Canadian Junior Mathematical Olympiad 2021

Official Solutions

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Note: Each problem starts on a new page.

Problem No. 1.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two concentric circles with $\mathcal{C}_1$ inside $\mathcal{C}_2$. Let P_1 and P_2 be two points on $\mathcal{C}_1$ that are not diametrically opposite. Extend the segment P_1P_2 past P_2 until it meets the circle $\mathcal{C}_2$ in Q_2. The tangent to $\mathcal{C}_2$ at Q_2 and the tangent to $\mathcal{C}_1$ at P_1 meet in a point X. Draw from X the second tangent to $\mathcal{C}_2$ which meets $\mathcal{C}_2$ at the point Q_1. Show that P_1X bisects angle $Q_1P_1Q_2$.

Solution. We will show that the angles $\angle Q_2P_1X$ and $\angle Q_1P_1X$ are congruent. Note that, if O denotes the centre of both circles, the points P_1, X, Q_2 and Q_1 lie on the circle of diameter XO since XP_1 is tangent to the circle thus $\angle OP_1X = \pi/2$, and similar for the other tangents XP_2, XQ_1, XQ_2. On the other hand, $m(\angle Q_2P_1X)$ is half the measure of the arc XQ_2 and $m(\angle Q_1P_1X)$ is half the measure of the arc XQ_1, and these two arcs are equal because $|XQ_2| = |XQ_1|$.

□

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Problem No. 2.

How many ways are there to permute the first n positive integers such that in the permutation, for each value of $k \leq n$, the first k elements of the permutation have distinct remainder mod k?

Solution. We show by induction that the first k elements of the permutation must be k consecutive integers from $1, \ldots, n$. It is trivially true that for $k = n$ all remainders $\mod n$ are distinct and we induct downwards to show that, with the above condition, it is true for all $k < n$ that first k elements have distinct remainders $\mod k$. Note that in any k consecutive integers, the only two with the same remainder $\mod (k-1)$ are the smallest and largest integers, so one of these two must be the k-th integer of the permutation. This completes the induction, and thus at every step taking away the k-th entry of the permutation, there are 2 choices to eliminate an integer (the largest or the smallest) and obtain a new permutation where the first $k-1$ entries have distinct remainders $\mod (k-1)$, so the answer is 2^{n-1}. $\square$

Problem No. 3.

Let $ABCD$ be a trapezoid with AB parallel to CD, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB, AC and BD, where A and I are on opposite sides of BD. Let J be the center of the circle tangent to lines CD, AC and BD, where D and J are on opposite sides of AC. Prove that $|IC| = |JB|$.

Solution. Let $\{P\} = AC \cap BD$ and let $\angle APB = 180 - 2a$. Since $ABCD$ is an isosceles trapezoid, APB is an isosceles triangle. Therefore $\angle PBA = a$, which implies that $\angle PBI = 90^\circ - a/2$ since I lies on the external bisector of $\angle PBA$. Since I lies on the bisector of $\angle CPB$, it follows that $\angle BPI = a$ and hence that IPB is isosceles with $|IP| = |PB|$. Similarly JPC is isosceles with $|JP| = |PC|$. So, in the triangles CPI and BPJ we have $PI \equiv PB$ and $PJ \equiv CP$. Since I and J both lie on the internal bisector of $\angle BPC$, it follows that triangles CPI and BPJ are congruent. Therefore $|IC| = |JB|$. $\square$

Problem No. 4.

Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1 + a_1} + \frac{a_3}{1 + a_1 + a_2} + \cdots + \frac{a_n}{1 + a_1 + a_2 + \cdots + a_{n-1}}.$$

Solution. We claim the the minimum possible value of this expression is n. Observe that by AM-GM, we have that

$$\begin{aligned}
&\frac{a_1}{1} + \frac{a_2}{1 + a_1} + \cdots + \frac{a_n}{1 + a_1 + a_2 + \cdots + a_{n-1}} \\
&\quad = \frac{1 + a_1}{1} + \frac{1 + a_1 + a_2}{1 + a_1} + \cdots + \frac{1 + a_1 + a_2 + \cdots + a_n}{1 + a_1 + a_2 + \cdots + a_{n-1}} - n \\
&\quad \geq n \cdot \sqrt[n]{\frac{1 + a_1}{1} \cdot \frac{1 + a_1 + a_2}{1 + a_1} \cdots \frac{1 + a_1 + a_2 + \cdots + a_n}{1 + a_1 + a_2 + \cdots + a_{n-1}}} - n \\
&\quad = n \cdot \sqrt[n]{1 + a_1 + a_2 + \cdots + a_n} - n \\
&\quad = 2n - n = n.
\end{aligned}$$

Furthermore, equality is achieved when $a_k = 2^{k-1}$ for each $1 \leq k \leq n$. □

Problem No. 5.

