The 2021 Canadian Junior Mathematical Olympiad

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

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Official Problem Set

1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two concentric circles with $\mathcal{C}_1$ inside $\mathcal{C}_2$. Let P_1 and P_2 be two points on $\mathcal{C}_1$ that are not diametrically opposite. Extend the segment P_1P_2 past P_2 until it meets the circle $\mathcal{C}_2$ in Q_2. The tangent to $\mathcal{C}_2$ at Q_2 and the tangent to $\mathcal{C}_1$ at P_1 meet in a point X. Draw from X the second tangent to $\mathcal{C}_2$ which meets $\mathcal{C}_2$ at the point Q_1. Show that P_1X bisects angle $Q_1P_1Q_2$.

2. How many ways are there to permute the first n positive integers such that in the permutation, for each value of $k \le n$, the first k elements of the permutation have distinct remainder mod k?

3. Let $ABCD$ be a trapezoid with AB parallel to CD, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB, AC and BD, where A and I are on opposite sides of BD. Let J be the center of the circle tangent to lines CD, AC and BD, where D and J are on opposite sides of AC. Prove that $|IC| = |JB|$.

4. Let $n \ge 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

   Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}.$$

5. A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\Big(f(f(x)), f(x+y)\Big) = \gcd(x, y)$$

for all pairs of positive integers x and y.

Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

Important!

Please do not discuss this problem set online for at least 24 hours.

