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MATHEMATTIC

No. 19

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by January 15, 2021.

MA91. The points $(1, 2, 3)$ and $(3, 3, 2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

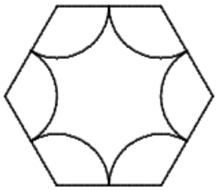
MA92. Eight students attend a soccer practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

MA93. Consider the system of equations

$$\log_4 x + \log_8(yz) = 2,$$
$$\log_4 y + \log_8(xz) = 4,$$
$$\log_4 z + \log_8(xy) = 5.$$

Given that xyz can be expressed in the form 2^k, compute k.

MA94. At each vertex of a regular hexagon, a sector of a circle of radius one-half of the side of the hexagon is removed. Find the fraction of the hexagon remaining.



MA95. Find the smallest and the largest prime factors of M, where

$$M = 1 + 2 + 3 + \cdots + 2017 + 2018 + 2019 + 2018 + 2017 + \cdots + 3 + 2 + 1.$$

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 janvier 2021.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA91. Les points $(1,2,3)$ et $(3,3,2)$ sont sommets d'un cube. Déterminer toutes les valeurs possibles pour le volume du cube.

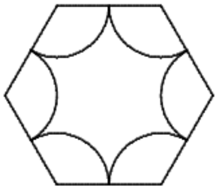
MA92. Huit étudiants ont assisté à un match de soccer et ont décidé de marquer l'occasion en prenant des autophotos. Chaque autophoto captera soit deux soit trois étudiants. Déterminer le nombre minimum d'autophotos faisant en sorte que chaque paire d'étudiants se retrouve dans exactement une autophoto.

MA93. Soit le système d'équations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2,\\ \log_4 y + \log_8(xz) &= 4,\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

Prenant pour acquis que xyz peut est écrit sous la forme 2^k, déterminer k.

MA94. On enlève de chaque coin d'un hexagone régulier une section en pointe de tarte de rayon égal à la moitié de la longueur de côté de l'hexagone. Déterminer la fraction de surface restante.



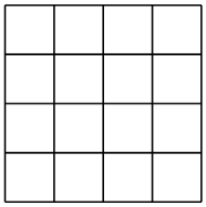
MA95. Déterminer le plus petit et le plus grand facteurs premiers de M, où

$$M = 1 + 2 + 3 + \cdots + 2017 + 2018 + 2019 + 2018 + 2017 + \cdots + 3 + 2 + 1.$$

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(4), p. 145–148.

MA66. The 16 small squares shown in the diagram each have a side length of 1 unit. How many pairs of vertices (intersections of lines) are there in the diagram whose distance apart is an integer number of units?



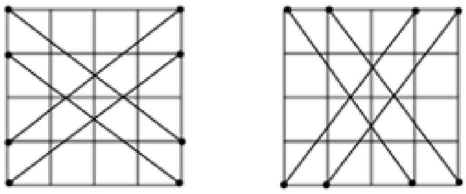
Originally Problem 19 of the 2013 UK Senior Mathematical Challenge.

We received 7 submissions of which 5 were correct and complete. We present the solution by Arya Kondur.

Each pair of vertices in the same row or column will be an integer number of units apart. There are five points in a given row or column. Thus, there are $\binom{5}{2} = 10$ distinct pairs of vertices in each row or column that have an integer distance between them. Since the figure in the problem statement has 5 rows and 5 columns, this gives us $(5+5)\cdot(10) = 100$ pairs of vertices.

The distance of vertices that are not in the same row or column can be found using the Pythagorean Theorem. Suppose the horizontal distance between the two points is x, the vertical distance is y, and the total distance is d. We want to find pairs of vertices such that $x^2 + y^2 = d^2$ and $x, y \leq 4$. We place this restriction on x and y since the side length of the large square is 4 units.

There is only one Pythagorean triplet in which the smaller two values are at most 4. This triplet is (3, 4, 5). Thus, we want to find pairs of vertices with a horizontal distance of 4 units and a vertical distance of 3 units (or vice versa). The first scenario is depicted in the left square of figure below and the second scenario is depicted in the right square of figure below.



Notice that from the figure, we have 8 more pairs of vertices that are an integer number of units apart. There are no more Pythagorean triples other than $(3, 4, 5)$ that fit our specifications. Thus, we can conclude that there are $100 + 8 = 108$ total pairs of vertices whose distance apart is an integer number of units.

MA67. Consider numbers of the form $10n + 1$, where n is a positive integer. We shall call such a number *grime* if it cannot be expressed as the product of two smaller numbers, possibly equal, both of which are of the form $10k + 1$, where k is a positive integer. How many grime numbers are there in the sequence $11, 21, 31, 41, \ldots, 981, 991$?

Originally Problem 22 of the 2013 UK Senior Mathematical Challenge.

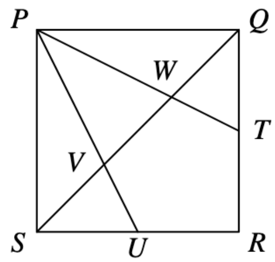
We received 9 submissions, all of which were correct and complete. We present the solution by Tianqi Jiang.

Call a number *special* if it is in our sequence $11, 21, \cdots, 991$. Note that $\sqrt{991} < 41$, so any non-grime in the sequence must have at least one special factor in the set $\{11, 21, 31\}$.

Note that $\left\lfloor \frac{991}{11} \right\rfloor = 90$, so there are $|\{11, 21, \cdots, 81\}| = 8$ non-grimes in the sequence whose smallest special factor is 11. Similarly, $\left\lfloor \frac{991}{21} \right\rfloor = 47$, so there are $|\{21, 31, 41\}| = 3$ non-grimes in the sequence whose smallest special factor is 21. Finally, since $31 \cdot 41 > 991$, $31^2 = 961$ is the only non-grime in the sequence whose smallest special factor is 31.

In total, we have $8+3+1 = 12$ non-grimes in the sequence, meaning that $99-12 = \boxed{87}$ must be grimes.

MA68. $PQRS$ is a square. The points T and U are the midpoints of QR and RS respectively. The line QS cuts PT and PU at W and V respectively. What fraction of the area of the square $PQRS$ is the area of the pentagon $RTWVU$?



Originally Problem 23 of the 2013 UK Senior Mathematical Challenge.

We received twelve submissions, all correct. We present the solution provided by Richard Hess.

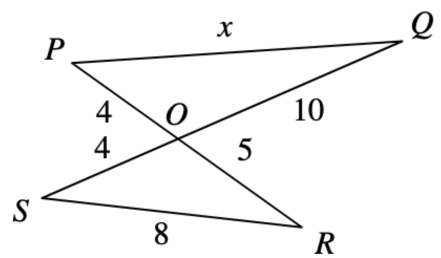
The pentagon $RTWVU$ covers 1/3 of the area of the square.

Scale the square so that $PQ = 1$. Let the points M and N be the midpoints of UT and VW respectively. Then $TU = NP = \frac{\sqrt{2}}{2}$ and $MN = MR = \frac{\sqrt{2}}{4}$, so that $MP = \frac{3\sqrt{2}}{4}$. Then $\frac{VW}{TU} = \frac{NP}{MP}$ leads to $VW = \frac{\sqrt{2}}{3}$.

From these lengths, we find the area of $RTWVU$ is

$$\frac{MN(VW+TU)}{2} + \frac{TU^2}{4} = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{2}\right)/2 + \frac{1}{8} = \frac{1}{3}.$$

MA69. The diagram shows two straight lines PR and QS crossing at O. What is the value of x?



Originally Problem 24 of the 2013 UK Senior Mathematical Challenge.

We received 15 submissions of which 14 were correct and complete.

We present the solution by Šefket Arslanagić. All solutions were essentially the same.

Let $\varphi = \angle SOR$. By the Law of Cosines applied to triangle SOR:

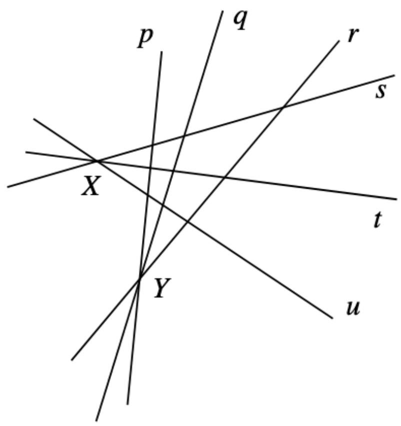
$$\cos\varphi = \frac{|OS|^2 + |OR|^2 - |RS|^2}{2|OS||OR|} = -\frac{23}{40}.$$

Since $\angle POQ = \angle SOR = \varphi$, we apply the Law of Cosines to triangle POQ and obtain

$$\begin{aligned} x^2 = |PQ|^2 &= |PO|^2 + |QO|^2 - 2|PO||OQ|\cos\varphi \\ &= 4^2 + 10^2 - 2\cdot 4\cdot 10\left(-\frac{23}{40}\right) \\ &= 162. \end{aligned}$$

It follows that $x = 9\sqrt{2}$.

MA70. Challengeborough's underground train network consists of six lines, p, q, r, s, t, u, as shown. Wherever two lines meet, there is a station which enables passengers to change lines. On each line, each train stops at every station. Jessica wants to travel from station X to station Y. She does not want to use any line more than once, nor return to station X after leaving it, nor leave station Y having reached it. How many different routes, satisfying these conditions, can she choose?



Originally 2013 UK Senior Mathematical Challenge, #25.

We received 3 submissions, all of which were correct and complete.

We present the solution by Dmitry Fleischman with a few edits.

Note that a route with an odd number of trains utilized will end on line s, t or u. Since these lines do not contain Y, Jessica must use an even number of lines.

A route with *two lines* can use any of the lines s, t, u first, followed by any of the lines p, q, r. This gives $3 \times 3 = 9$ routes.

A route with *four lines* can use any of the lines s, t, u first, followed by any of the lines p, q, r second, then there are two choices for the third line, and two for the fourth. This gives a total of $3 \times 3 \times 2 \times 2 = 36$ routes.

A route with *six lines* can use any of the lines s, t, u first, followed by any of the lines p, q, r second, then there are two choices for the third line, and two for the fourth. Now the fifth line belongs to the set of s, t, u, but since we've used two such lines already, there is only one choice. Similarly, there is only one choice for the sixth line. This gives a total of $3 \times 3 \times 2 \times 2 \times 1 \times 1 = 36$ routes.

Summing up gives a total of 81 routes.

TEACHING PROBLEMS

No.12

John McLoughlin

"Insight" Problems: Mental Mathematical Problem Solving

Discuss these four problems in small groups for at most 15 minutes in total. Ideally write as little as possible in solving these problems. They are referred to as "Insight" Problems in that an observation in each case is likely to make the (method of) solution to be both efficient and evident.

1. Abner wrote down three different prime numbers. The sum of these primes was 40. What was the product of these primes?

2. A school has 77 girls and 23 boys. If N girls leave the school, then the percentage of girls will become exactly 54%. Determine the value of N. (Assume that no boys left the school.)

3. In isosceles triangle ABC, side AB is twice as long as side AC. If the perimeter is 200 cm, how long is side BC?

4. Circle A has a radius of 3 cm and Circle B has a radius of 4 cm. If Circle C has the same total area as Circles A and B combined, determine the radius (in cm.) of Circle C.

Context

These problems are examples specifically intended to provoke one of two things: small group discussion or individual reflection with a focus on thinking without writing. You could think of these as examples of mental mathematical problems to solve, with the emphasis on the mental math as being part of the problem. That is, the tendency to begin writing and setting up equations or drawing a diagram are to be resisted in an effort to sharpen attention with the idea of identifying the insight.

Discussion of the Insights

Beginning with Question 4, it may be that the idea of $3-4-5$ jumps out when one realizes that there will be squaring and sums. So in fact the radius of Circle C is 5 cm. The problem is Pythagorean in nature without the triangles.

