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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

This Volume marks the first year of *Crux* as an open access publication and we owe a big thank you to Intact Foundation for their support.

Unsurprisingly, as a result of open access, we have been witnessing a continuous increase in the number of submissions and have been adjusting to the new volume of materials coming in. With our increased reach, we also decided to bring back materials that target wider audiences, which resulted in the creation of a new section *MathemAttic* aimed at secondary high school students and teachers. We welcome feedback and submissions to the section; you can reach its editors at `mathemattic@cms.math.ca`. As usual, we are aways looking for article submissions and I'd like to draw your attention to Robert Dawson's "How To Write A *Crux* Article... Revisited!" in this issue to familiarize yourself with our new expectations and larger variety of acceptable pieces. We also moved to a shorter problems-to-solutions cycle and, as a result, tightened our submission deadlines (watch out for those!).

Every year when I look back at the latest completed Volume, I am amazed by the amount of materials and resources that goes through this publication. I am also constantly humbled by the expertise and impressed with the flexibility and responsiveness of my editors. I am immensely grateful for their support. In light of the increased volume of materials, we will be looking to expand the Editorial Board, so look out for the CMS call for *Crux* editors or email me directly at `crux.eic@gmail.com` if you would like to apply to join this outstanding group of individuals.

Here is to many more mathematics to be discovered and re-discovered on the pages of *Crux*.

Kseniya Garaschuk

MathemAttic

No. 10

The problems featured in this section are intended for students at the secondary school level.

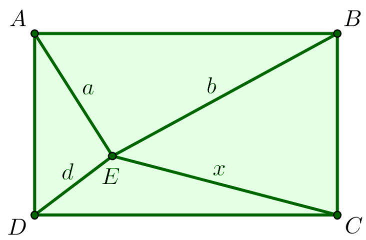
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2020**.

MA46. If both x and y are integers, determine all solutions (x, y) for the equation

$$(x-8)\cdot(x-10) \;=\; 2^y.$$

MA47. Let E be any point in rectangle $ABCD$.



Express x in terms of a, b and d.

MA48. Given any triangle ABC where AD is a median of length m, prove that $4m^2 = b^2 + c^2 + 2bc\cos A$.

MA49. Given that the perimeters of an equilateral triangle T and a square S are equal, determine the ratio of the area of the equilateral triangle T to the area of the square S.

MA50. A family of straight lines is determined by the condition that the sum of the reciprocals of the x and y intercepts is a constant k. Show that all members of the family are concurrent and find the coordinates of their point of intersection.

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

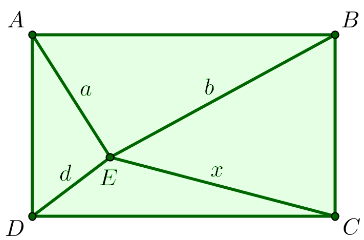
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 février 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA46. Déterminer toutes les solutions entières (x, y) à l'équation

$$(x-8)\cdot(x-10) \;=\; 2^y.$$

MA47. Soit E un point à l'intérieur d'un rectangle $ABCD$.



Exprimer x en termes de a, b et d.

MA48. Si m est la longueur de la médiane AD dans un triangle ABC, démontrer que $4m^2 = b^2 + c^2 + 2bc\cos A$.

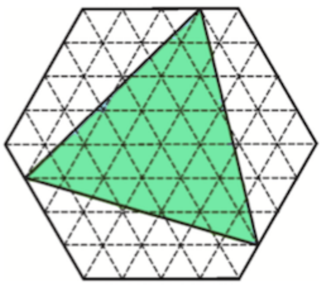
MA49. Sachant que le périmètre d'un triangle équilatéral T et d'un carré S sont égaux, déterminer le ratio de la surface du triangle équilatéral T par rapport à la surface du carré S.

MA50. Une famille de lignes droites est définie par la condition que la somme des réciproques d'interceptions avec l'axe des x et l'axe des y soit une constante k. Démontrer que les lignes de cette famille sont concourantes et trouver les coordonnées de ce point commun d'intersection.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(5), p. 226–229.

MA21. An equilateral triangle is inscribed into a regular hexagon as shown below. If the area of the hexagon is 96, find the area of the inscribed triangle.



Originally Problem 11 of the 2017 Savin contest.

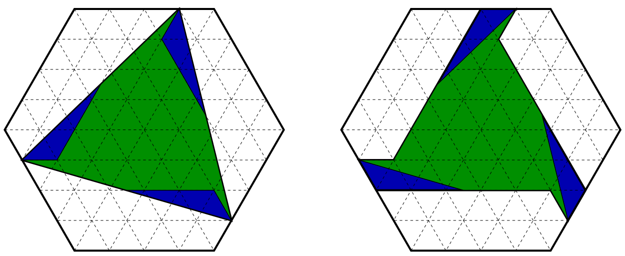
We received 5 solutions. We present the solution by Richard Hess and a graphical solution by the editor.

Solution 1, by Richard Hess.

The area of a small equilateral triangle is 1, since there are 96 such triangles in the hexagon. Let the side of each small equilateral triangle be s. Then $\sqrt{3}s^2/4 = 1$. Let the coloured equilateral triangle have side length d. From the figure we can see that there exists a triangle with side lengths d, $5s$, and $2s$ with a 120° angle opposite d. From the cosine law we obtain $d^2 = (2s)^2 + (5s)^2 + (2s)(5s) = 39s^2$. The area of the coloured triangle is then $\sqrt{3}d^2/4 = 39\sqrt{3}s^2/4 = 39$.

