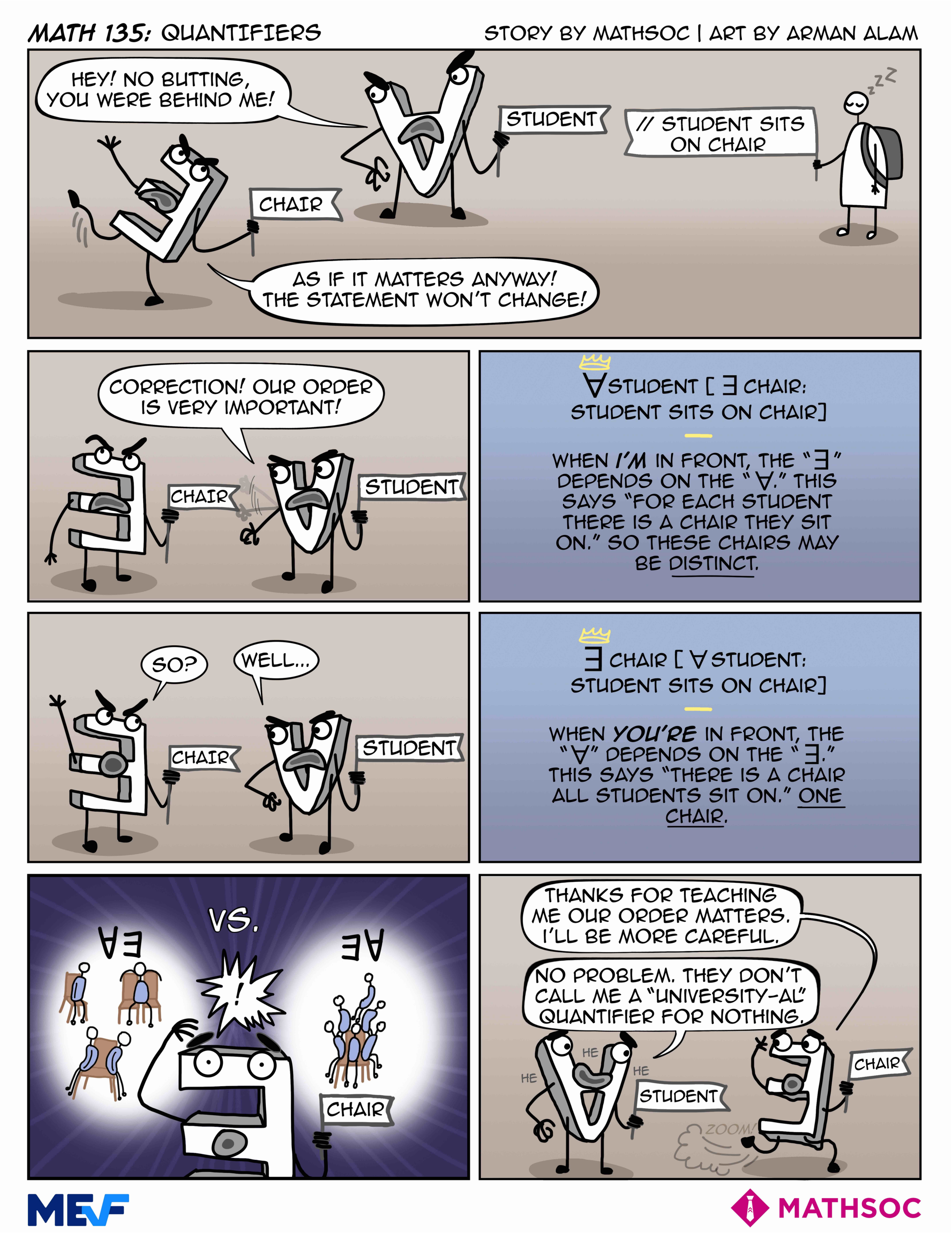
$$v\left(\ell_1(\pi)\right) = \sum_{k=1}^{p}\left(a_k + 1\right)\left(a_{k+1} + 1\right) = v(\pi) + \frac{1}{2}p(p+1) = v(\pi) + 0 = v(\pi).$$

Thus, $v(\pi)$ is constant for all π in a common equivalence class.

For $1 \leq m \leq p-1$ consider the permutations $\pi_m = (0, m, 2m, 3m, \ldots, (p-1)m)$. The equivalence class of π_m has p elements and value

$$v(\pi_m) = m \sum_{k=0}^{p-1} (k^2 + k) = m \left(\frac{1}{3}(p-1)p(2p-1) + \frac{1}{2}(p-1)p \right) = 0.$$

It remains to consider $\pi \notin \bigcup_{m=1}^{p-1} [\pi_m]_{\approx}$. Since p is prime, all the p differences $d\left(s_0(\pi)\right)$, $d\left(s_1(\pi)\right)$, ..., $d\left(s_{p-1}(\pi)\right)$ are distinct. Hence, $[\pi]_{\approx}$ has p^2 elements. If among these equivalence classes there are C classes with value 0, then the total number of permutations $\pi \in \mathcal{S}_p$ with value $v(\pi) = 0$ is just $(p-1)p + Cp^2$. The required result follows.


MATH 135: QUANTIFIERS
STORY BY MATHSOC | ART BY ARMAN ALAM
HEY! NO BUTTING, YOU WERE BEHIND ME!
STUDENT
// STUDENT SITS ON CHAIR
CHAIR
AS IF IT MATTERS ANYWAY! THE STATEMENT WON'T CHANGE!
CORRECTION! OUR ORDER IS VERY IMPORTANT!
CHAIR
STUDENT
∀STUDENT [∃ CHAIR: STUDENT SITS ON CHAIR]
WHEN I'M IN FRONT, THE "∃" DEPENDS ON THE "∀." THIS SAYS "FOR EACH STUDENT THERE IS A CHAIR THEY SIT ON." SO THESE CHAIRS MAY BE DISTINCT.
SO?
WELL...
CHAIR
STUDENT
∃ CHAIR [∀ STUDENT: STUDENT SITS ON CHAIR]
WHEN YOU'RE IN FRONT, THE "∀" DEPENDS ON THE "∃." THIS SAYS "THERE IS A CHAIR ALL STUDENTS SIT ON." ONE CHAIR.
∀∃
VS.
∃∀
!
CHAIR
THANKS FOR TEACHING ME OUR ORDER MATTERS. I'LL BE MORE CAREFUL.
NO PROBLEM. THEY DON'T CALL ME A "UNIVERSITY-AL" QUANTIFIER FOR NOTHING.
HE
HE
HE
STUDENT
CHAIR
ZOOM
MATHSOC

Using Conway's triangle notation to solve algebraic problems

Daniel Sitaru

In this paper is presented a method for proving algebraic inequalities, based on a coordinate system for triangles invented by John H. Conway.