Let us revert to the opening problem. The sum of three prime numbers is an even number. How can that be when primes are odd numbers? Of course, there is the even prime number 2 that can be used. Then it turns out that only 7 and 31 will complete the sum. Typically the question may ask for the three numbers, but instead we are asked for their product giving an answer of 434.

Curiously there is a reason for this unusual "ask" in that this question was featured in a math league game in Newfoundland and Labrador. Since groups of students from different schools are simultaneously working in groups at tables in a common area like a library or cafeteria, it is sensible to have an answer that if overheard may not seem to be obviously recognized as such. Hence, 434 seemed preferable to 2, 7 and 31.

Percentages and a representation of the remaining students could be combined to make equivalent ratios as a method for solving Question 2. However, without writing out such an expression observe that the number of boys must make up 46% of the students remaining in the school. That makes each person (with there being 23 boys) worth 2% and hence the 54% corresponds to 27 girls. Therefore, $77 - 27$ or 50 girls left the school. Some may have thought of 54/100 as being equivalent to 27/50 and that would have led to the same result, namely $N = 50$.

Finally there is the isosceles triangle in Question 3. It is surprising how many decent math students get this question wrong as it seems that 50, 50, and 100 represents the instinctive way of having a repeated number and a double to sum to 100. However, these numbers with centimetres as lengths do not work to make a triangle. Rather it is 80, 80 and 40 that are required. Since BC is the side not involved in the statement that AB is twice as long as AC it follows that BC must have the length that is repeated, as in 80 cm. This problem has offered richness with respect to reinforcing the importance of context. Further, it seems that a $1 : 1 : 2$ ratio is more natural for us to see than the $2 : 2 : 1$ or $1 : 2 : 2$ ratio required to satisfy the triangular constraints.

Closing Comments

In summary, the idea of offering problems to be thought about rather than written about has potential to unearth rich discussion. Also it is likely that some of the less formal problem solvers may step up to gain attention. There is the implicit problem-solving element of not being able to solve a problem the usual way that adds value to the experience. Further value comes from recognizing that the mathematics does not need to be difficult to create a problem. In fact, it is important that the problem setter provide well thought out scenarios or numbers to ensure that the focal point remains getting the insight rather than messing around when that has been found. The combination of familiar ideas nestled in unfamiliar forms with easy to use values brings attention to the thinking and the identification of the critical points in the problems. The four questions that appeared in this piece have come from *Shaking Hands in Corner Brook*, a collection of problems that appeared over a period of several years in Newfoundland and Labrador senior high school math league games. The complete reference appears below.

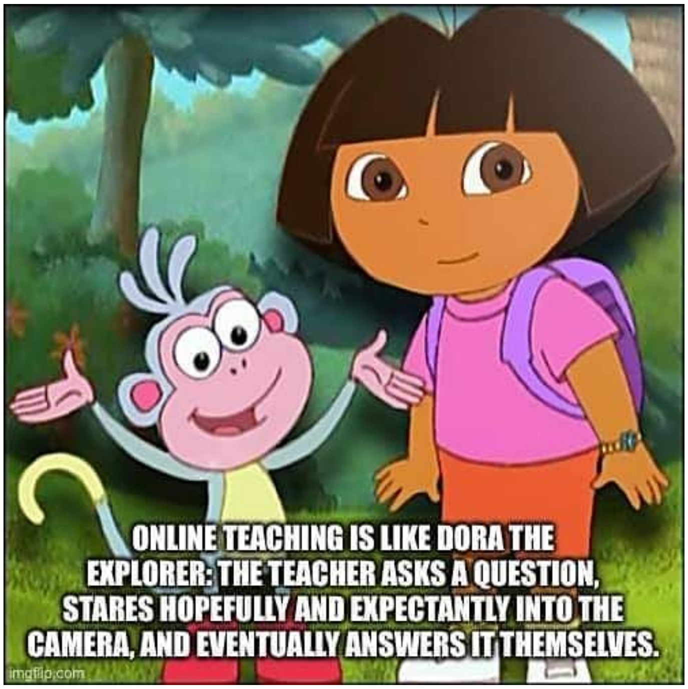
Finally, a problem introduced to me by Ed Barbeau has been adapted to close this feature. Know that you are welcome to share any of your insight problems by sending a note along to me via `johngm@unb.ca`

Determine the value of $x + y + z$ given that

$$\frac{x}{9-y} = \frac{z}{5-x} = \frac{y}{10-z} = 2.$$

Reference

Shawyer, B., Booth, P., & Grant McLoughlin, J. (1995). *Shaking Hands in Corner Brook and Other Math Problems.* Waterloo: Waterloo Mathematics Foundation.



OLYMPIAD CORNER

No. 387

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by January 15, 2021.

OC501. Pavel alternately writes crosses and circles in the cells of a rectangular table (starting with a cross). When the table is completely filled, the resulting score is calculated as the difference $O - X$, where O is the total number of rows and columns containing more circles than crosses and X is the total number of rows and columns containing more crosses than circles.

(a) Prove that for a $2 \times n$ table the resulting score will always be 0.

(b) Determine the highest possible score achievable for the table $(2n+1)\times(2n+1)$ depending on n.

OC502. Find the largest possible number of elements of a set M of integers having the following property: from each three different numbers from M you can select two of them whose sum is a power of 2 with an integer exponent.

OC503. Let ABC be a non-isosceles acute-angled triangle with centroid G. Let M be the midpoint of BC, let Ω be the circle with center G and radius GM, and let N be the intersection point between Ω and BC that is distinct from M. Let S be the symmetric point of A with respect to N, that is, the point on the line AN such that $AN = NS$ $(A \neq S)$. Prove that GS is perpendicular to BC.

OC504. Let $\mathcal{F}$ be the set of continuous functions $f : [0, 1] \to \mathbb{R}$ such that $\max_{0\leq x\leq 1} |f(x)| = 1$ and let $I : \mathcal{F} \to \mathbb{R}$,

$$I(f) = \int_0^1 f(x)\ dx - f(0) + f(1).$$

(a) Prove that $I(f) < 3$ for all $f \in \mathcal{F}$.

(b) Determine $\sup\{I(f) \mid f \in \mathcal{F}\}$.

OC505. Let n be a positive integer. We will say that a set of positive integers is *complete of order* n if the set of all remainders obtained by dividing an element in A by an element in A is $\{0, 1, 2, \ldots, n\}$. For example, the set $\{3, 4, 5\}$ is a complete set of order 4. Determine the minimum number of elements of a complete set of order 100.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 janvier 2021.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC501. Dans les cases d'un damier rectangulaire, on trace des cercles et des croix. On commence avec une croix et on alterne jusqu'à ce que le damier soit rempli. On compte alors le nombre de rangées et de colonnes ayant plus de cercles que de croix et on dénote ceci par O; de façon similaire, on compte le nombre de rangées et de colonnes ayant plus de croix que de cercles et on dénote ceci par X. Le score est alors $O - X$.

(a) Démontrer que pour un damier $2 \times n$ le score est toujours 0.

(b) Déterminer, selon n, le plus gros score atteignable sur un damier $(2n + 1) \times (2n + 1)$.

OC502. Un ensemble M est formé d'entiers tels que parmi trois quelconques de ces entiers, on peut en choisir deux dont la somme sera une puissance entière de 2. Déterminer le plus grand nombre possible d'éléments que peut avoir M.

OC503. Soit ABC un triangle acutangle non isocèle de centroïde G. Soit M le point milieu de BC, Ω le cercle de centre G et rayon GM, puis N le point d'intersection entre Ω et BC différent de M. Enfin, soit S le point symétrique à A par rapport à N, c'est-à-dire le point sur la ligne AN tel que $AN = NS$ ($A \neq S$). Démontrer que GS est perpendiculaire à BC.

OC504. Soit F l'ensemble de fonctions continues $f : [0,1] \to \mathbb{R}$ telles que $\max_{0 \le x \le 1} |f(x)| = 1$ et soit $I : \mathcal{F} \to \mathbb{R}$,

$$I(f) = \int_0^1 f(x)\ dx - f(0) + f(1).$$

(a) Démontrer que $I(f) < 3$ pour tout $f \in \mathcal{F}$.

(b) Déterminer $\sup\{I(f) \mid f \in \mathcal{F}\}$.

OC505. Soit n un entier positif. Un ensemble d'entiers positifs A est dit *complet d'ordre* n si les restes lors de division d'un élément de A par un élément de A donnent $\{0, 1, 2, \ldots, n\}$. Par exemple, l'ensemble $\{3, 4, 5\}$ est complet d'ordre 4. Déterminer le plus petit nombre d'éléments possible pour un ensemble complet d'ordre 100.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(4), p. 159–160.

OC476. Let x be a real number such that both sums $S = \sin 64x + \sin 65x$ and $C = \cos 64x + \cos 65x$ are rational numbers. Prove that in one of these sums, both terms are rational.

Originally Russia Math Olympiad, 1st Problem, Grade 11, Final Round 2017.

We received 14 submissions of which 13 were complete and correct. We present the submission by UCLan Cyprus Problem Solving Group.

We have that

$$\begin{aligned}
&S^2 + C^2\\
&= (\sin 64x + \sin 65x)^2 + (\cos 64x + \cos 65x)^2\\
&= (\sin^2 64x + \cos^2 64x) + (\sin^2 65x + \cos^2 65x) + 2(\sin 64x \sin 65x + \cos 64x \cos 65x)\\
&= 2 + 2\cos(65x - 64x) = 2 + 2\cos x
\end{aligned}$$

is rational. So $\cos x = S^2/2 + C^2/2 - 1$ is rational.

It follows that $\cos 2x = 2\cos^2 x - 1$ is rational, as well. Inductively we get that $\cos 64x = \cos(2^6 x)$ is also rational. Then so is $\cos 65x = C - \cos 64x$. So both terms of C are rational.

Another way to conclude that $\cos 64x$ and $\cos 65x$ are rational, is to use the fact that a rational $\cos x$ implies that $\cos(nx)$ is rational, for any integer n. This follows from either inductively $\cos((n+1)x) = 2\cos x \cos(nx) - \cos(n-1)x$ or the existence of Chebyshev polynomials that relate $\cos(nx)$ to $\cos x$.

Editor's Comment. All correct submissions follow along the lines of the presented solution.

OC477. Let $A = \{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Prove that $(|z+1| - \sqrt{2})(|z-1| - \sqrt{2}) \leq 0 \ \forall z \in A$.

(b) Prove that for any $z_1, z_2, \ldots, z_{12} \in A$, there is a choice of signs "±" so that

$$\sum_{k=1}^{12} |z_k \pm 1| < 17.$$

Originally Romania Math Olympiad, 4th Problem, Grade 10, District Round 2017.

We received 18 correct submissions. We present two solutions.

Solution 1, by Joel Schlosberg.

(a) Since $|z| = 1$,

$$\begin{aligned} |z \pm 1|^2 &= (z \pm 1)(\overline{z \pm 1}) \\ &= (z \pm 1)(\bar{z} \pm 1) \\ &= z\bar{z} + 1 \pm (z + \bar{z}) \\ &= |z|^2 + 1 \pm \mathrm{Re}(z) \\ &= 2 \pm 2\mathrm{Re}(z). \end{aligned}$$

If $\mathrm{Re}(z) > 0$ then $|z+1| - \sqrt{2} > 0$ and $|z-1| - \sqrt{2} < 0$. If $\mathrm{Re}(z) < 0$ then $|z+1| - \sqrt{2} < 0$ and $|z-1| - \sqrt{2} > 0$. In either case, their product is negative. If $\mathrm{Re}(z) = 0$ then $|z+1| - \sqrt{2} = 0$ and $|z-1| - \sqrt{2} = 0$, so their product is zero.

