Official Problem Set

1. Let $C_1$ and $C_2$ be two concentric circles with $C_1$ inside $C_2$. Let $P_1$ and $P_2$ be two points on $C_1$ that are not diametrically opposite. Extend the segment $P_1P_2$ past $P_2$ until it meets the circle $C_2$ in $Q_2$. The tangent to $C_2$ at $Q_2$ and the tangent to $C_1$ at $P_1$ meet in a point $X$. Draw from $X$ the second tangent to $C_2$ which meets $C_2$ at the point $Q_1$. Show that $P_1X$ bisects angle $Q_1P_1Q_2$.

2. How many ways are there to permute the first $n$ positive integers such that in the permutation, for each value of $k \leq n$, the first $k$ elements of the permutation have distinct remainder mod $k$?

3. Let $ABCD$ be a trapezoid with $AB$ parallel to $CD$, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let $I$ be the center of the circle tangent to lines $AB$, $AC$ and $BD$, where $A$ and $I$ are on opposite sides of $BD$. Let $J$ be the center of the circle tangent to lines $CD$, $AC$ and $BD$, where $D$ and $J$ are on opposite sides of $AC$. Prove that $|IC| = |JB|$.

4. Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1 + a_1} + \frac{a_2}{1 + a_1} + \frac{a_3}{1 + a_1 + a_2} + \cdots + \frac{a_n}{1 + a_1 + a_2 + \cdots + a_{n-1}}.$$
5. A function $f$ from the positive integers to the positive integers is called *Canadian* if it satisfies
\[
\gcd \left( f(f(x)), f(x+y) \right) = \gcd (x,y)
\]
for all pairs of positive integers $x$ and $y$.

Find all positive integers $m$ such that $f(m) = m$ for all Canadian functions $f$.

Important!

*Please do not discuss this problem set online for at least 24 hours.*