1 Conway's Coordinates

Let $x, y, z \in \mathbb{R}$ be such that

$$x + y > 0; y + z > 0; z + x > 0; xy + yz + zx > 0; \tag{1}$$

and denote

$$a = \sqrt{x+y},\ b = \sqrt{y+z},\ c = \sqrt{z+x} \tag{2}$$

We will prove that a, b, c are edge lengths of a nondegenerate triangle ΔABC. We show that $a + b > c$; the other two cases of the triangle inequality follow by symmetry. Using (2), this becomes $\sqrt{x+y} + \sqrt{y+z} > \sqrt{z+x}$.

Squaring, we see (both sides being positive) that this is equivalent to

$$\begin{aligned} & x + y + y + z + 2\sqrt{(x+y)(y+z)} > z + x \\ \Leftrightarrow \quad & y + \sqrt{(x+y)(y+z)} > 0 \\ \Leftrightarrow \quad & y + \sqrt{xz + yz + xy + y^2} > y + |y| \geq 0 \end{aligned}$$

proving the claim. We now derive expressions for various values associated with this triangle.

The semiperimeter is clearly given by

$$s = \frac{1}{2}(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x})\ . \tag{3}$$

The area is given by

$$F = \frac{1}{2}\sqrt{xy + yz + zx}. \tag{4}$$

Exercise. Prove this by plugging (3) into Heron's formula

$$F^2 = s(s-a)(s-b)(s-c).$$

Let r, R be the inradius and circumradius of ΔABC. Using (3), we obtain:

$$r = \frac{F}{s} = \frac{\sqrt{xy + yz + zx}}{\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}} \tag{5}$$

$$R = \frac{abc}{4F} = \frac{\sqrt{(x+y)(y+z)(z+x)}}{2\sqrt{xy+yz+zx}} \tag{6}$$

Let m_a, m_b, m_c be the medians of ΔABC; we have

$$m_a^2 = \frac{4x+y+z}{4}; \quad m_b^2 = \frac{x+4y+z}{4}; \quad m_c^2 = \frac{x+y+4z}{4}. \tag{7}$$

Exercise. Prove this using (2) and the familiar formula

$$m_a^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2.$$

Exercise. Let h_a, h_b, h_c be the altitudes of ΔABC. Show that

$$h_a = \sqrt{\frac{xy+yz+xz}{y+z}} = \sqrt{x + \frac{yz}{y+z}} \tag{8}$$

and similarly for h_b and h_c. (What geometric formula would you start with?)

To the problem solver, these formulae will have a familiar feel: they are very much like the cyclic sums that turn up in algebraic inequalities! Of course, triangle geometry is also a rich source of inequalities. If we were given a triangle inequality hidden in this way, could we unmask it? Here's a problem from a national Olympiad that seems to be built from the right parts. In particular, the given value for $xy+yz+zx$ and the existence of the square roots justify the use of Conway's coordinates - this could be a strong hint to the alert solver! If we translate it back into geometry, what do we find?

Example 1 (Turkey NMO - 2006). If x, y, z are positive numbers with $xy + yz + zx = 1$ then:

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq (\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x})^2 \geq 6\sqrt{3}$$

Proof. Substituting (2), the first inequality to be proved becomes

$$\frac{27}{4}a^2b^2c^2 \geq (a+b+c)^2. \tag{9}$$

Using the familiar geometric formulae $R = abc/4F$ and $2s = a+b+c$, this becomes

$$27R^2F^2 \geq s^2. \tag{10}$$

But (4) gives $F^2 = (xy + yz + zx)/4$ which is by hypothesis $1/4$. Plugging this in and taking square roots, (10) is equivalent to

$$\frac{3\sqrt{3}R}{2} \geq s,$$

one of a familiar pair of inequalities due to Mitrinović.

The second inequality to be proved becomes $4s^2 \geq 6\sqrt{3}$ or $s^2 \geq \frac{3}{2}\sqrt{3}$. Introducing $F = rs = 1/2$ (as we were given), this is equivalent to $s^2 \geq 3\sqrt{3}rs$, or

$$s \geq 3\sqrt{3}r,$$

the other Mitrinović inequality. □

Application 2 (IMO - DataBase - 2008). If x, y, z are three reals such that the numbers $y + z, z + x, x + y$ and $yz + zx + xy$ are all nonnegative, then:

$$\sum_{cyc} \sqrt{(z+x)(x+y)} \geq x + y + z + \sqrt{3(xy + yz + zx)}$$

Proof.
We will use Conway's substitutions:

$$a = \sqrt{y+z}; b = \sqrt{z+x}; c = \sqrt{x+y}$$

$$F = \frac{1}{2}\sqrt{xy + yz + zx}$$

The nequality can be written as:

$$\sum_{cyc} \sqrt{(z+x)(x+y)} \geq \frac{1}{2}\sum_{cyc}(\sqrt{x+y})^2 + \sqrt{3} \cdot \sqrt{xy + yz + zx}$$

$$\sum_{cyc} ab \geq \frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}F$$

$$\frac{1}{2}(a^2 + b^2 + c^2) \geq a^2 + b^2 + c^2 - (ab + bc + ca) + 2\sqrt{3}F$$

$$a^2 + b^2 + c^2 \geq (a-b)^2 + (b-c)^2 + (c-a)^2 + 4\sqrt{3}F$$

which is the well known Hadwiger-Finsler's inequality in triangle. □

2 Problems

If $x, y, z \in \mathbb{R}$ are such that: $x + y > 0; y + z > 0; z + x > 0; xy + yz + zx > 0$ then:

1. $\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)} \geq 4(xy + yz + zx)$

2. $(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 6\sqrt{3(xy+yz+zx)}$
3. $2(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\sqrt{xy+yz+zx} \leq 3\sqrt{3(x+y)(y+z)(z+x)}$
4. $8(x+y+z)(xy+yz+zx) \leq 9(x+y)(y+z)(z+x)$
5. $x+y+z \geq \sqrt{3(xy+yz+zx)}$
6. $\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \geq 2\sqrt{\frac{3(xy+yz+zx)}{(x+y)(y+z)(z+x)}}$
7. $\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \leq \frac{\sqrt{3}(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})}{2\sqrt{xy+yz+zx}}$
8. $\frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}} \geq \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)}$
9. $\frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}} \leq \frac{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2}{4(xy+yz+zx)}$
10. $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \geq \frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{\sqrt{(x+y)(y+z)(z+x)}}$
11. $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \leq \frac{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2}{4(xy+yz+zx)}$
12. $\sqrt{\frac{x+y}{x+z}}+\sqrt{\frac{x+z}{x+y}} \leq \frac{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\sqrt{(x+y)(y+z)(z+x)}}{2(xy+yz+zx)}$
13. $(x+y+z)(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 18(xy+yz+zx)$
14. $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)} \geq \frac{36(xy+yz+zx)}{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2}$
15. $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)} \leq \frac{9(x+y)(y+z)(z+x)}{4(xy+yz+zx)}$
16. $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)} \geq 2\sqrt{3(xy+yz+zx)}$
17. $x^2+y^2+z^2 \geq xy+yz+zx$
18. $(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y) \geq 4(xy+yz+zx)$
19. $\frac{9\sqrt{(x+y)(y+z)(z+x)}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}} \geq 2\sqrt{3(xy+yz+zx)}$
20. $\sqrt{3}(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}) \leq 4\sqrt{\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}}+\frac{2\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$
21. $\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}} \geq \frac{9\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$
22. $\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}} \leq \frac{9\sqrt{(x+y)(y+z)(z+x)}}{4\sqrt{xy+yz+zx}}$
23. $\frac{3\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}} \leq \sqrt[4]{\frac{3}{4}(xy+yz+zx)}$