(b) Define $y_1, y_2, \ldots, y_{12}$ as follows:

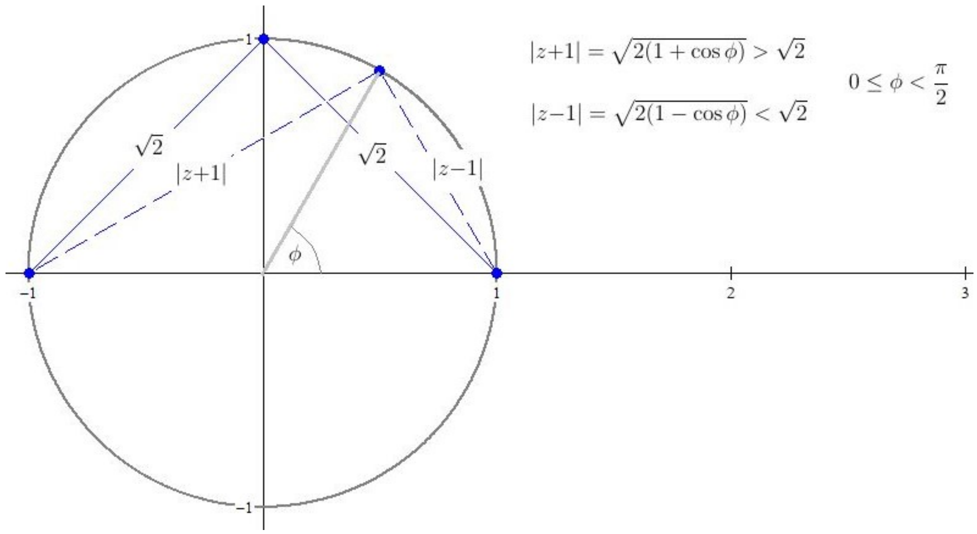
$$y_k = 1 \text{ if } \mathrm{Re}(z_k) < 0 \text{ and } y_k = -1 \text{ if } \mathrm{Re}(z_k) \geq 0.$$

Based on inequalities described in part (a), $|z_k + y_k| \leq \sqrt{2}$ for all $k = 1, \ldots, 12$. Therefore,

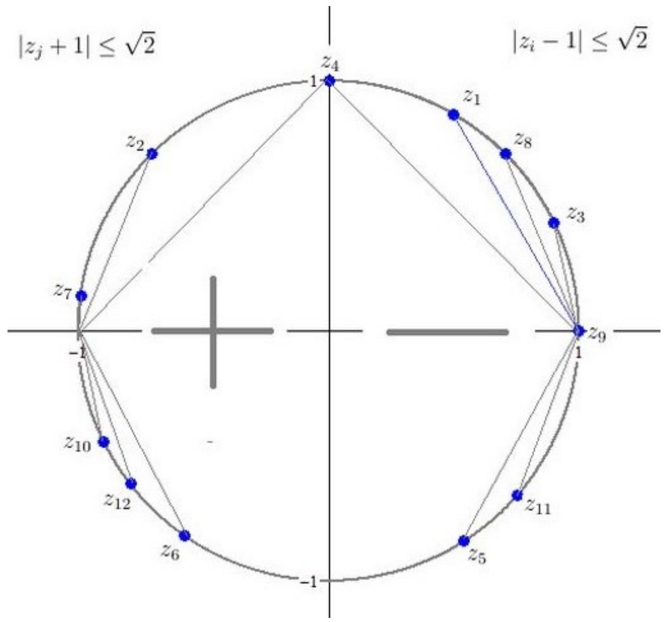
$$\sum_{k=1}^{12} |z_k + y_k| \leq 12\sqrt{2} = \sqrt{288} < \sqrt{289} = 17.$$

Solution 2, proof without words, by Zoltan Retkes.

(a)



(b)



$$\left(\sum_{m=1}^{12} |z_m \pm 1|\right)^2 \leq (12\sqrt{2})^2 = 288 < 289 = 17^2.$$

OC478. Consider two non-commuting matrices $A, B \in \mathcal{M}_2(\mathbb{R})$.

(a) Knowing that $A^3 = B^3$, prove that A^n and B^n have the same trace for any nonzero natural number n.

(b) Give an example of two noncommuting matrices $A, B \in \mathcal{M}_2(\mathbb{R})$ such that for any nonzero $n \in \mathbb{N}$, $A^n \neq B^n$, and A^n and B^n have different traces.

Originally Romania Math Olympiad, 3rd Problem, Grade 11, District Round 2017.

We received 6 submissions of which 5 were correct and complete. We present the solution by Oliver Geupel.

(a) Let d_A and t_A, respectively, denote the determinant and the trace of the matrix A. By the Cayley-Hamilton theorem, A satisfies the characteristic equation: $A^2 - t_A A + d_A I = 0$. Hence,

$$A^3 = t_A A^2 - d_A A = t_A(t_A A - d_A I) - d_A A = (t_A^2 - d_A)A - t_A d_A I.$$

Similarly, with d_B and t_B denoting the determinant and trace of the matrix B, it holds

$$B^3 = (t_B^2 - d_B)B - t_B d_B I.$$

By hypothesis,

$$(t_A^2 - d_A)A - (t_B^2 - d_B)B - (t_A d_A - t_B d_B)I = A^3 - B^3 = O_2.$$

It follows that

$$(t_A^2 - d_A)AB = (t_B^2 - d_B)B^2 + (t_A d_A - t_B d_B)B = (t_A^2 - d_A)BA.$$

Since A and B do not commute, we deduce that $d_A = t_A^2$. Analogously, $d_B = t_B^2$. Thus, $t_A^3 = t_A d_A = t_B d_B = t_B^3$. As traces of real matrices, t_A and t_B are real numbers. Hence $t_A = t_B$ and $d_A = d_B$. Now the eigenvalues λ_1 and λ_2 of either matrices A and B are constraint to the conditions $\lambda_1 + \lambda_2 = t_A$ and $\lambda_1\lambda_2 = d_A$, which determine them uniquely. Therefore A and B have identical eigenvalues λ_1 and λ_2, and $\mathrm{tr}(A^n) = \mathrm{tr}(B^n) = \lambda_1^n + \lambda_2^n$.

(b) Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

A straight forward computation shows that:

A and B do not commute

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, \qquad BA = \begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix},$$

and A^n and B^n are not equal for any $n \geq 1$

$$A^n = \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}, \qquad B^n = \begin{bmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{bmatrix}.$$

Moreover A and B have the same eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 2$. Hence A^n and B^n have identical traces.

Editor's Comments. As some solvers pointed out, part (b) was interesting when asking for examples of matrices A and B with non-equal powers but with powers of identical trace.

OC479. We say that the function $f : \mathbb{Q}_+^* \to \mathbb{Q}$ has the property $\mathcal{P}$ if

$$f(xy) = f(x) + f(y) \qquad \forall x, y \in \mathbb{Q}_+^*.$$

(a) Prove that there do not exist injective functions with property $\mathcal{P}$.

(b) Do there exist surjective functions with property $\mathcal{P}$?

Originally Romania Math Olympiad, 2nd Problem, Grade 10, Final Round 2017.

We received 12 submissions of which 11 were correct and complete. We present the solution by Kathleen Lewis.

(a) The property P of the function implies first that $f(1) = f(1 \times 1) = f(1) + f(1)$, so $f(1) = 0$. Secondly, $f(x) = f((x/y) \times y) = f(x/y) + f(y)$, so we have that $f(x/y) = f(x) - f(y)$ for all $x, y \in \mathbb{Q}^*_+$.

In particular, $f(y^{-1}) = f(1/y) = f(1) - f(y) = -f(y)$.

Moreover, $f(x^2) = f(x \times x) = f(x) + f(x) = 2f(x)$.

By induction on n, we can see that $f(x^n) = nf(x)$ for any integer n.

Let $f(2) = a/b$ and $f(3) = p/q$, for some integers a, $b \neq 0$, p, $q \neq 0$. Then $f(2^{bp}) = bpf(2) = ap$ and $f(3^{aq}) = aqf(3) = ap$. So $f(2^{bp}) = f(3^{aq})$. If $2^{bp} \neq 3^{aq}$ then f is not injective. If $2^{bp} = 3^{aq}$ then $bp = aq = 0$. In this case $f(2) = f(3) = 0$. Again, f is not injective.

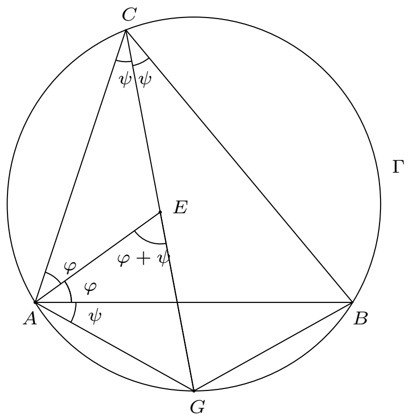
(b) From the properties described above, we see that the function f is uniquely defined by its values on prime numbers. Next we select the values of f on the prime numbers to be: $f(2) = 1/2$, $f(3) = 1/3$, $f(5) = 1/4$, and so on. In summary, if p_n is the n^{th} prime, then $f(p_n) = 1/(n+1)$. For this specific f, its range is $\mathbb{Q}$. To see this, let $a/b \in \mathbb{Q}$ with integers a and $b \neq 0$. Note we can assume that $b > 1$. If $b = 1$ then we can rewrite the rational number as $2a/2$. Then $f(p_{b-1}) = 1/b$, so $f(p^a_{b-1}) = a/b$. In conclusion, the example constructed is a surjective function.

OC480. In the plane, there are points C and D on the same region with respect to the line defined by the segment AB so that the circumcircles of triangles ABC and ABD are the same. Let E be the incenter of triangle ABC, let F be the incenter of triangle ABD and let G be the midpoint of the arc AB not containing the points C and D. Prove that points A, B, E, F are on a circle with center G.

Originally Hungary Math Olympiad, 2nd Problem, Category I, Final Round 2017.

We received 12 submissions of which 11 were correct and complete. We present two solutions.

Solution 1, by Miguel Amengual Covas.



Let Γ denote the common circumcircle of $\triangle ABC$ and $\triangle ABD$. Since G is the midpoint of the arc AB of Γ not containing C, we have

$$\angle ACG = \text{arc}AG/2 = \text{arc}GB/2 = \angle GCB.$$

Thus, G lies on the angle bisector of $\angle ACB$. As the incentre of $\triangle ABC$, E lies on the angle bisector of $\angle ACB$ and the angle bisector of $\angle CAB$. Thus, C, E, and G are co-linear.

Let $\varphi = \angle A/2$ and $\psi = \angle C/2$. We notice that $\angle BAG = \angle BCG = \psi$. So, $\angle EAG = \varphi + \psi$. In addition $\angle AEG = \varphi + \psi$, as the exterior angle of $\triangle AEC$ at E.

In conclusion, triangle AGE is isosceles with $GA = GE$. Similarly, from $\triangle ABD$, $GA = GF$. In turn, the equality of arcs AG and GB implies the equality of the chords GA and GB.

Thus, $GB = GA = GE = GF$, placing A, B, E, F on a circle centered at G.

Solution 2, by Corneliu Manescu-Avram.

Choose a complex system of coordinates with the circumcircle of $\triangle ABC$ as the unit circle. Take $A(a^2)$, $B(b^2)$, $C(c^2)$, $D(d^2)$ with $a, b, c, d \in \mathbb{C}$, $|a| = |b| = |c| = |d| = 1$ and denote the complex coordinates of remaining points by the same letter: $E(e)$, $F(f)$, $G(g)$. Then we have

$$e = -ab - bc - ca, f = -ab - bd - da, g = -ab.$$

Moreover, it is easy to verify that

$$AG = BG = EG = FG = |a + b|.$$

FOCUS ON...