A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\Big(f(f(x)), f(x+y)\Big) = \gcd(x, y)$$

for all pairs of positive integers x and y.

Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

Solution. Define an $m \in \mathbb{N}$ to be *good* if $f(m) = m$ for all such f. It will be shown that m is good if and only if m has two or more distinct prime divisors. Let $P(x, y)$ denote the assertion

$$\gcd(f(f(x)), f(x+y)) = \gcd(x, y)$$

for a pair $x, y \in \mathbb{N}$. Let x be a positive integer with two or more distinct prime divisors and let p^k be largest power of one of these prime divisors such that $p^k \mid x$. If $x = p^k \cdot q$, then p^k and q are relatively prime and $x > p^k, q > 1$. By $P(q, x-q)$,

$$\gcd(f(f(q)), f(x-q+q)) = \gcd(f(f(q)), f(x)) = \gcd(q, x-q) = q$$

which implies that $q \mid f(x)$. By $P(p^k, x-p^k)$,

$$\gcd(f(f(p^k)), f(x-p^k+p^k)) = \gcd(f(f(p^k)), f(x)) = \gcd(p^k, x-p^k) = p^k$$

which implies that $p^k \mid f(x)$. Since p^k and q are relatively prime, $x = p^k \cdot q$ divides $f(x)$, which implies that $f(x) \geq x$. Now assume for contradiction that $f(x) > x$. Let $y = f(x) - x > 0$ and note that, by $P(x, y)$, it follows that

$$f(f(x)) = \gcd(f(f(x)), f(x + f(x) - x)) = \gcd(x, f(x) - x) = \gcd(x, f(x)).$$

Therefore $f(f(x)) \mid x$ and $f(f(x)) \mid f(x)$. By $P(x, x)$, it follows that

$$\gcd(f(f(x)), f(2x)) = \gcd(x, x) = x.$$

This implies that $x \mid f(f(x))$, which when combined with the above result, yields that $f(f(x)) = x$. Since $x \mid f(x)$ and x is divisible by at least two distinct prime numbers, $f(x)$ is also divisible by at least two distinct prime numbers. As shown previously, this implies that $f(x) \mid f(f(x)) = x$, which is a contradiction since $f(x) > x$. Therefore $f(x) = x$ for all positive integers x with two or more distinct prime divisors.

Now it will be shown that all $m \in \mathbb{N}$ such that either m has one prime divisor or $m = 1$ are not good. In either case, let $m = p^k$ where $k \geq 0$ and p is a prime number and consider the function satisfying that $f(p^k) = p^{k+1}$, $f(p^{k+1}) = p^k$ and $f(x) = x$ for all $x \neq p^k, p^{k+1}$. Note that this function also satisfies that $f(f(x)) = x$ for all positive integers x. If $x + y \neq p^k, p^{k+1}$, then $P(x, y)$ holds by the Euclidean algorithm since $f(f((x)) = x$ and $f(x+y) = x+y$. If $x+y = p^{k+1}$, then $P(x,y)$ is equivalent to $\gcd(x, p^k) = \gcd(x, p^{k+1} - x) = \gcd(x, p^{k+1})$ for all $x < p^{k+1}$ which holds since the greatest power of p that can divide x is p^k. If $x + y = p^k$, then $P(x,y)$ is equivalent to $\gcd(x, p^{k+1}) = \gcd(x, p^k - x) = \gcd(x, p^k)$ for all $x < p^k$ which holds as shown above. Note that if $m = 1$ then this case cannot occur. Since this function satisfies $P(x,y)$, m is good if and only if m has two or more distinct prime divisors. $\square$