3 Hints

With Conway's substitutions, the inequalities 1-23 are well known:

1. $R \geq 2r$ (Euler)
2. $s \geq 3\sqrt{3}r$ (Mitrinović I)
3. $s \leq \frac{3\sqrt{3}}{2}R$ (Mitrinović II)
4. $a^2 + b^2 + c^2 \leq 9R^2$ (Leibniz)
5. $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ (Ionescu-Weitzenbock)
6. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{\sqrt{3}}{R}$ (Leuenberger I)
7. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}$ (Leuenberger II)
8. $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}$ (Leuenberger III)
9. $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{1}{4r^2}$ (Leuenberger IV)
10. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{2Rr}$ (Steinig I)
11. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}$ (Steinig II)
12. $\frac{b}{c} + \frac{c}{b} \leq \frac{R}{r}$ (BĂndilĂ)
13. $a^2 + b^2 + c^2 \geq 36r^2$ (Neuberg)
14. $ab + bc + ca \geq 36r^2$ (Leuenberger V)
15. $ab + bc + ca \leq 9R^2$ (Leuenberger VI)
16. $ab + bc + ca \geq 4\sqrt{3}F$ (Gordon)
17. $a^4 + b^4 + c^4 \geq 16F^2$ (Goldner I)
18. $a^2b^2 + b^2c^2 + c^2a^2 \geq 16F^2$ (Goldner II)
19. $\frac{9abc}{a+b+c} \geq 4F\sqrt{3}$ (Curry)
20. $s\sqrt{3} \leq 4R + r$ (Doucet)
21. $m_a + m_b + m_c \geq 9r$ (Gotman I)
22. $m_a + m_b + m_c \leq \frac{9R}{2}$ (Gotman II)
23. $\frac{3}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} \leq \sqrt[4]{3F^2}$ (Makowski)

. . . .

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2022**.

4731. *Proposed by Michel Bataille.*

Let n be a nonnegative integer. Evaluate

$$\sum_{k=0}^{\infty} \binom{2n+k}{n} \frac{1}{2^k}.$$

4732. *Proposed by Arben Ajredini.*

Let φ denote Euler's totient function. Find all polynomials $p \in \mathbb{Z}[x]$ such that

$$\varphi(x) \mid p(x), \ \forall x \in \mathbb{Z}_{\geq 1}.$$

4733. *Proposed by Dong Luu.*

Let ABC be a triangle with circumcircle (O) and $AB < AC$. Denote by M and N the midpoints of the segments BC, OA, respectively. For any point D on the side BC, let E, F be the respective projections of the vertices B, C on the line AD. The line from E, parallel to AB, meets the line from F, parallel to AC, at G. Denote by I the center of the circle (GEF). Prove that $NI = NM$.

4734. *Proposed by Paolo Perfetti.*

Let $a_k > 0$, $k = 1, 2, \dots$ be a monotonic strictly increasing sequence such that $\lim\limits_{k\to\infty} a_k = +\infty$. Determine the convergence or not of the series

$$\sum_{k=1}^{\infty} \left(\tan \frac{\pi}{2} \frac{a_k}{a_{k+1}} \right)^{-1}.$$

4735. *Proposed by Florentin Visescu, modified by the Editorial Board.*

Given triangle ABC with circumradius R, let P be either point on the line AB for which there exist points S and T on the line AC for which PST is an equilateral triangle whose sides have length BC. Determine the length of AP as a function of R.

4736. *Proposed by Marius Stănean.*

Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} + 6xyz \geq 9.$$

4737. *Proposed by George Stoica.*

Let α, β be real numbers. Find all convergent sequences $(a_n)_{n\geq 1}$ satisfying

$$\alpha(a_1 + \cdots + a_n) + \beta(a_1 \cdot \ldots \cdot a_n) = 1$$

for all $n \geq 1$.

4738. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with $\measuredangle A = 20^\circ$ and $\measuredangle B = 80^\circ$. The sides AC and BC are extended so that $C \in AE$ and $C \in BD$. On the extensions of the sides AC and BC, the points D and E are taken so that $\measuredangle ADC = 50^\circ$ and $\measuredangle DEC = 70^\circ$. Show that the bisector of $\angle DCE$ is parallel to BE.

4739. *Proposed by Michel Bataille.*

Let α, β, γ be the angles of a triangle and let

$$m = \frac{\cos\alpha}{\sin\beta\sin\gamma}, \quad n = \frac{\cos\beta}{\sin\gamma\sin\alpha}, \quad p = \frac{\cos\gamma}{\sin\alpha\sin\beta}.$$

Prove that

$$\begin{vmatrix} m\cos^2\alpha & m\cos^2\beta - 1 & m\cos^2\gamma - 1 \\ n\cos^2\alpha - 1 & n\cos^2\beta & n\cos^2\gamma - 1 \\ p\cos^2\alpha - 1 & p\cos^2\beta - 1 & p\cos^2\gamma \end{vmatrix} = -2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

4740. *Proposed by Daniel Sitaru.*

If $0 < a \leq b < 1$ then:

$$\exp\left(\int_a^b \int_a^b \frac{x+y+2}{1-xy} dxdy\right) \leq \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2022**.

4731. *Soumis par Michel Bataille.*

Soit n un entier non négatif. Calculez l'expression suivante

$$\sum_{k=0}^{\infty} \binom{2n+k}{n} \frac{1}{2^k}.$$

4732. *Soumis par Arben Ajredini.*

Soit φ l'indicatrice d'Euler. Trouvez tous les polynômes $p \in \mathbb{Z}[x]$ pour lesquels

$$\varphi(x) \mid p(x),\ \forall x \in \mathbb{Z}_{\geq 1}.$$

4733. *Soumis par Dong Luu.*

Soit ABC un triangle de cercle circonscrit (O) et qui vérifie $AB < AC$. Notons par M et N respectivement les points milieux des segments de droite BC et OA. Pour tout point D du côté BC, on note par E et F respectivement les projections des sommets B et C sur la droite AD. La droite parallèle à AB et passant par E rencontre la droite parallèle à AC et passant par F en un point G. Notons par I le centre du cercle (GEF). Montrez que $NI = NM$.