No. 43

Michel Bataille

Solutions to Exercises from Focus On... No. 37 - 41

From Focus On... No. 37

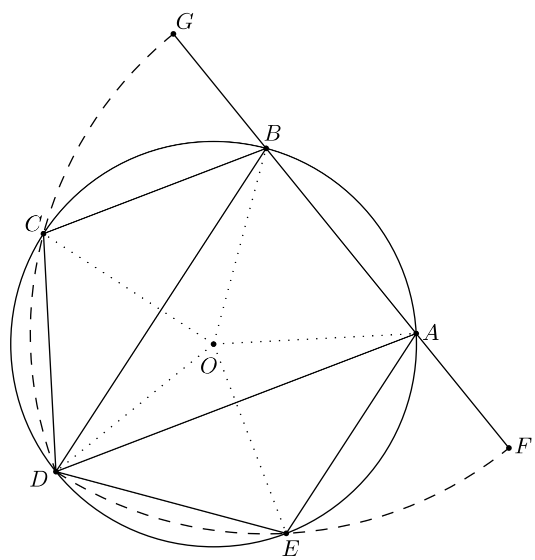
1. *Let C be a point distinct from the vertices of a triangle OAB. Suppose that ΔOCD and ΔCAE are directly similar to ΔOAB. Prove that $CDBE$ is a parallelogram.*

We take O as the origin and denote by a, b, c, d, e the affixes of A, B, C, D, E, respectively. From the hypotheses, we have that $c = ua$, $d = ub$ and $a = va+c$, $e = vb + c$ for some non-zero complex numbers u, v. Then, we obtain $d - c = u(b - a)$ and

$$b - e = b - vb - ua = b(1 - v) - ua = b \cdot \frac{c}{a} - ua = ua \cdot \frac{b}{a} - ua = u(b - a).$$

Thus, $b - e = d - c$ so that $\overrightarrow{EB} = \overrightarrow{CD}$ and $CDBE$ is a parallelogram.

2. *On the extension of the side AB of the regular pentagon $ABCDE$, let the points F and G be placed in the order F, A, B, G so that $AG = BF = AC$. Compare the area of triangle FGD to the area of pentagon $ABCDE$.*



Without loss of generality, we suppose that A, B, C, D, E are the points of the complex plane with respective affixes $a = 1, b = \omega, c = \omega^2, d = \omega^3, e = \omega^4$ where

$\omega = e^{2\pi i/5}$. Let O be the centre of the circumcircle of $ABCDE$ (the unit circle). The area $[ABCDE]$ of the pentagon is $5 \times [AOB] = \frac{5}{2} \cdot \sin\frac{2\pi}{5}$. We show that this area equals the area $[FGD]$ of the triangle FGD.

Note that $DA = DB = AC$, $\angle ADB = \frac{1}{2}\angle AOB = \frac{\pi}{5}$, so $\angle BAD = \angle ABD = \frac{2\pi}{5}$.

Let f and g denote the affixes of F and G. We have $g - a = e^{-2\pi i/5}(d - a)$ and $f - b = e^{2\pi i/5}(d - b)$, hence $g = 1 - e^{-2\pi i/5} + e^{4\pi i/5}$ and $f = 2\cos\frac{2\pi}{5} - e^{4\pi i/5}$. We deduce

$$g - d = 1 + 2i\sin\frac{4\pi}{5} - e^{-2\pi i/5} \quad \text{and} \quad \overline{f} - \overline{d} = 2\cos\frac{2\pi}{5} - 2\cos\frac{4\pi}{5} = \sqrt{5}.$$

(Recall that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\cos\frac{4\pi}{5} = \frac{-\sqrt{5}-1}{4}$.)
Observing that the imaginary part of $g - d$ is

$$\operatorname{Im}(g - d) = 2\sin\frac{4\pi}{5} + \sin\frac{2\pi}{5} = \sin\frac{2\pi}{5}(4\cos\frac{2\pi}{5} + 1) = \sqrt{5}\sin\frac{2\pi}{5},$$

we obtain

$$[FGD] = \frac{1}{2}\left|\operatorname{Im}((g - d)(\overline{f} - \overline{d}))\right| = \frac{\sqrt{5}}{2}\,|\operatorname{Im}(g - d)| = \frac{5}{2} \cdot \sin\frac{2\pi}{5} = [ABCDE].$$

From Focus On... No. 39

1. *Prove the relations*

$$a^4S_A + b^4S_B + c^4S_C - 3S_AS_BS_C = 2(a^2 + b^2 + c^2)F^2 = S_AS_BS_C + a^2b^2c^2$$

and deduce a condition on a, b, c for the nine-point centre N to lie on the circumcircle of ΔABC.

From $a^4S_A = S_A(S_B + S_C)^2 = S_AS_B^2 + S_AS_C^2 + 2S_AS_BS_C$ and similar relations, we obtain

$$\begin{aligned}
&a^4S_A + b^4S_B + c^4S_C \\
&= c^2S_AS_B + b^2S_CS_A + a^2S_BS_C + 6S_AS_BS_C \\
&= c^2(4F^2 - c^2S_C) + b^2(4F^2 - b^2S_B) + a^2(4F^2 - a^2S_A) + 6S_AS_BS_C
\end{aligned}$$

and so $2(a^4S_A + b^4S_B + c^4S_C) = 4F^2(a^2 + b^2 + c^2) + 6S_AS_BS_C$. The first equality follows.

As for the second one, we use the identities:

$$\begin{aligned}
&(x + y - z)(y + z - x)(z + x - y) \\
&= xy^2 + x^2y + yz^2 + y^2z + zx^2 + z^2x - (x^3 + y^3 + z^3) - 2xyz \\
&= (x + y + z)(xy + yz + zx) - 5xyz - [3xyz + (x + y + z)(x^2 + y^2 + z^2 - (xy + yz + zx))] \\
&= (x + y + z)(2xy + 2yz + 2zx - x^2 - y^2 - z^2) - 8xyz
\end{aligned}$$

to obtain

$$8S_AS_BS_C = (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2) = (a^2+b^2+c^2)16F^2-8a^2b^2c^2,$$

and the second equality.

Since the equation of the circumcircle Γ of ΔABC is $a^2yz+b^2zx+c^2xy=0$, the point N is on Γ if and only if

$$a^2(S_CS_A+4F^2)(S_AS_B+4F^2)+b^2(S_AS_B+4F^2)(S_BS_C+4F^2)+c^2(S_BS_C+4F^2)(S_CS_A+4F^2)=0,$$

which easily rewrites first as

$$2S_AS_BS_C+a^4S_A+b^4S_B+c^4S_C+4F^2(a^2+b^2+c^2)=0$$

and finally (with the help of the relations above)

$$5a^2b^2c^2=16(a^2+b^2+c^2)F^2.$$

2. *Use S_A, S_B, S_C to show that O, H and the incenter I are collinear if and only if the triangle ABC is isosceles.*

The points O, H, I are collinear if and only if $\delta=0$ where

$$\delta = \begin{vmatrix} S_AS_B+S_AS_C & S_BS_C & a \\ S_BS_C+S_AS_B & S_CS_A & b \\ S_AS_C+S_BS_C & S_AS_B & c \end{vmatrix}.$$

Calculating δ as follows:

$$\delta = (S_BS_C+S_CS_A+S_AS_B)\begin{vmatrix} 1 & S_BS_C & a \\ 1 & S_CS_A & b \\ 1 & S_AS_B & c \end{vmatrix} = 4F^2\begin{vmatrix} 1 & S_BS_C & a \\ 0 & S_C(b^2-a^2) & b-a \\ 0 & S_B(c^2-a^2) & c-a \end{vmatrix},$$

we obtain

$$\delta = 4F^2(b-a)(c-a)((a+b)S_C-(a+c)S_B) = 2F^2(b-a)(c-a)(b-c)(a+b+c)^2.$$

Thus, O, H, I are collinear if and only if ΔABC is isosceles.

3. *Find the point at infinity of the perpendiculars to OI, where O and I are the circumcentre and the incentre of a scalene triangle ABC.*

From $2sI = aA+bB+cC$ and $(8F^2)O = (a^2S_A)A+(b^2S_B)B+(c^2S_C)C$, we deduce that

$$8sF^2\overrightarrow{OI} = (4aF^2-a^2sS_A)A+(4bF^2-b^2sS_B)B+(4cF^2-c^2sS_C)C$$

so that the point at infinity of the line OI is $(f:g:h)$ with

$$f=a(4F^2-asS_A),\ g=b(4F^2-bsS_B),\ h=c(4F^2-csS_C).$$

It follows that $gS_B - hS_C = 4F^2(bS_B - cS_C) - s(bS_B - cS_C)(bS_B + cS_C)$.

But we know that $bS_B - cS_C = 2s(s-a)(c-b)$ and a short calculation gives

$$2(bS_B + cS_C) = b(c^2 + a^2 - b^2) + c(a^2 + b^2 - c^2) = 4(b+c)(s-b)(s-c).$$

As a result, we obtain $4(gS_B - hS_C) = 2s(s-a)(c-b)[16F^2 - 8(b+c)s(s-b)(s-c)]$ and using $16F^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$, we readily get

$$4(gS_B - hS_C) = -2as(s-a)(c-b)(a+b+c)(c+a-b)(a+b-c) = a(b-c)(16sF^2).$$

By cyclic permutation,

$$4(hS_C - fS_A) = b(c-a)(16sF^2),\ 4(fS_A - gS_B) = c(a-b)(16sF^2)$$

and we conclude that the point at infinity of the perpendiculars to OI is

$$(a(b-c) : b(c-a) : c(a-b)).$$

4. *If* $M_1 = (x_1 : y_1 : z_1),\ M_2 = (x_2 : y_2 : z_2)$ *with* $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 1$, *show that*

$$M_1M_2^2 = S_A(x_2 - x_1)^2 + S_B(y_2 - y_1)^2 + S_C(z_2 - z_1)^2.$$

We remark that $x_2 - x_1 = -(y_2 - y_1) - (z_2 - z_1)$ and deduce that

$$\overrightarrow{M_1M_2} = (x_2 - x_1)A + (y_2 - y_1)B + (z_2 - z_1)C = (y_2 - y_1)\overrightarrow{AB} + (z_2 - z_1)\overrightarrow{AC}.$$

Thus,

$$\begin{aligned} M_1M_2^2 &= (y_2 - y_1)^2c^2 + (z_2 - z_1)^2b^2 + 2(y_2 - y_1)(z_2 - z_1)\overrightarrow{AB}\cdot\overrightarrow{AC} \\ &= (y_2 - y_1)^2(S_A + S_B) + (z_2 - z_1)^2(S_A + S_C) + 2S_A(y_2 - y_1)(z_2 - z_1) \\ &= S_A(y_2 - y_1 + z_2 - z_1)^2 + S_B(y_2 - y_1)^2 + S_C(z_2 - z_1)^2 \\ &= S_A(x_2 - x_1)^2 + S_B(y_2 - y_1)^2 + S_C(z_2 - z_1)^2. \end{aligned}$$

From Focus On... No. 40

1. *Let* $n \in \mathbb{N}$ *and* $\Delta(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i - \prod_{i=1}^{n} x_i$. *If* $a_1, a_2, \dots, a_n \in (0, 1]$ *prove that* $\Delta(a_1, a_2, \dots, a_n) \geq \Delta\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)$.

We use induction. Let (R_n) denote the desired result. Since $\Delta(a_1) = 0 = \Delta(1/a_1)$, (R_1) holds. Assume that (R_n) holds for some positive integer n and consider $a_1, \dots, a_n, a_{n+1}$ in $(0, 1]$. We introduce the auxiliary function f defined on $(0, 1]$ by

$$f(x) = x + a_1 + \cdots + a_n - xa_1 \cdot \cdots \cdot a_n - \left(\frac{1}{x} + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) + \frac{1}{xa_1 \cdot \cdots \cdot a_n},$$

that is, $f(x) = \Delta(a_1, \dots, a_n, x) - \Delta\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}, \frac{1}{x}\right)$.
The derivative of f satisfies:

$$f'(x) = 1 - a_1 \cdot \dots \cdot a_n + \frac{1}{x^2} - \frac{1}{x^2 a_1 \cdot \dots \cdot a_n} = (1 - a_1 \cdot \dots \cdot a_n)\left(1 - \frac{1}{x^2 a_1 \cdot \dots \cdot a_n}\right).$$

Clearly $f'(x) \leq 0$ for all $x \in (0, 1]$, hence f is nonincreasing on the interval $(0, 1]$.
As a result,

$$f(a_{n+1}) \geq f(1) = \Delta(a_1, a_2, \dots, a_n) - \Delta\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right) \geq 0$$

where the last inequality follows from (R_n). Thus, (R_{n+1}) holds and the proof is complete.

2. *For $y \in (0, 1]$, let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = y^x + x^y - 1$ and $g : (0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - \frac{x}{y} \cdot f'(x)$. From the study of g deduce that $f(x) > 0$ for $x \in (0, 1]$.*

Note that the result holds if $y = 1$ so we will suppose that $y \in (0, 1)$. First, we calculate

$$f'(x) = (\ln(y))y^x + yx^{y-1}, \qquad f''(x) = (\ln(y))^2 y^x + y(y-1)x^{y-2}$$

and deduce that

$$g'(x) = \frac{y-1}{y} \cdot f'(x) - \frac{x}{y} \cdot f''(x) = \frac{y^x(\ln(y))^2}{y}\left(\frac{y-1}{\ln(y)} - x\right).$$

Since $\ln(y) < y - 1 < 0$, we have $0 < \frac{y-1}{\ln(y)} < 1$ and the variations of g on $(0, 1]$ give $g(x) > \min(\lim_{x\to 0} g, g(1))$ for any $foranyx \in (0, 1)$. Since $\lim_{x\to 0} g = 0$ and $g(1) = y - 1 - \ln(y) > 0$, we conclude that $g(x) > 0$ if $x \in (0, 1]$.
Now f, being continuous on $[0, 1]$, attains its minimum, say at x_0. Should we have $x_0 \in (0, 1)$, we would then have $f'(x_0) = 0$ and so

$$f(0) \geq f(x_0) > \frac{x_0}{y} f'(x_0) = 0$$

(using $g(x_0) > 0$). But this contradicts $f(0) = 0$, hence the minimum of f on $[0, 1]$ must be attained at 0 or 1. Since $f(1) > f(0)$, this minimum is $f(0) = 0$, attained only at 0 and we can conclude that $f(x) > 0$ for all $x \in (0, 1)$.

3. *Let m be an integer with $m \geq 2$ and r a real number in $[1, \infty)$. Prove that*

$$\left(\frac{1 + r^m}{1 + r^{m-1}}\right)^{m+1} \geq \frac{1 + r^{m+1}}{2}.$$

[Hint: first determine the sign of $u(x) = (m-1)(1 + x^{m+1}) - x(1 + x^{m-1})$ for $x \geq 1$.]

First, we study the sign of $u(x)$. Straightforward calculations lead to

$$u'(x) = (m^2-1)x^m - mx^{m-1} - 1, \qquad u''(x) = m(m-1)x^{m-2}[(m+1)x-1].$$

Since $u''(x) > 0$ for $x \geq 1$, the function u' is increasing on $[1, \infty)$ and so $u'(x) \geq u'(1) = (m+1)(m-2)$.

Therefore $u'(x) \geq 0$ for $x \in [1, \infty)$ and, similarly, $u(x) \geq u(1) = 2(m-2)$. We conclude that $u(x) \geq 0$ if $x \geq 1$.

Taking logarithms, we see that the desired inequality holds if $f(r) \leq \ln(2)$ where

$$f(x) = \ln(1+x^{m+1}) - (m+1)[\ln(1+x^m) - \ln(1+x^{m-1})].$$

Since $f(1) = \ln(2)$, it is sufficient to show that $f'(x) \leq 0$ when $x \geq 1$. We calculate

$$f'(x) = (m+1)x^{m-2}\left(\frac{x^2}{1+x^{m+1}} - \frac{mx}{1+x^m} + \frac{m-1}{1+x^{m-1}}\right),$$

which has the same sign as

$$N(x) = x^2(1+x^{m-1})(1+x^m) - mx(1+x^{m-1})(1+x^{m+1}) + (m-1)(1+x^m)(1+x^{m+1}).$$

Expanding and arranging yield

$$\begin{aligned} N(x) &= (mx^{m+1} - x^m - (m-1)x^{m+2}) + (x^2 - mx + (m-1)) \\ &= x^m(x-1)(1-(m-1)x) + (x-1)(x-(m-1)) \end{aligned}$$

and finally $N(x) = (1-x)u(x)$. Thus $N(x) \leq 0$ if $x \geq 1$ and we are done.

From Focus On... No. 41

1. *Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that*

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \geq \frac{3}{4}.$$

The function $f : x \mapsto (x+x^2)^{-1}$ is convex on $(0, \infty)$ (its second derivative f'', given by $f''(x) = 2(x+x^2)^{-3}(3x^2+3x+1)$, is positive). Since $a+b+c = 1$, Jensen's inequality yields

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} = af(b) + bf(c) + cf(a) \geq f(\sigma)$$

where we set $ab+bc+ca = \sigma$. It follows that

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \geq \sigma f(\sigma) = \frac{1}{\sigma+1}.$$

But $1 = (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$, hence $\sigma \leq \frac{1}{3}$ and $\frac{1}{\sigma+1} \geq \frac{3}{4}$. The result follows.

2. *Prove, for real numbers a, b, x, y with $a > b > 1$ and $x > y > 1$, that*

$$\frac{a^x - b^y}{x - y} > \left(\frac{a+b}{2}\right)^{\frac{x+y}{2}} \log\left(\frac{a+b}{2}\right).$$

(Hint: first apply Hadamard's inequality to the function $t \mapsto m^t$ on $[y, x]$, where $m = \frac{a+b}{2}$).

[We slightly extend the result by supposing only $x > y > 0$.]

Let $m = \frac{a+b}{2}$. The function $t \mapsto m^t$ is continuous and convex on $(0, \infty)$, hence for $x > y > 0$, we have

$$m^{\frac{x+y}{2}} \leq \frac{1}{x-y}\int_y^x m^t \, dt$$

(from Hadamard's inequality). Since $m > 1$, we have $\log(m) > 0$ and so

$$\left(\frac{a+b}{2}\right)^{\frac{x+y}{2}} \log\left(\frac{a+b}{2}\right) = (\log(m))m^{\frac{x+y}{2}} \leq \frac{1}{x-y}\int_y^x (\log(m))m^t \, dt = \frac{m^x - m^y}{x-y}.$$

Thus, it is sufficient to prove that $m^x - m^y < a^x - b^y$, that is,

$$b^y - m^y < a^x - m^x \qquad (1).$$

Now, since $x, y > 0$, the functions $t \mapsto t^x$ and $t \mapsto t^y$ are strictly increasing on $(0, \infty)$, therefore $b^y < m^y$ and $m^x < a^x$ (since $b < m < a$) and (1) clearly holds.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by January 15, 2021.

4581. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle, with $AB < AC$ and with circumcircle Γ, circumcenter O and incenter I. Denote $AI \cap \Gamma = A_1,\ BI \cap A_1O = B_1,\ CI \cap A_1O = C_1$. Prove that

$$\frac{BC_1 - C_1I}{B_1I + B_1C} = \frac{BI \cdot A_1C_1}{CI \cdot A_1B_1}.$$

4582. *Proposed by Leonard Giugiuc.*

Let $k > 9$ be a fixed real number. Consider the following system of equations with $a \leq b \leq c \leq d$:

$$\begin{cases} a+b+c+d = 3+k, \\ a^2+b^2+c^2+d^2 = 3+k^2, \\ abcd = k. \end{cases}$$

a) Find all solutions in positive reals.

b) Determine the number of real solutions.

4583. *Proposed by Daniel Sitaru.*

Let

$$A = \begin{pmatrix} \frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2} \\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2} \\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2} \end{pmatrix},$$

where a, b and c are positive real numbers. Find the value of the sum of all the entries of A^n, where n is a natural number, $n \geq 2$.

4584. *Proposed by Michel Bataille.*

For $n \in \mathbb{N}$, let

$$S_n = \sum_{k=1}^{n} \frac{k}{n + \sqrt{k+n^2}}.$$

Find real numbers a, b such that $\lim_{n\to\infty} (S_n - an) = b$.

4585. *Proposed by George Stoica.*

Let $P(x)$ be a real polynomial of degree n whose n roots are all real. Then for all $k = 0, \dots, n-2$, prove that, for $c \in \mathbb{R}$:

$$P^{(k)}(c) \neq 0, P^{(k+1)}(c) = 0 \Rightarrow P^{(k+2)}(c) \neq 0.$$

4586. *Proposed by Nguyen Viet Hung.*

Find all triples (m, n, p) where m, n are two non-negative integers and p is a prime, satisfying the equation

$$m^4 = 4(p^n - 1).$$

4587. *Proposed by Kai Wang.*

Find an elementary proof of $\tan \frac{2\pi}{7} = -\sqrt{7} + 4 \sin \frac{4\pi}{7}$.

4588. *Proposed by the Editorial Board.*

If a, b, c are positive real numbers such that $ab+bc+ca = 3$, prove that $a^n + b^n + c^n \geq 3$ for all integers n.

4589. *Proposed by Lorian Saceanu.*

Let a, b, c be real numbers, not all zero, such that $a^2 + b^2 + c^2 = 2(ab + bc + ca)$. Prove that

$$2 \leq \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \leq \frac{12}{5}.$$

4590. *Proposed by George Apostolopoulos.*

Let ABC be an acute angled triangle. Prove that

$$\sum \frac{\sin^2 A}{\cos^2 B + \cos^2 C} \leq \frac{9}{2},$$

where the sum is taken over all cyclic permutations of (A, B, C).

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposeś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 janvier 2021.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4581. *Proposeé par Mihaela Berindeanu.*

Soit ABC un triangle tel que $AB < AC$ et soit Γ son cercle circonscrit, de centre O; soit aussi I le centre de son cercle inscrit. Dénotons

$$AI \cap \Gamma = A_1,\ BI \cap A_1O = B_1,\ CI \cap A_1O = C_1.$$

Démontrer que

$$\frac{BC_1 - C_1I}{B_1I + B_1C} = \frac{BI \cdot A_1C_1}{CI \cdot A_1B_1}.$$

4582. *Proposeé par Leonard Giugiuc.*

Soit $k > 9$ un nombre réel quelconque. Considérons le système d'équations qui suit, où $a \leq b \leq c \leq d$:

$$\begin{cases} a + b + c + d = 3 + k, \\ a^2 + b^2 + c^2 + d^2 = 3 + k^2, \\ abcd = k. \end{cases}$$

a) Déterminer toutes les solutions réelles positives.

b) Déterminer le nombre de solutions réelles.

4583. *Proposeé par Daniel Sitaru.*

Soit

$$A = \begin{pmatrix} \frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2} \\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2} \\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2} \end{pmatrix},$$

où a, b et c sont des nombres réels positifs. Déterminer la somme de toutes les valeurs dans A^n, où n est un nombre naturel $n \geq 2$.

4584. *Proposeé par Michel Bataille.*

Pour $n \in \mathbb{N}$, soit

$$S_n = \sum_{k=1}^{n} \frac{k}{n + \sqrt{k + n^2}}.$$

Déterminer des nombres réels a et b tels que $\lim_{n\to\infty}(S_n - an) = b$.

4585. *Proposeé par George Stoica.*

Soit $P(x)$ un polynôme réel de degré n dont les n racines sont réelles. Démontrer que pour $k = 0, \ldots, n-2$ et $c \in \mathbb{R}$ la suivante tient:

$$P^{(k)}(c) \neq 0, P^{(k+1)}(c) = 0 \Rightarrow P^{(k+2)}(c) \neq 0.$$

4586. *Proposeé par Nguyen Viet Hung.*

Déterminer tous les triplets (m, n, p), où m et n sont des entiers non négatifs et p est premier, tels que

$$m^4 = 4(p^n - 1).$$

4587. *Proposeé par Kai Wang.*

Donner une preuve élémentaire de $\tan \frac{2\pi}{7} = -\sqrt{7} + 4 \sin \frac{4\pi}{7}$.

4588. *Propose par le comit ditorial.*

Soient a, b, c des nombres réels positifs tels que $ab + bc + ca = 3$. Démontrer que $a^n + b^n + c^n \geq 3$ pour tout entier n.