4734. *Soumis par Paolo Perfetti.*

Soit $a_k > 0$, $k = 1, 2, \ldots$ une suite strictement monotone croissante telle que $\lim\limits_{k\to\infty} a_k = +\infty$. Déterminez si la série $\sum\limits_{k=1}^{\infty} \left(\tan \frac{\pi}{2} \frac{a_k}{a_{k+1}}\right)^{-1}$ converge.

4735. *Soumis par Florentin Visescu, modifié par le comité de rédaction.*

Soit ABC un triangle. On note le rayon de son cercle circonscrit par R. Soit P l'un ou l'autre des points de la droite AB pour lesquels il existe des points S et T sur la droite AC tels que PST soit un triangle équilatéral dont les côtés mesurent BC. Déterminez la longueur de AP comme fonction de R.

4736. *Soumis par Marius Stănean.*

Soit x, y et z des nombres réels positifs vérifiant $x^2 + y^2 + z^2 + xyz = 4$. Montrez que

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} + 6xyz \geq 9.$$

4737. *Soumis par George Stoica.*

Soit α et β des nombres réels. Trouvez toutes les suites convergentes $(a_n)_{n\geq 1}$ vérifiant

$$\alpha(a_1 + \ldots + a_n) + \beta(a_1 \cdot \ldots \cdot a_n) = 1$$

pour tout $n \geq 1$.

4738. *Soumis par Mihaela Berindeanu.*

Soit ABC un triangle avec $\measuredangle A = 20^\circ$ et $\measuredangle B = 80^\circ$. On prolonge les côtés AC et BC de sorte à avoir $C \in AE$ et $C \in BD$. Sur le prolongement des côtés AC et BC, les points D et E sont choisis de sorte que $\measuredangle ADC = 50^\circ$ et $\measuredangle DEC = 70^\circ$. Montrez que la bissectrice de $\angle DCE$ est parallèle à BE.

4739. *Soumis par Michel Bataille.*

Soit α, β et γ les angles d'un triangle et soit

$$m = \frac{\cos\alpha}{\sin\beta\sin\gamma}, \quad n = \frac{\cos\beta}{\sin\gamma\sin\alpha}, \quad p = \frac{\cos\gamma}{\sin\alpha\sin\beta}.$$

Montrez que

$$\begin{vmatrix} m\cos^2\alpha & m\cos^2\beta - 1 & m\cos^2\gamma - 1 \\ n\cos^2\alpha - 1 & n\cos^2\beta & n\cos^2\gamma - 1 \\ p\cos^2\alpha - 1 & p\cos^2\beta - 1 & p\cos^2\gamma \end{vmatrix} = -2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

4740. *Soumis par Daniel Sitaru.*

Si $0 < a \leq b < 1$ alors :

$$\exp\left(\int_a^b \int_a^b \frac{x+y+2}{1-xy}\,dxdy\right) \leq \left(\frac{1-a}{1-b}\right)^{2(b-a)}$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(9), p. 449–452.

4681. *Proposed by Michel Bataille.*

Let n be a positive integer and

$$P_n(x) = \prod_{k=1}^{n} \left(x + 4\sin^2 \frac{2k\pi}{2n+1} \right).$$

Evaluate $P_n(0)$ and $P_n'(0)$.

We received 12 solutions, all of which were correct, and we present the solution by Mehra Vivek.

It is easily verified that

$$P_n(0) = 4^n \cdot \prod_{k=1}^{n} \left[\sin\left(\frac{2k\pi}{2n+1} \right) \right]^2$$

and

$$P_n'(0) = \frac{P_n(0)}{4} \cdot \sum_{k=1}^{n} \left[\csc\left(\frac{2k\pi}{2n+1} \right) \right]^2.$$

Using $|\sin(2\pi - v)| = |\sin v|$, we have

$$P_n(0) = 4^n \cdot \left| \prod_{k=1}^{2n} \sin\left(\frac{2k\pi}{2n+1} \right) \right|,$$

and from $\left| e^{2iu} - 1 \right| = |2\sin u|$ for real u, it follows that

$$P_n(0) = \left| \prod_{k=1}^{2n} \left(e^{4i\left(\frac{k\pi}{2n+1}\right)} - 1 \right) \right|.$$

Noting that the set

$$\left\{ e^{\frac{2ik\pi}{2n+1}} : k = 1, 2, ..., 2n \right\}$$

is a permutation of the set

$$\left\{ e^{\frac{4ik\pi}{2n+1}} : k = 1, 2, ..., 2n \right\},$$

we obtain

$$P_n(0) = \left| \prod_{k=1}^{2n} \left(e^{\left(\frac{2ik\pi}{2n+1}\right)} - 1 \right) \right|.$$

Now, for $k = 1, 2, ..., 2n$, $e^{\frac{2ik\pi}{2n+1}}$ are the roots of

$$f(x) = \frac{x^{2n+1} - 1}{x - 1} = 1 + x + \cdots + x^{2n},$$

so

$$\left| \prod_{k=1}^{2n} \left(e^{\left(\frac{2ik\pi}{2n+1}\right)} - 1 \right) \right| = |f(1)| = 2n + 1,$$

implying that

$$P_n(0) = 2n + 1.$$

Now we calculate

$$\sum_{k=1}^{n} \left[\csc\left(\frac{2k\pi}{2n+1} \right) \right]^2 = \sum_{k=1}^{n} \left[\cot\left(\frac{2k\pi}{2n+1} \right) \right]^2 + n.$$

By DeMoivre's Theorem, $\cot\left(\frac{2k\pi}{2n+1}\right)$ for $k = 1, 2, ..., 2n$ are the roots of

$$(x - i)^{2n+1} - (x + i)^{2n+1}.$$

Expanding this last expression and setting $y = x^2$, we find that $\cot^2\left(\frac{2k\pi}{2n+1}\right)$ for $k = 1, 2, ..., n$ are the roots of the polynomial

$$2(2n + 1)y^n - \frac{2n(2n - 1)}{3} y^{n-1} + \cdots .$$

By Vieta's formulae we therefore have

$$\sum_{k=1}^{n} \left[\cot\left(\frac{2k\pi}{2n+1} \right) \right]^2 = \frac{n(2n - 1)}{3},$$

so that

$$P_n'(0) = \frac{2n + 1}{4} \cdot \left[\frac{n(2n - 1)}{3} + n \right] = \frac{n(n + 1)(2n + 1)}{6}.$$

4682. *Proposed by Goran Conar.*

Determine the infimum and supremum (if they exist) of the set

$$\left\{ \sqrt[mn]{m + n} : m, n \in \mathbb{N} \right\}.$$

We received 18 submissions and they were all correct. We present two solutions.

We claim that $\inf A = 1$ and $\sup A = 2$.

Solution 1, by the majority of solvers, slightly modified by the editor.

It is clear that $\sqrt[mn]{m + n} \geq 1$ holds for $m, n \in \mathbb{N}$ and thus $\inf A \geq 1$. Note that for each $n \in \mathbb{N}$, $x_n = \sqrt[n]{n + 1} \in A$ by setting $m = 1$. Using squeeze theorem, it is easy to show that $\lim_{n\to\infty} x_n = 1$. Thus, $\inf A = 1$.