4589. *Proposeé par Lorian Saceanu.*

Soient a, b, c des nombres réels, pas tous zéro, tels que $a^2+b^2+c^2 = 2(ab+bc+ca)$. Démontrer que

$$2 \leq \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \leq \frac{12}{5}.$$

4590. *Proposeé par George Apostolopoulos.*

Soit ABC un triangle acutangle. Démontrer que

$$\sum \frac{\sin^2 A}{\cos^2 B + \cos^2 C} \leq \frac{9}{2},$$

où la somme est prise par rapport à toutes les permutations cycliques de (A, B, C).

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(4), p. 175–180.

4531. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let a, b and c be positive real numbers and let x, y and z be real numbers. Suppose that $a + b + c = 2$ and $xa + yb + zc = 1$. Prove that

$$x + y + z - (xy + yz + zx) \geq \frac{3}{4}.$$

We received 10 submissions, including the one from the proposers. As it turned out, the statement of the problem as printed was incorrect due to a small but inadvertent modification. Except for the proposers, 7 submissions all gave simple counter-examples, and 2 solvers actually gave wrong "proofs" for the incorrect version. Furthermore, two solvers showed that the claimed inequality holds under various additional assumptions. We present below a complete solution with added condition.

Solution by Michel Bataille to the modified and correct statement.

The problem, as stated, was incorrect. For example, if $(a, b, c) = (\frac{3}{2}, -\frac{5}{2}, -\frac{5}{2})$, then it is readily verified that $(x + y + z) - (xy + yz + zx) = -\frac{9}{4}$. We now show that the claim is true if we add the condition that a, b, c are the side lengths of a triangle.

Let (I) denote the inequality to be proven, namely

$$x + y + z - (xy + yz + zx) \geq \frac{3}{4}. \tag{I}$$

We set $u = x - \frac{1}{2}$, $v = y - \frac{1}{2}$, and $w = z - \frac{1}{2}$. Then

$$ua + vb + wc = xa + yb + zc - \frac{a + b + c}{2} = 1 - 1 = 0, \tag{1}$$

and (I) becomes

$$-(uv + vw + wu) \geq 0. \tag{II}$$

From (1) we deduce that $auw + bvw = -cw^2$, $bvu + cwu = -au^2$, $cwv + auv = bv^2$, so

$$ac^2wu + bc^2vw = -c^3w^2, \tag{2}$$

$$abcuv + ac^2wu = -ca^2u^2, \tag{3}$$

$$bc^2vw + abcuv = -b^2cv^2. \tag{4}$$

From (3) + (4) - (2) we obtain

$$2abcuv = c(c^2w^2 - a^2u^2 - b^2v^2).$$

Similarly, we get

$$2abcvw = a(a^2u^2 - b^2v^2 - c^2w^2) \quad \text{and} \quad 2abcwu = b(b^2v^2 - c^2w^2 - a^2u^2).$$

Hence

$$-2abc(uv + vw + wu) = (au)^2(b + c - a) + (bv)^2(c + a - b) + (cw)^2(a + b - c) \geq 0$$

from which it follows that $-(uv + vw + wu) \geq 0$ so (II) holds and we are done.

4532. *Proposed by Marius Stănean.*

Let ABC be a triangle with circumcircle Γ and let M, N, P be points on the sides BC, CA, AB, respectively. Let M', N', P' be the intersections of AM, BN, CP with Γ different from the vertices of the triangle. Prove that

$$MM' \cdot NN' \cdot PP' \leq \frac{R^2r}{4},$$

where R and r are the circumradius and the inradius of triangle ABC.

We received 12 solutions, 11 of which were correct. We present 2 solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

By power of the point M we have $(AM)(MM') = (BM)(MC)$. Let $A_1 = \angle BAM$ and $A_2 = \angle CAM$. By the Sine Rule in the triangle BAM and CAM we have

$$BM = AM\frac{\sin A_1}{\sin B} \quad \text{and} \quad CM = AM\frac{\sin A_2}{\sin C}.$$

So we get

$$MM' = \frac{(BM)(MC)}{AM} = \sqrt{(BM)(MC)} \cdot \sqrt{\frac{\sin A_1 \sin A_2}{\sin B \sin C}}.$$

By Cauchy-Schwarz we have

$$(BM)(MC) \leq \frac{(BM + MC)^2}{4} = \frac{a^2}{4} = R^2 \sin^2 A.$$

We also have

$$\begin{aligned} \sin A_1 \sin A_2 &= \frac{\cos(A_1 - A_2) - \cos(A_1 + A_2)}{2} \\ &\leq \frac{1 - \cos(A_1 + A_2)}{2} = \sin^2\left(\frac{A_1 + A_2}{2}\right) = \sin^2\left(\frac{A}{2}\right). \end{aligned}$$

So

$$MM' \leq R \sin\left(\frac{A}{2}\right)\sqrt{\frac{\sin^2 A}{\sin B \sin C}}\,.$$

Multiplying with the analogous inequalities for NN' and PP' we get

$$(MM')(NN')(PP') \leq R^3 \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{R^2 r}{4}\,.$$

Here we have used the formula

$$r = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\,.$$

Solution 2, by Sorin Rubinescu.

The power of M with respect to Γ is $AM \cdot MM' = R^2 - OM^2$, with O being the circumcenter of ABC. The power of M is also

$$R^2 - OM^2 = BM \cdot MC \leq \left(\frac{BM + MC}{2}\right)^2 = \frac{BC^2}{4} = \frac{a^2}{4},$$

and the equality holds for $AM = m_a$ (when $[AM]$ is a median). But

$$m_a^2 = \frac{2\left(b^2+c^2\right)-a^2}{4} \geq \frac{(b+c)^2 - a^2}{4} = \frac{(b+c-a)(b+c+a)}{4} = s(s-a)\,,$$

with s being the semiperimeter. We have that $m_a \geq \sqrt{s(s-a)}$, so $MM' \leq \frac{a^2}{4m_a}$. By writing the analogous relations for NN' and PP' we get that

$$MM' \cdot NN' \cdot PP' \leq \frac{a^2}{4m_a}\cdot\frac{b^2}{4m_b}\cdot\frac{c^2}{4m_c} = \frac{a^2b^2c^2}{64m_am_bm_c} \leq \frac{(abc)^2}{64s\sqrt{s(s-a)(s-b)(s-c)}}.$$

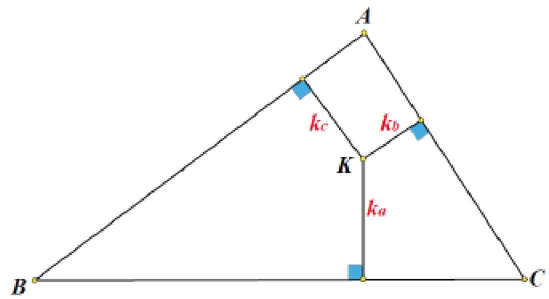
If $S = sr$ is the area of $\triangle ABC$, then $S = \sqrt{s(s-a)(s-b)(s-c)}$ and $abc = 4RS$, so it follows that

$$MM' \cdot NN' \cdot PP' \leq \frac{16R^2r^2s^2}{64rs^2} = \frac{R^2 r}{4}.$$

4533. *Proposed by Leonard Giugiuc and Kadir Altintas.*

Let K be the symmedian point of ABC. Let k_a, k_b and k_c be the lengths of the altitudes from K to the sides BC, AC and AB, respectively. If r is the inradius and s is the semiperimeter, prove that

$$\left(\frac{1}{r}\right)^2 + \left(\frac{3}{s}\right)^2 \geq \frac{2}{k_a^2 + k_b^2 + k_c^2}.$$



We received 16 submissions, all of which were correct and complete.

On the lefthand side of the inequality to be proved, a '+' was printed instead of a '-', what was intended by the proposers. This gives a strictly weaker result. Many solvers suspected this misprint and solved the intended problem. The techniques used in the solutions of the misprinted problem are enough (with minor tweaks) to prove the intended problem, and so we count such solutions as correct.

We present the solution by Debmalya Biswas. Almost all solutions used similar ideas.

Let a, b, c be the lengths of the sides BC, AC, AB, respectively. Also let Δ denote the area of the triangle ABC. We use the following well-known fact about the symmedian point (also called the Lemoine point), namely

$$\frac{k_a}{a} = \frac{k_b}{b} = \frac{k_c}{c} = t. \tag{1}$$

Since $\Delta KAB + \Delta KBC + \Delta KAC = \Delta$, we have

$$ak_a + bk_a + ck_c = 2\Delta,$$

and so from (1):

$$(a^2 + b^2 + c^2)t = 2\Delta, \quad \text{or} \quad t = \frac{2\Delta}{a^2 + b^2 + c^2}.$$

This gives

$$k_a^2 + k_b^2 + k_c^2 = (at)^2 + (bt)^2 + (ct)^2 = \frac{4\Delta^2}{a^2 + b^2 + c^2}. \tag{2}$$

Now the inequality to be proved:

$$\left(\frac{1}{r}\right)^2 - \left(\frac{3}{s}\right)^2 \geq \frac{2}{k_a^2 + k_b^2 + k_c^2},$$

by (2) is equivalent to

$$2s^2 - 18r^2 \geq a^2 + b^2 + c^2. \tag{3}$$

By Heron's area formula, we have that

$$r^2 s^2 = s(s-a)(s-b)(s-c).$$

Upon substitution of r^2, (3) becomes

$$2s^2 - \frac{18(s-a)(s-b)(s-c)}{s} \geq a^2 + b^2 + c^2. \qquad (4)$$

Define the positive reals $x = s-a, y = s-b, z = s-c$. Now (4) is equivalent to

$$2(x+y+z)^2 - \frac{18}{xyz}x+y+z \geq (x+y)^2 + (y+z)^2 + (x+z)^2,$$

which simplified and rearranged is

$$(x+y+z)(xy+yz+xz) \geq 9xyz.$$

The above holds by the AM-GM inequality applied to both factors on the lefthand side above. Equality holds if and only if $x = y = z$, or $a = b = c$, i.e. ABC is an equilateral triangle.

4534. *Proposed by Michel Bataille.*

For $n \in \mathbb{N}$, evaluate

$$\frac{\sum\limits_{k=0}^{\infty} \frac{1}{k!(n+k+1)}}{\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!}}.$$

We received 20 submissions, all correct. We present the solution by Marie-Nicole Gras, enhanced slightly by the editor.

Let $S_n = \sum\limits_{k=0}^{\infty} \frac{1}{k!(n+k+1)}$ and $T_n = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!}$. We prove that

$$\frac{S_n}{T_n} = n!e. \qquad (1)$$

Since $\sum_{k=0}^{\infty} \frac{1}{k!} = e$, we have

$$\begin{aligned}(n+1)S_n =& \sum_{k=0}^{\infty} \frac{(n+1+k)-k}{k!(n+k+1)} \\ =& \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=1}^{\infty} \frac{k}{k!(n+k+1)} = e - \sum_{k=1}^{\infty} \frac{1}{(k-1)!(n+k+1)} \\ =& e - \sum_{k=0}^{\infty} \frac{1}{k!(n+1+k+1)} = e - S_{n+1}. \qquad (2)\end{aligned}$$

Next, we derive some relation between S_n and S_{n+1} as well as some relation between T_n and T_{n+1}. We have

$$\begin{aligned}(n+1)S_n &= \sum_{k=0}^{\infty} \frac{(n+1+k)-k}{k!(n+k+1)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=1}^{\infty} \frac{1}{(k-1)!\big[(n+1)+(k-1)+1\big]} \\ &= e - \sum_{k=0}^{\infty} \frac{1}{k!(n+1+k+1)} = e - S_{n+1}. \end{aligned} \tag{3}$$

Hence,

$$\frac{S_{n+1}}{(n+1)!} + \frac{S_n}{n!} = \frac{e}{(n+1)!}. \tag{4}$$

Also,

$$\begin{aligned}T_n &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k+1)!} \\ &= \frac{1}{(n+1)!} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\big[(n+1)+(k-1)+1\big]!} \\ &= \frac{1}{(n+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{((n+1)+(k+1))!} = \frac{1}{(n+1)!} - T_{n+1}. \end{aligned} \tag{5}$$

Now we are ready to prove (1) by induction on n.

When $n = 0$, we have

$$S_0 = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} = e - 1$$

and

$$T_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} = 1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} = 1 - e^{-1},$$

so

$$\frac{S_0}{T_0} = \frac{e-1}{1-e^{-1}} = e = 0!e.$$

Suppose $\dfrac{S_n}{T_n} = n!e$ for some $n \in \mathbb{N} \cup \{0\}$. Then

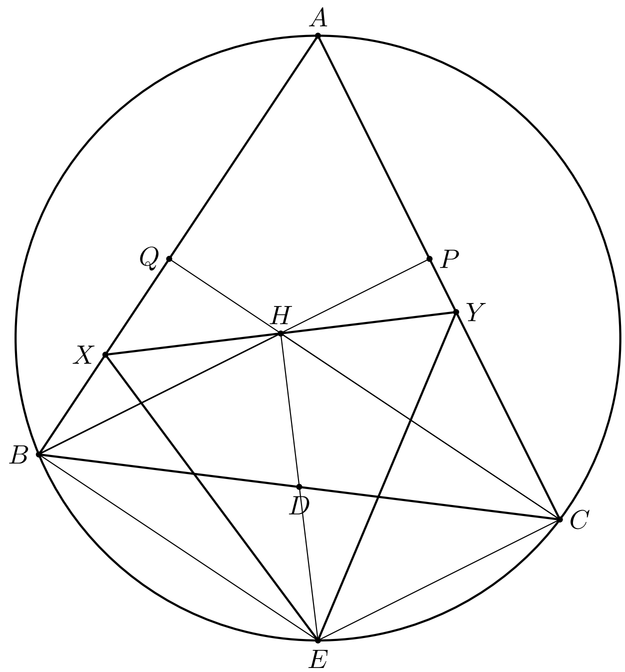
$$\frac{S_n}{n!} = eT_n \implies \frac{S_{n+1}}{(n+1)!} = \frac{e}{(n+1)!} - \frac{S_n}{n!} = \frac{e}{(n+1)!} - eT_n = eT_{n+1}$$

so $\dfrac{S_{n+1}}{T_{n+1}} = n!e$, completing the induction and the proof.

4535. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with orthocenter H, and let E be the reflection of H in the midpoint D of side BC. If the perpendicular to DE at H intersects AB at X and AC at Y, prove that $HX \cdot EC + YC \cdot HE = EX \cdot BE$.

15 correct solutions were received. A further solution which made use of a computer package resulting in unnecessarily complicated workings was not accepted, in view of the transparent arguments that were available.



Solution 1, by Michel Bataille, Prithwijit De, Dimitrić Ivko, Madhav Modak, Ion Patrascu, Vedaant Srivastava, Muhammad Thoriq, and the proposer (independently).

Since $BD = DC$ and $HD = DE$, then $BECH$ is a parallelogram, so that BE is equal and parallel to HC. Since $BE \| CH$ and $CH \perp AB$, then $BE \perp AB$ and $\angle ABE = 90°$. Similarly, $\angle ACE = 90°$. Thus, $ABEC$ is concyclic and E lies on the circumcircle of ABC.

Since the opposite angles at H and C are right, $ECYH$ is concyclic and $\angle HEY = \angle HCY = 90° - \angle BAC$. Similarly, $\angle HEX = \angle HBX = 90° - \angle BAC$, so that $\angle HEY = \angle HEX$. Thus right triangles EHX and EHY are congruent and $EX = EY$, $HX = HY$.

Applying Ptolemy's theorem to $ECYH$, we obtain that

$$HX \cdot EC + YC \cdot HE = HY \cdot EC + YC \cdot HE = EY \cdot HC = EX \cdot BE.$$

Solution 2, by Jiahao Chen, Todor Zaharinov, and UCLan Cyprus Problem Solving Group (independently).

Let BP and CQ be altitudes (with common point H). Then B, C, P, Q lie on a circle with centre D. Since $DH \perp XY$, H is the midpoint of the chord through X and Y. Applying the Butterfly theorem to the chords BP and CQ through H, we find that $HX = HY$. Since HE right bisects XY, $EY = EX$. As in Solution 1, we show that $HC = BE$ and obtain the desired result.

Solution 3, by C.R. Pranesachar.

Suppose that O is the circumcentre of triangle ABC. Then $AH \| OD$ and $AH = 2OD$ (consider the central similarity through the centroid with factor -2 taking O to H and D to A). Let AO and HD intersect in F. Then, since triangles FOD and FAH are similar, $HF = 2DF$ and $AF = 2OF$, so that $HD = DF$ and $OF = AO$, a radius of the circumcircle. Thus $E = F$, E lies on the circumcircle and AE is a diameter. Hence $\angle YCE = \angle ACE = 90^\circ = \angle EHY$ and $HECY$ is concyclic. As in Solution 1, it can be shown that $XH = HY$ and Ptolemy's theorem can be used to obtain the result.

Solution 4, by Cristóbal Sánchez-Rubio.

Since $AX \perp CH$ and $XY \perp EH$, $\angle AXY = \angle EHC$. Since $AB \perp CH$ and $AC \perp EC$, $\angle BAC = \angle ECH$. Therefore, triangle AXY and CHE are similar. Also $\angle HAX = \angle HCD = 90^\circ - \angle ABC$, so that in the similarity, AH and CD correspond. Since CD is a median of triangle CHE, then AH is a median of triangle AXY, so that $XH = HY$ and so $EX = EY$. We can now apply Ptolemy's theorem to the concyclic quadrilateral to achieve the desired result.

Solution 5, by Marie-Nicole Gras.

As in the foregoing solutions, we show that $HECY$ is concyclic. Triangles AHY and CDE are similar, as are triangles HAX and DCH. This is because (1) $\angle HYA = 180^\circ - \angle HYC = \angle DEC$; (2) $\angle HAY = 90^\circ - \angle ACB = \angle DCE$; (3) $\angle AHX = 180^\circ - \angle AHY = 180^\circ - \angle CDE = \angle CDH$; and (4) $\angle HAX = 90^\circ - \angle ABC = \angle DCH$. Hence $HY : DE = HA : DC = HX : DH$, so that $HY = HX$. Therefore $EX = EY$ and we can apply Ptolemy's theorem to get the result.

Comment by the editor. It is interesting to explore when the configuration is possible, *i.e.*, when XY is internal to the triangle. Place the triangle in the coordinate plane: $A \sim (1, a)$, $B \sim (0, 0)$, $C \sim (2c, 0)$, where $b, c > 0$. Then $D \sim (c, 0)$ and $H \sim (1, (2c-1)/a)$. The triangle is right if and only if either (1) $c = 1$ or (2) $a = (2c-1)/a$ or $a^2 = 2c - 1$. It is acute if and only if $c > 1$ and $a^2 > 2c - 1$.

The slope of AC is $-a/(2c-1)$ and of HD is $-(2c-1)/(a(c-1))$. The segment XY perpendicular to HD will lie in the triangle if and only if the line HD is at least as steep as AC, or

$$\frac{2c-1}{a(c-1)} \geq \frac{a}{2c-1} \quad \Leftrightarrow \quad a^2 \leq \frac{(2c-1)^2}{c-1}.$$

The situation can be illustrated by some special cases. When $(a,c) = (3,5)$, we get a right triangle. When $(a,c) = (3,2)$, we find that $HD \| AC$, $X = B$, $Y = P$ and $BH = HP$. When $(a,c) = (4,3)$, then X falls outside AB.

4536. *Proposed by Leonard Giugiuc and Rovsan Pirkuliev.*

Let ABC be a triangle with $\angle ABC = 60^\circ$. Consider a point M on the side AC. Find the angles of the triangle, given that

$$\sqrt{3}BM = AC + \max\{AM, MC\}.$$

Of the 16 submissions, all but one were complete and correct. We feature the solution by Fahreezan Sheraz Diyaldin, slightly shortened by the editor.

We shall see that the angles of ΔABC are either $90^\circ, 60^\circ, 30^\circ$, or they are all 60°. Denote the circumcircle by Ω, with circumcenter O and circumradius R. Since point M is inside Ω, the power of M with respect to Ω equals $-AM \cdot MC = OM^2 - R^2$. By the triangle inequality we have

$$BM \leq BO + OM = R + \sqrt{R^2 - AM \cdot MC},$$

so that

$$\frac{AC + \max\{AM, MC\}}{\sqrt{3}} \leq R + \sqrt{R^2 - AM \cdot MC}. \tag{1}$$

By the Law of Sines,

$$R = \frac{AC}{2\sin\angle ABC} = \frac{AC}{2\sin 60^\circ} = \frac{AC}{\sqrt{3}}. \tag{2}$$

Because the vertices A and C play symmetric roles, we may assume (without loss of generality) that $AM \geq MC$. Substituting equation (2) into the inequality (1), we get the equivalent inequalities

$$\begin{aligned}
\frac{AC + AM}{\sqrt{3}} &\leq \frac{AC}{\sqrt{3}} + \sqrt{\frac{AC^2}{3} - AM \cdot MC} \\
\frac{AM}{\sqrt{3}} &\leq \sqrt{\frac{(AM + MC)^2}{3} - AM \cdot MC} \\
AM^2 &\leq AM^2 - AM \cdot MC + MC^2 \\
AM \cdot MC &\leq MC^2.
\end{aligned}$$

There arise two cases:

- Case 1: $MC = 0$.
  $MC = 0$ indicates that the points M and C coincide, in which case

$$BC = BM = \frac{AC + AM}{\sqrt{3}} = \frac{2AC}{\sqrt{3}} = 2R.$$

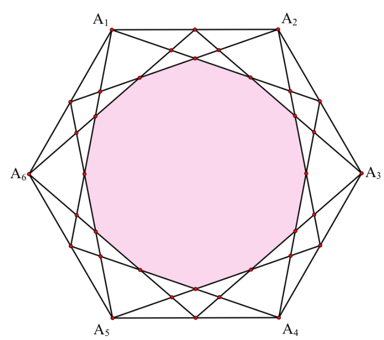
  This implies that BC is the diameter of Ω, whence $\angle BAC$ must be a right angle and $\angle BCA = 30°$.

- Case 2: $MC \neq 0$.
  Now the last inequality further reduces to $AM \leq MC$; but we also have $AM \geq MC$, whence $AM = MC$. In other words, M is the midpoint of AC. Moreover, the inequalities are all equalities; in particular, $BM = BO + OM$, which implies that M is on the line BO, and $BM \perp AC$. Because M is the midpoint of AC as well as the foot of the perpendicular from B, we deduce that $BA = BC$ which, together with $\angle ABC = 60°$, implies that ABC is an equilateral triangle (and all angles are $60°$). This completes the argument.

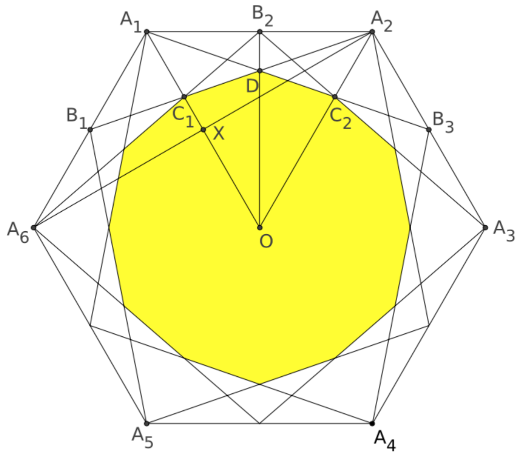
Editor's comment. Some geometers exclude the vertices in their definition for the side of a triangle. Readers who used that convention concluded that only the equilateral triangle satisfied all the requirements of the problem.