Setting $m = n = 1$, we get $\sqrt[mn]{m+n} = \sqrt[1\cdot 1]{1+1} = 2 \in A$ so is $2 \le \sup A$. To show that $\sup A = 2$, it suffices to show that $\sqrt[mn]{m+n} \le 2$ holds for all $m, n \in \mathbb{N}$. Equivalently, it suffices to show that

$$m + n \le 2^{mn} \tag{1}$$

holds for all $m, n \in \mathbb{N}$. This follows from the binomial theorem:

$$2^{mn} = (1+1)^{mn} = 1 + mn \ge m + n$$

since $1 + mn - m - n = (m-1)(n-1) \ge 0$.

Solution 2, by Ferdi Ferdi and Mehra Vivek (independently).

Using the AM-GM inequality, we get

$$\sqrt[mn]{m+n} \le \frac{(m+n) + \underbrace{1 + 1 + \cdots + 1}_{mn-1}}{mn} = \frac{m+n+mn-1}{mn};$$

using the GM-HM inequality, we get

$$\sqrt[mn]{m+n} \ge \frac{mn}{\frac{1}{m+n} + \underbrace{\frac{1}{1} + \frac{1}{1} + \cdots + \frac{1}{1}}_{mn-1}} = \frac{mn}{\frac{1}{m+n} + mn - 1}.$$

Therefore,

$$\inf A \ge \inf_{m,n\in\mathbb{N}} \frac{mn}{\frac{1}{m+n} + mn - 1} = 1, \quad \sup A \le \sup_{m,n\in\mathbb{N}} \frac{m+n+mn-1}{mn} = 2.$$

Similar to Solution 1, we conclude that $\inf A = 1$ and $\sup A = 2$.

Editor's Comment. There are other ways to prove inequality (1). Some solvers differentiated the function $f(x,y) = \sqrt[xy]{x+y}$ or the function $g(x,y) = \log f(x,y) = \dfrac{\log(x+y)}{xy}$, and some solvers used induction.

4683. *Proposed by Warut Suksompong.*

For a non-negative integer n, let $S(n)$ be the sum of digits in the decimal representation of n. Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that for any real number r, there exists an integer k such that $S(|P(k)|) > r$.

We received 5 submissions and they were all correct. We present three solutions. Solution 1 and Solution 2 show the existence of such a k, and Solution 3 gives an explicit construction of such a k.

Solution 1, by Amit Kumar Basistha, slightly modified by the editor.

Without loss of generality, we can assume that the leading coefficient of the polynomial $P(x)$ is positive; otherwise we can consider the polynomial $-P(x)$. Now since the leading term of $P(x)$ is positive, there exists $x_0 > 0$ such that $P(x)$ is increasing when $x > x_0$ and $P(x) > 0$ when $x > x_0$. It suffices to show that the set

$$A := \{S(P(n)) : n \in \mathbb{Z}, n > x_0\}$$

is unbounded.

Suppose that the set A is bounded. Then there is $r, t \in \mathbb{Z}$ such that $S(P(t)) = r$ with $t > x_0$, and $S(P(n)) \leq r$ for all $n > x_0$. Let k be the number of digits in the decimal representation of $P(t)$; then $P(t) < 10^k$. Note that we have

$$P(x + 10^k) - P(x) = 10^k Q(x),$$

where $Q(x) \in \mathbb{Z}[x]$; moreover, $Q(x) > 0$ when $x > x_0$ since $P(x)$ is increasing when $x > x_0$. In particular, we have $P(t + 10^k) = 10^k Q(t) + P(t)$ and thus $S(P(t+10^k)) = S(P(t)) + S(Q(t)) > S(P(t))$, a contradiction. Thus, we conclude that A is unbounded, establishing the claim.

Solution 2, by UCLan Cyprus Problem Solving Group.

Given positive integers n, m let $A(n, m)$ be the set of all natural numbers k with at most m digits and $S(k) \leqslant n$. Then $|A(n, m)|$ is the number of solutions of $x_1 + \cdots + x_m \leqslant n$ in non-negative integers. Since each x_i can take at most 9 non-zero values and at most n of the x_i's are non-zero, then $|A(n, m)| \leqslant \binom{m}{n} 9^n$.

Suppose $P(x)$ has degree d and that the sum of the absolute values of the coefficients of $P(x)$ is M. Then $|P(k)| \leqslant MN^d$ for each positive integer $k \leqslant N$. Amongst $|P(1)|, \ldots, |P(N)|$ at most $2d$ are equal, so the set $\{|P(1)|, \ldots, |P(N)|\}$ contains at least $N/2d$ distinct values, all of them having at most $m = 1 + \lfloor \log_{10}(MN^d) \rfloor$ digits.

From the above

$$|A(r, m)| \leqslant \binom{m}{r} 9^r \leqslant (9m)^r \leqslant (C_1 + C_2 \log N)^r$$

for some constants C_1, C_2 depending on d, M (i.e. on the polynomial P) but not on N.

Since powers of logarithms grow slower than linear, then $|A(r, m)| < N/2d$ when N is large enough. In particular, there is a $k \in \{1, 2, \ldots, N\}$ such that $|P(k)| \notin A(r, m)$. But since $|P(k)|$ has at most m digits, that $S(|P(k)|) > r$ as required.

Solution 3, by Oliver Geupel, slightly modified by the editor.

Let

$$P(x) = \sum_{i=0}^{m} a_i x^i$$

where $m \geq 1$. We suppose with no loss of generality that $a_m > 0$. Next, denote $a = \max\{|a_i| : 0 \leq i \leq m\}$. Let N and Q be positive integers such that

$$r + 1 < N \qquad \text{and} \qquad a(m+1)^2(N+m)^{m+1} < 10^Q.$$

In the following we prove that

$$k = \sum_{\ell=0}^{N} B^{\ell} \qquad \text{with} \qquad B = 10^Q$$

has the required property.

For nonnegative integers i and j, let $b(i,j)$ denote the total number of i-tuples $(k_1, k_2, \ldots, k_i) \in \{0, 1, 2, \ldots, N\}^i$ such that $\sum_{\ell=1}^{i} k_\ell = j$. A trivial general upper bound for $b(i,j)$ is $(N+1)^i$. For $1 \leq j \leq N$, note that

$$b(m, (m-1)N + j) = \binom{N - j + m - 1}{m - 1}$$

since it is well-known that the number of non-negative integral solutions to equation $\sum_{\ell=1}^{m} y_\ell = N - j$ is $\binom{N-j+m-1}{m-1}$, where $y_\ell = N - k_\ell$. In particular, we can see that $b(m, mN) = 1$; moreover, for $1 \leq j \leq N$, we have

$$b(m, (m-1)N + j) \leq (N+m)^{m-1} < B.$$

For $0 \leq i \leq m$, we have the identity

$$k^i = \left(\sum_{\ell=0}^{N} B^{\ell}\right)^i = \sum_{j=0}^{iN} b(i,j) B^j.$$

Hence,

$$\begin{aligned} P(k) &= \sum_{i=0}^{m} a_i k^i = \sum_{i=0}^{m} a_i \sum_{j=0}^{iN} b(i,j)\, B^j = \sum_{j=0}^{mN} \sum_{i=\lceil \frac{j}{N} \rceil}^{m} a_i\, b(i,j)\, B^j \\ &= \sum_{j=0}^{(m-1)N} \sum_{i=\lceil \frac{j}{N} \rceil}^{m} a_i\, b(i,j) B^j + a_m \sum_{j=1}^{N-1} b(m, (m-1)N + j) B^{(m-1)N+j} + a_m\, B^{mN}. \end{aligned}$$

By the triangle inequality and by hypothesis,

$$\begin{aligned} \left| \sum_{j=0}^{(m-1)N} \sum_{i=\lceil \frac{j}{N} \rceil}^{m} a_i\, b(i,j) B^j \right| &< \sum_{j=0}^{mN} \sum_{i=0}^{m} a(N+1)^m B^{(m-1)N} \\ &< a(m+1)^2 (N+1)^{m+1} B^{(m-1)N} \\ &< B^{(m-1)N+1}. \end{aligned}$$

Putting

$$R = \sum_{j=0}^{(m-1)N} \sum_{i=\lceil \frac{j}{N} \rceil}^{m} a_i\, b(i,j)B^j + a_m b(m,(m-1)N+1)B^{(m-1)N+1},$$

we obtain

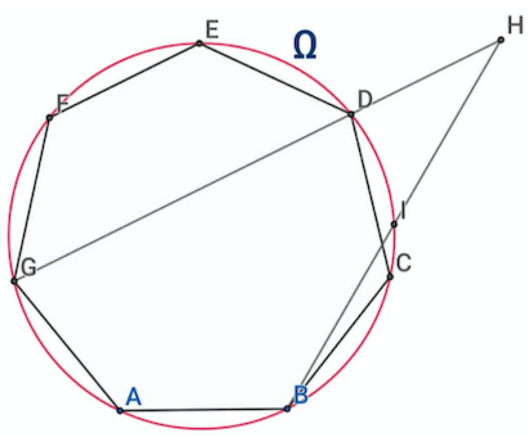
$$P(k) = a_m\, B^{mN} + a_m \sum_{\ell=1}^{N-2} b(m, mN-\ell)B^{mN-\ell} + R, \qquad 0 < R < B^{mN-(N-2)}.$$

Consequently, the decimal representation of $P(k)$ includes $N-1$ nonoverlapping nonzero segments, one segment consisting of the decimal representation of a_m and $N-2$ segments containing the decimal representation of a nonzero number. Since by hypothesis $N-1 > r$, the proof is complete.

Editor's Comment. The basic idea for the proposed problem is to construct special integers k such that the decimal representation of $P(k)$ admits large blocks of consecutive zeros so that one can estimate $S(P(k))$ by estimating the contribution from each nonzero block. Solution 1 gives a recursive algorithm on how to construct such a k, which provides some motivation for the explicit construction given in Solution 3.

4684. *Proposed by Alin Creţu.*

Let $ABCDEFG$ be a regular heptagon with the vertices on the circle Ω. Suppose that $BH \cap \Omega = \{I\}$ and that G, D, H are collinear. If $DH = DC$, show that $HI = IB$.



We received 17 submissions, of which 15 were complete and correct; the remaining 2 were missing documents or document contents. We present the solution by Didier Pinchon.

For a given Cartesian coordinate system in the plane, a point X with coordinates x and y will be denoted by $X = (x, y)$. A Cartesian coordinate system is chosen such that Ω is the unit circle. In this way,

$$E = (0, 1), \quad D = (\cos(3\pi/14), \sin(3\pi/14)), \quad B = (\cos(5\pi/14), -\sin(5\pi/14)).$$

Lines FE and GD are parallel and, also line FD and the line of equation $y = 1$. So, if H is the intersection point of GD and the line of equation $y = 1$, $HEFD$ is a parallelogram and $DH = FE = DC$, which is the hypothesis. Because $ED = DH$, EDH is an isosceles triangle, and the abscissa of H is twice the abscissa of D, i.e. $H = (2\cos(3\pi/14), 1)$.

The power of H with respect to the circle Ω is equal to HE^2 and also to $HB.HI$. So I is the middle of segment HB if $HB^2 = 2HE^2$. As

$$\begin{aligned} HE^2 &= 4\cos^2\frac{3\pi}{14}, \\ HB^2 &= \left(2\cos\frac{3\pi}{14} - \cos\frac{5\pi}{14}\right)^2 + \left(1 + \sin\frac{5\pi}{14}\right)^2 = 4\left(1 + \cos\frac{3\pi}{7}\right), \end{aligned}$$

the last equality being obtained by usual trigonometric manipulations, and the result is proved.

Solver's remark. It is equivalent to compute the coordinates of the middle I of segment HB,

$$I = \left(\frac{1}{2}\cos\frac{5\pi}{14} + \cos\frac{3\pi}{14}, \frac{1}{2} - \frac{1}{2}\sin\frac{5\pi}{14}\right),$$

and to verify that I is on Ω, i.e. satisfies $x^2 + y^2 = 1$.

4685. *Proposed by Abdollah Zohrabi.*

There are 24 students in a school with two classrooms, each with capacity to sit 12 people. Every student goes to school every day, and goes to exactly one of the classrooms. Prove that the students can attend school for 14 days in such a way that each pair of students are present in the same classroom at least once.

We received 9 solutions, all of which were correct. We present the solution by UCLan Cyprus Problem Solving Group.

In fact, just 3 days are enough. Partition the students into 4 groups A, B, C, D of 6 students each. On day 1, students in groups A, B go to the first class and the rest to the second. On day 2, students in groups A, C go to the first class and the rest to the second. On day 3, students in groups A, D go to the first class and the rest to the second.

Of course 2 days are not possible since every student meets only 11 students per day and therefore cannot meet the other 23 students in 2 days.

4686. *Proposed by Nguyen Viet Hung.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{a}{2}+\sqrt[3]{\frac{b^3+c^3}{2}}.$$

We received 10 submissions of which 8 were correct and complete. We present the solution by Oliver Geupel.