4537. *Proposed by Arsalan Wares.*

Let A be a regular hexagon with vertices A_1, A_2, A_3, A_4, A_5 and A_6. The six midpoints on the six sides of hexagon A are connected to the six vertices with 12 line segments as shown. The dodecagon formed by these 12 line segments has been shaded. What part of hexagon A has been shaded?



We received 21 submissions, out of which twelve were correct and complete. The incorrect submissions all overlooked that the dodecagon in the question is not regular. We present the solution by the Missouri State Problem Solving Group, lightly edited.



Let O be the center of the hexagon and B_1, B_2, B_3 be the midpoints of A_6A_1, A_1A_2 and A_2A_3. By symmetry, the lines A_6B_2, A_2B_1, and OA_1 intersect in a point, which we call C_1. Similarly we define C_2 as the intersection of A_1B_3, A_3B_2, and OA_2 and we let D be the intersection of A_2B_1, A_1B_3, and OB_2. Finally we define X as the intersection of A_6A_2 and OA_1. Since $OA_6A_1A_2$ is a rhombus, X is also the midpoint of OA_1.

The hexagon can be partitioned into twelve triangles that are all congruent to ΔOA_1B_2, whereas the dodecagon can be partitioned into twelve triangles, all congruent to ΔOC_1D. Therefore the ratio of the area of the dodecagon to that of the hexagon is equal to the ratio of the area of ΔOC_1D to that of ΔOA_1B_2.

Consider the triangle $A_1A_2A_6$. Since A_2B_1 and A_6B_2 are medians, C_1 is the centroid. Therefore $|A_1C_1| = \frac{2}{3}|XA_1| = \frac{1}{3}|OA_1|$. Similarly $|A_2C_2| = \frac{1}{3}|OA_2|$. Consider the triangle OA_1A_2. The point D is the intersection of the three cevians OB_2, A_1C_2, and A_2C_1. Therefore

$$\frac{OD}{DB_2} = \frac{OC_1}{C_1A_1} + \frac{OC_2}{C_2A_2} = 2 + 2 = 4,$$

and thus $|OD|/|OB_2| = 4/5$. Finally, using the sine law,

$$\begin{aligned} \frac{[OC_1D]}{[OA_1B_2]} &= \frac{\frac{1}{2} \cdot |OC_1| \cdot |OD| \cdot \sin \angle C_1OD}{\frac{1}{2} \cdot |OA_1| \cdot |OB_2| \cdot \sin \angle A_1OB_2} \\ &= \frac{|OC_1}{|OA_1|} \cdot \frac{|OD|}{|OB_2|} \\ &= \frac{2}{3} \cdot \frac{4}{5} \\ &= \frac{8}{15}. \end{aligned}$$

Therefore the ratio of the area of the dodecagon to the area of the hexagon is $8/15$.

4538. *Proposed by Nguyen Viet Hung.*

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Prove that

$$\sum_{1\le i\le n} \sqrt{1+{a_i}^2} + \sum_{1\le i<j\le n} a_i a_j \ge n-1+\sqrt{1+(\sum_{1\le i\le n} a_i)^2}.$$

When does equality occur?

We received 16 submissions, of which 13 were correct and complete. We present two solutions.

Solution 1 by Jiahao Chen, slightly edited.

We will prove the given inequality by induction on n, and show that equality occurs if and only if $n-1$ of the numbers $a_1, \ldots, a_n$ are zero.

When $n = 1$, equality holds trivially. Let us consider the case $n = 2$; to simplify the presentation, we use x and y instead of a_1 and a_2. That is, we have to prove that

$$\sqrt{1+x^2} + \sqrt{1+y^2} + xy \ge 1 + \sqrt{1+(x+y)^2},$$

which by rearranging is equivalent to

$$\sqrt{1+x^2} - 1 + xy \ge \sqrt{1+(x+y)^2} - \sqrt{1+y^2} \qquad \Leftrightarrow$$
$$\frac{x^2 + xy(\sqrt{1+x^2}+1)}{\sqrt{1+x^2}+1} \ge \frac{x^2+2xy}{\sqrt{1+(x+y)^2}+\sqrt{1+y^2}}. \tag{1}$$

The condition $x, y \ge 0$ allows us to note that the numerator on the left-hand side is greater than or equal to the numerator on the right-hand side, while the denominator on the left-hand side is smaller than or equal to the denominator on the right-hand side; moreover, when x and y are both non-zero, these inequalities are strict. Hence the inequality in (1) holds.

Consider non-negative real numbers $a_1, \ldots, a_n$ for $n = k+1$, where $k \ge 2$. Let $x = \sum_{j=1}^{k} a_j$. From the case $n = 2$ we know that

$$\sqrt{1+a_{k+1}^2} + \sqrt{1+x^2} + a_{k+1}x \ge 1 + \sqrt{1+(a_{k+1}+x)^2},$$

with equality if at least one of a_{k+1} and x are zero (note that $x = 0$ implies $a_j = 0$ for all $1 \le j \le k$). On the other hand, by the induction hypothesis, for $a_1, \ldots, a_k$ we have

$$\sum_{j=1}^{k} \sqrt{1+a_j^2} + \sum_{1\le i<j\le k} a_i a_j \ge (k-1) + \sqrt{1+x^2},$$

with equality if and only if $k-1$ of the a_j's are zero.

Combining these two inequalities gives us the desired result.

Solution 2 by Marian Dincă, completed and corrected by the editor.

Note that

$$\sum_{1\le i<j\le n} a_i a_j = \frac{1}{2}\left((\sum_{i=1}^n a_i)^2 - \sum_{i=1}^n a_i^2 \right).$$

The inequality we want to prove can thus be rewritten as

$$\sum_{i=1}^n \left(\sqrt{1+a_i^2} - \frac{a_i^2}{2} - 1 \right) \ge \sqrt{1+\left(\sum_{i=1}^n a_i\right)^2} - \frac{1}{2}\left(\sum_{i=1}^n a_i\right)^2 - 1.$$

Define $f : [0,\infty) \to \mathbb{R}$ by $f(x) = \sqrt{1+x^2} - \frac{x^2}{2} - 1$; note $f(0) = 0$. We need to show that

$$\sum_{i=1}^n f(a_i) \ge f(\sum_{i=1}^n a_i).$$

We will use Jensen's Inequality. First we will show that the function f is concave. We have

$$f'(x) = \frac{x}{\sqrt{1+x^2}} - x \text{ and}$$
$$f''(x) = \frac{1}{(1+x^2)^{3/2}} - 1,$$

so $f''(x) < 0$ for $x > 0$, concluding the proof that f is concave. For $1 \le j \le n$ we have $0 \le a_j \le \sum\limits_{i=1}^n a_i$, so there exists $\lambda_j \in [0,1]$ such that $a_j = \lambda_j \cdot \sum\limits_{i=1}^n a_i$. Note that if $\sum\limits_{i=1}^n a_i \ne 0$ we must have $\sum\limits_{j=1}^n \lambda_j = 1$; we can ensure this is always the case by choosing $\lambda_1 = \ldots = \lambda_n = \frac{1}{n}$ when $a_i = 0$ for all i. Applying Jensen's Inequality to each term we have

$$\begin{aligned}\sum_{i=1}^n f(a_i) &= \sum_{i=1}^n f\left(\lambda_i \sum_{j=1}^n a_j + (1-\lambda_i)0\right)\\ &\ge \sum_{i=1}^n \lambda_i f(\sum_{j=1}^n a_j) + (1-\lambda_i) f(0)\\ &= f\left(\sum_{j=1}^n a_j\right).\end{aligned}$$

Clearly if $a_1 = \cdots = a_n = 0$ then equality holds. Suppose there exists a k such that $a_k \ne 0$. From the definition of the λ_j's, we must also have $\lambda_k \ne 0$. Then Jensen's Inequality will give us a strict inequality for that term (and hence for the overall inequality) unless $\lambda_k = 1$. If $\lambda_k = 1$ then $\lambda_j = 0$ for all other j (and thus $a_j = 0$ for $j \ne k$), which means that equality holds for all the other terms as well. In conclusion, equality holds for the given inequality if and only if at most one of the a_n's are non-zero.

4539. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle with centroid G, incircle ω, circumradius R and semiperimeter s. Show that $24R\sqrt{6} \geq 25s$ given that G lies on ω.

We received 12 submissions, all correct, and feature the solution by Theo Koupelis.

Let a, b, c, F be the side lengths and area, respectively, of the triangle. It is known (see, for example, page 51 of O. Bottema et al., *Geometric Inequalities* (1968)), that the distance between the centroid G and incenter I is given by

$$9GI^2 = 5r^2 - 16Rr + s^2,$$

where r is the radius of ω. Because $GI = r$ is given, it follows that $4r^2+16Rr-s^2 = 0$ or, recalling that the area is given by $F = rs = \frac{abc}{4R}$, we have

$$4F^2 + 4sabc - s^4 = 0.$$

Making the substitutions $a = x + y, b = y + z$, and $c = z + x$ (where x, y, z are positive for a nondegenerate triangle), we get

$$s = x + y + z, \quad F = \sqrt{(x + y + z)xyz},$$

$$abc = (x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) - xyz.$$

Therefore, the given condition becomes

$$(x + y + z)^2 = 4(xy + yz + zx), \tag{1}$$

or

$$x^2 + y^2 + z^2 = 2(xy + yz + zx).$$

Treating the last equation as a quadratic in z, we find that G lies on the incircle if and only if

$$z_{\pm} = (\sqrt{x} \pm \sqrt{y})^2 = x + y \pm 2\sqrt{xy}. \tag{2}$$

We observe that the required inequality, namely $24R\sqrt{6} \geq 25s$, is equivalent to $Fs \leq \frac{6\sqrt{6}}{25}abc$. Squaring and using $4abc = (x + y + z)^3 - 4xyz$ (from equation (1)), the inequality becomes

$$\left[(x + y + z)^3 - 54xyz\right]\left[(x + y + z)^3 - \frac{8}{27}xyz\right] \geq 0.$$

However by AM-GM, $(x + y + z)^3 \geq 27xyz > \frac{8}{27}xyz$, so that finally, the given inequality becomes

$$(x + y + z)^3 \geq 54xyz.$$

This last inequality clearly holds when $G \in \omega$ because (from (2))

$$(x + y + z_{\pm})^3 - 54xyz_{\pm} = 2(\sqrt{x} \mp \sqrt{y})^2(2\sqrt{x} \pm \sqrt{y})^2(\sqrt{x} \pm 2\sqrt{y})^2.$$

Equality occurs if and only if $(x, y, z) = (t, t, 4t)$ with $t > 0$, or its cyclic permutations, which leads to an isosceles triangle with side lengths $2t, 5t$, and $5t$.

4540. *Proposed by Prithwijit De.*

Given a prime p and an odd natural number k, do there exist infinitely many natural numbers n such that p divides $n^k + k^n$? Justify your answer.

We received 26 submissions of which 25 were correct and complete. We present the solution by the Missouri State University Problem Solving Group and Roy Barbara (done indepedently), slightly modified.

The answer is positive. We show the existence by constructing n explicitly. There are the following two cases.

- If $p \mid k$, then one can take n to be any multiple of p.
- If $p \nmid k$, then one can take any positive integer

  $$n \equiv p - 1 \pmod{p(p-1)}.$$

  Since $n \equiv -1 \pmod{p}$ and k is odd, we have $n^k \equiv (-1)^k \equiv -1 \pmod{p}$. Since $n \equiv 0 \pmod{p-1}$ and $p \nmid k$, by Fermat's little theorem, $k^n \equiv 1 \pmod{p}$. Hence, $n^k + k^n \equiv -1 + 1 \equiv 0 \pmod{p}$.

Editor's Comment. As pointed out by UCLan Cyprus Problem Solving Group, the condition that k is odd is necessary. For example, if $k = 4$ and $p = 3$, then $n^4 + 4^n$ is never a multiple of 3 since $n^4 + 4^n \equiv 2 \pmod{3}$ when $3 \nmid n$ and $n^4 + 4^n \equiv 1 \pmod{3}$ when $3 \mid n$.