By the AM-GM inequality,

$$\begin{aligned}\frac{b^2+c^2}{b+c} &= \frac{1}{2(b+c)}\cdot\frac{2(b+c)^2+\left(4b^2-4bc+4c^2\right)}{3}\\ &\geq \frac{1}{2(b+c)}\cdot\sqrt[3]{(b+c)^4\left(4b^2-4bc+4c^2\right)}=\sqrt[3]{\frac{(b+c)(b^2-bc+c^2)}{2}}\\ &= \sqrt[3]{\frac{b^3+c^3}{2}}.\end{aligned}$$

Thus, it suffices to prove

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{a}{2}+\frac{b^2+c^2}{b+c}.$$

Multiplying all terms by $2(a+b)(b+c)(c+a)$, and subtracting the terms on the right hand side from those on the left, we have only to apply the AM-GM inequality four times, to obtain

$$\begin{aligned}&2\sum_{\text{cyc}} a^2(c+a)(a+b)-a(a+b)(b+c)(c+a)-2(b^2+c^2)(c+a)(a+b)\\ &= 2\left(a^4+b^4+c^4\right)+a(a^2-bc)(b+c)-3a^2(b^2+c^2)\\ &= \left(a^2-b^2\right)^2+\left(a^2-c^2\right)^2+\left(\frac{b^4+2a^3b}{3}-a^2b^2\right)+\left(\frac{c^4+2a^3c}{3}-a^2c^2\right)\\ &\quad+\left(\frac{5b^4+4a^3b+3c^4}{12}-ab^2c\right)+\left(\frac{5c^4+4a^3c+3b^4}{12}-abc^2\right)\\ &\geq 0.\end{aligned}$$

Editor's comment. Theo Koupelis noted that

$$\left(\frac{b^2+c^2}{b+c}\right)^3 \geq \frac{b^3+c^3}{2}$$

since, for $x = b/c$, it is equivalent to

$$\left(\frac{x^2+1}{x+1}\right)^3 \geq \frac{x^3+1}{2} \iff (x-1)^4(x^2+x+1) \geq 0.$$

4687. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate
$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}-\ln 2+\frac{1}{4n}\right).$$

We received 13 submissions of which 12 are correct, and one is incomplete. Many of the solutions are fairly complicated and quote various different known results. We present the solution by Theo Koupelis.

Let $S_k=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}-\ln 2+\frac{1}{4k}$ for positive integers k. Then $S_1=\frac{1}{2}-\ln 2+\frac{1}{4}=\frac{3}{4}-\ln 2$, and

$$S_k-S_{k+1}=\frac{1}{1+k}+\frac{1}{4k}-\frac{1}{2k+1}-\frac{1}{2(k+1)}-\frac{1}{4(k+1)}=\frac{1}{4k(k+1)(2k+1)}.$$

Writing similar expressions for $k=1,2,\cdots,n$ and adding, we then get for $n\geq 1$ that

$$\begin{aligned}S_{n+1}&=S_1-\sum_{k=1}^{n}\frac{1}{4k(k+1)(2k+1)}\\&=S_1-\sum_{k=1}^{n}\left[\frac{1}{4}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\left(\frac{1}{2k+2}-\frac{1}{2k+1}\right)\right]\\&=S_1-\frac{1}{4}\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n+1}-\frac{1}{2n+2}\right).\qquad(1)\end{aligned}$$

Since it is well known that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\ln 2$, we have

$$\lim_{n\to\infty}S_{n+1}=S_1-\frac{1}{4}+\lim_{n\to\infty}\frac{1}{4(n+1)}+\left(\ln 2-\frac{1}{2}\right)=\lim_{n\to\infty}\frac{1}{4(n+1)}=0.$$

Thus, S_n is a sequence with positive terms such that $\lim_{n\to\infty}S_n=0$. Then from (1) and summing $n=1$ to $n=N$, we get

$$\begin{aligned}\sum_{n=1}^{N}S_{n+1}&=N\cdot S_1-\frac{1}{4}\sum_{n=1}^{N}\sum_{k=1}^{n}\frac{1}{k(k+1)(2k+1)}\\&=N\cdot S_1-\frac{1}{4}\left[\frac{N}{1\cdot 2\cdot 3}+\frac{N-1}{2\cdot 3\cdot 5}+\frac{N-2}{3\cdot 4\cdot 7}+\cdots+\frac{1}{N(N+1)(2N+1)}\right]\\&=N\cdot\left[S_1-\frac{1}{4}\cdot\sum_{k=1}^{N}\frac{1}{k(k+1)(2k+1)}\right]+\frac{1}{4}\sum_{k=2}^{N}\frac{k-1}{k(k+1)(2k+1)}\\&=N\cdot S_{N+1}+\sum_{k=2}^{N}\frac{k-1}{4k(k+1)(2k+1)},\end{aligned}$$

and therefore

$$\sum_{n=1}^{N+1} S_n = S_1 + N \cdot S_{N+1} + \frac{1}{4}\sum_{k=2}^{N}\left[\left(\frac{1}{k+1} - \frac{1}{k}\right) + 6\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)\right]$$
$$= S_1 + N \cdot S_{N+1} + \frac{1}{4}\left(\frac{1}{N+1} - \frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2N+1} - \frac{1}{2N+2}\right).$$

Letting $N \to \infty$, we finally get

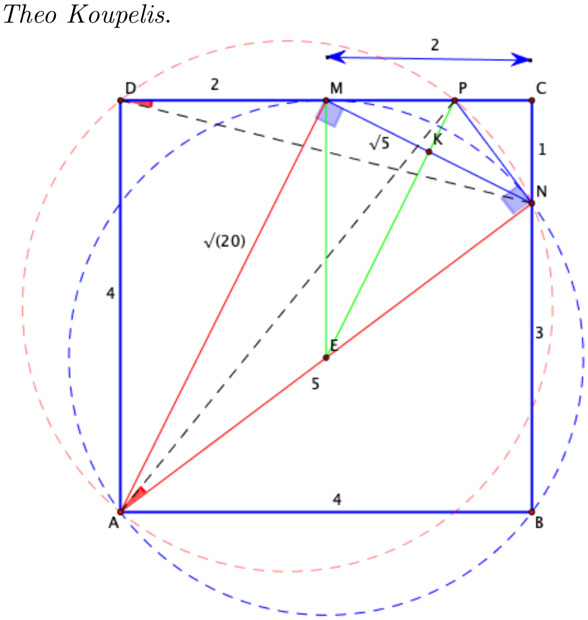
$$\sum_{n=1}^{\infty} S_n = \left(\frac{3}{4} - \ln 2\right) + \lim_{N\to\infty}\left(\frac{N}{4(N+1)} + \frac{1}{4(N+1)}\right) - \frac{1}{8} + \frac{3}{2}\left(\ln 2 - \frac{1}{2} - \frac{1}{12}\right)$$
$$= \frac{2\ln 2 - 1}{4}.$$

4688. *Proposed by Mihaela Berindeanu.*

Let $ABCD$ be a square, with $M, P \in (CD)$ and $N \in (BC)$ so that $DM = MC$, $NB = 3NC$ and $PN \perp AN$. If K is the middle point of segment MN, show that $\measuredangle(PAM) \equiv \measuredangle(KAN)$.

We received 22 submissions, all of which were correct, and we sample three of the half dozen interesting methods used.

Solution 1, by Theo Koupelis.



If we take the side length of the square to be 4, then

$AN^2 = AB^2 + BN^2 = 4^2 + 3^2 = 5^2 = 20 + 5 = (4^2 + 2^2) + (2^2 + 1^1) = AM^2 + MN^2;$

consequently, ΔAMN is a right triangle with $\angle AMN = 90^\circ$. If E is the midpoint of the hypotenuse AN, then the circle Γ with center E and radius EN is the circumcircle of triangle AMN; moreover, $EK \perp MN$ (because K is the midpoint of the chord MN). But points M, E are the midpoints of DC, AN, respectively, and so $EM \perp DC$, whence DC is a tangent to Γ at M. Since we are given that $PN \perp EN$, the point P is the intersection of the tangents to Γ at points M and N, and thus AP is the A-symmedian of triangle AMN. But AK is a median of the triangle, and therefore $\angle PAM = \angle KAN$.

Solution 2, by Titu Zvonaru.

Because $\angle PAM = \angle KAM - \angle KAP$, while $\angle NAK = \angle NAP - \angle KAP$, it suffices to prove that $\angle KAM = \angle NAP$. Because of the right angles at D and at N, both D and N are on the circle with diameter AP. Consequently,

$$\angle NAP = \angle NDP = \angle NDC.$$

As in solution 1, $\angle AMN = 90^\circ$, so we can compare the legs of the right triangles AMK and DCN. We have

$$\frac{MK}{MA} = \frac{\frac{\sqrt{5}}{2}}{\sqrt{20}} = \frac{1}{4} = \frac{CN}{CD};$$

hence, $\angle KAM = \angle NDC = \angle NAP$, as desired.

Solution 3, by Xicheng Peng.

We represent the vertices of the square $ABCD$ by the complex numbers $0, 4, 4 + 4i, 4i$. Thus, for some real number x,

$$N = 4 + 3i, \quad M = 2 + 4i, \quad P = x + 4i, \quad \text{and} \quad K = \frac{M+N}{2} = \frac{6+7i}{2}.$$

From $PN \perp AN$ we have $A - N = ki(P - N)$ for some real number k; that is,

$$4 + 3i = ki(4 - x - i), \qquad \text{or} \qquad (4 - k) + (3 - 4k + kx)i = 0.$$

Setting the real and imaginary parts to zero, we find that $k = 4, x = \frac{13}{4}$, and finally, $P = \frac{13}{4} + 4i$. It remains to show that the rotation about the origin A that takes the ray AP to AM also takes AN to AK; that is, we must show that $\dfrac{M-A}{P-A}$ is a real multiple of $\dfrac{K-A}{N-A}$:

$$\frac{M/P}{K/N} = \frac{2+4i}{\frac{13}{4}+4i} \cdot \frac{(4+3i)2}{6+7i} = \frac{16 \cdot (1+2i)(4+3i)}{(13+16i)(6+7i)} = \frac{16(-2+11i)}{17(-2+11i)} = \frac{16}{17},$$

which is a real number as desired.

4689. *Proposed by Daniel Sitaru.*

Solve for positive real numbers x, y and z such that $x+y+z=3$:

$$x^{x^2}\cdot y^{y^2}\cdot z^{z^2}=\frac{1}{(x^2)^{xz}\cdot(y^2)^{yx}\cdot(z^2)^{zy}}.$$

Due to a small typo, the statement "... positive real numbers" was typed as "... positive natural numbers" and as a result, the problem becomes trivial, but all solvers noticed this error. There were 15 submissions, 7 of which gave the trivial solution, and 7 others corrected the typo and then gave a valid solution. There was one incorrect solution. We present the solution by Oliver Geupel.

We show that the only solution is the obvious one: $(x,y,z)=(1,1,1)$. Let $f(t)=\log t+\frac{1}{t}-1$ for $t>0$. Then $f'(t)=\frac{t-1}{t^2}$ is negative for $0<t<1$ and positive for $t>1$, so f has a unique global maximum point $(0,1)$. Suppose $x,y,z>0$ with $x+y+z=3$ and $(x,y,z)\neq(1,1,1)$. Then

$$\begin{aligned}0&=(x+y+z)^2-3(x+y+z)\\&=\left(x^2+2xz\right)\left(1-\frac{1}{x}\right)+\left(y^2+2yx\right)\left(1-\frac{1}{y}\right)+\left(z^2+2zy\right)\left(1-\frac{1}{z}\right)\\&<\left(x^2+2xz\right)\log x+\left(y^2+2yx\right)\log y+\left(z^2+2zy\right)\log z\\&=\log\left(x^{x^2+2xz}\cdot y^{y^2+2yx}\cdot z^{z^2+2zy}\right).\end{aligned}$$

Hence,

$$x^{x^2+2xz}\cdot y^{y^2+2yx}\cdot z^{z^2+2zy}>1,\qquad\text{or}\qquad x^{x^2}\cdot y^{y^2}\cdot z^{z^2}>\frac{1}{(x^2)^{xz}\cdot(y^2)^{yx}\cdot(z^2)^{zy}}.$$

So (x,y,z) can't be a solution, completing the proof.

4690. *Proposed by Leonard Giugiuc.*

Let x,y and z be nonnegative real numbers such that $xy+yz+zx>3$ and $xy+yz+zx+xyz<4$. Prove that

$$x+y+z>xy+yz+zx.$$

We received 11 submissions and all are correct. We feature the solution by C.R. Pranesachar.

From the 1st and 2nd given inequalities, we infer respectively that $xy+yz+zx=3+a$ and $3+a+xyz=4-b$, for some $a>0$ and $b>0$. Thus, $xyz=1-a-b$. Since $xyz\geq 0$, we have $0<a<1$.

We may further assume that at least one of x,y, and z is greater than 1, since if $x\leq 1,\ y\leq 1$, and $z\leq 1$, then $xy+yz+zx\leq 3$, a contradiction.

Let $x > 1$. We set $y + z = A$ and $yz = B$. Then from the relations stated above we have $xA + B = 3 + a$ and $xB = 1 - a - b$. Solving for A and B, we obtain $B = \frac{1-a-b}{x}$ and $A = \frac{(a+3)x+a+b-1}{x^2}$. Now

$$\begin{aligned} x + y + z - (xy + yz + zx) &= x + A - (a + 3) \\ &= x + \frac{(a+3)x + a + b - 1}{x^2} - (a + 3) \\ &= \frac{x^3 - (3+a)x^2 + (3+a)x + a + b - 1}{x^2}. \end{aligned}$$

Let M denote the numerator of the expression above. By rewriting the coefficients of the terms x^k for $k = 0, 1, 2, 3$ and noting that $a + b - 1 = -(1 - a) + b$, $a + 3 = 4a + 3(1 - a)$, $-(a + 3) = -4a - 3(1 - a)$ and $1 = a + (1 - a)$ we deduce that $M = ax(x - 2)^2 + (1 - a)(x - 1)^3 + b$. Since $x > 1$, $1 > a$ and $b > 0$, we get $M > 0$, from which $x + y + z > xy + yz + zx$ follows and our proof is complete.