Solution 2, by the editor.

The area of each small triangle is 1. By cutting and rearranging the large triangle (see Figure), we observe that the area of the large triangle is equal to the area of 39 small triangles.



MA22. Do there exist three positive integers a, b and c for which both $a+b+c$ and abc are perfect squares? Justify your answer.

Originally Problem 1 of grade 8 of 2018 LXXXI Moscow.

We received 10 solutions. We showcase several of them here.

Among the solutions were the following infinite sequences of numbers (a, b, c):

- $(1, 2^{2k+2}, 2^{4k+2})$ with $k \in \mathbb{N}$ [Šefket Arslanagić];
- $((2mp - 2nq)^2, (2mq + 2np)^2, (m^2 + n^2 - p^2 - q^2)^2)$, with $m, n, p, q \in \mathbb{N}$ [Lorenzo Benedetti];
- $(1, y^2, y^2)$, where y satisfies $x^2 - 2y^2 = 1$ (Pell's equation) [Joel Schlosberg];
- $(1, 2n^2 + 2n, 2n^2 + 2n)$ with $n \in \mathbb{N}$ [Digby Smith];
- $(4x^2, 4y^2, (x^2 + y^2 - 1)^2)$, with $x, y \in \mathbb{N}$ [Konstantin Zelator].

MA23. Integer numbers were placed in squares of a 4×4 grid so that the sum in each column and the sum in each row are all equal. Seven of the sixteen numbers are known as shown below, while the rest are hidden.

1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Show that it is possible to determine at least one of the missing numbers. Is it possible to determine more than one of the missing numbers?

Originally Problem 2 of Spring Junior A-level of XXXIX Tournament of Towns.

We received 4 solutions, 3 of which were correct. We present the solution by Joel Schlosberg, modified by the editor.

We first show that it is possible to determine at least one of the missing squares. Assign variables to the hidden numbers as follows:

1	a	b	2
c	4	5	d
e	6	7	f
3	g	h	i

Then

$$\begin{aligned} 0 &= 1 + a + b + 2 - [c + 4 + 5 + d] - [e + 6 + 7 + f] + [3 + g + h + i] \\ &\quad + [1 + c + e + 3] - [a + 4 + 6 + g] - [b + 5 + 7 + h] + [2 + d + f + i] \\ &= 2i - 32, \end{aligned}$$

so the number in the lower-right corner must be 16.

The following two possibilities show that the values of all of the other hidden squares can differ while satisfying the given requirements, and so none of them can be determined without additional information:

1	15	16	2
14	4	5	11
16	6	7	5
3	9	6	16

1	16	15	2
16	4	5	9
14	6	7	7
3	8	7	16

MA24. You have 5 cards with numbers 3, 4, 5, 6 and 7 written on their backs. How many five digit numbers are divisible by 55 with the digits 3, 4, 5, 6, and 7 each appearing once in the number (the card with 6 cannot be rotated to be used as a 9)?

Originally Problem 17 of the 2019 Savin contest.

We received 4 solutions, all of which were correct. We present the joint solution by Sarvin Malekaninejad and Shashwat Mookherjee, completed independently, modified by the editor.

There are 4 numbers which satisfy the conditions. To see this, consider the number $a_4a_3a_2a_1a_0$ which is divisible by 55 with the digits 3, 4, 5, 6, and 7 each appearing once in the number. By hypothesis, it is clear that $a_4a_3a_2a_1a_0$ is divisible by 5 and 11. Since $a_4a_3a_2a_1a_0$ is divisible by 5, we infer that $a_0 = 5$ (0 not being an available digit). On the other hand, since $a_4a_3a_2a_1a_0$ is divisible by 11, by the test of divisibility, we conclude that

$$(a_4 + a_2 + 5) - (a_3 + a_1) \equiv 0 \pmod{11}.$$

One can easily check that the numbers 74635, 73645, 64735 and 63745 are all the numbers that satisfy the conditions.

MA25.

1. Find four consecutive natural numbers such that the first is divisible by 3, the second is divisible by 5, the third is divisible by 7 and the fourth is divisible by 9.
2. Can you find 100 consecutive natural numbers such that the first is divisible by 3, the second is divisible by 5, . . . the 100th is divisible by 201?

Originally Problem 19 of the 2018 Savin contest.

We received 3 correct and complete submissions. Presented is the one by Corneliu Manescu-Avram.

1. We have $159 = 3 \cdot 53$, $160 = 5 \cdot 32$, $161 = 7 \cdot 23$, and $162 = 9 \cdot 18$.
2. If
$$n = \frac{lcm(3, 5, 7, \ldots, 201) + 3}{2},$$

then n is a positive integer and

$$n + k - 1 = \frac{lcm(3, 5, 7, \ldots, 201) + 2k + 1}{2}$$

is divisible by $2k + 1$ for $1 \le k \le 100$. Taking an odd multiple of the least common multiple instead we obtain infinitely many n with this property.

Math Quotes

The biologist can push it back to the original protist, and the chemist can push it back to the crystal, but none of them touch the real question of why or how the thing began at all. The astronomer goes back untold million of years and ends in gas and emptiness, and then the mathematician sweeps the whole cosmos into unreality and leaves one with mind as the only thing of which we have any immediate apprehension. Cogito ergo sum, ergo omnia esse videntur. All this bother, and we are no further than Descartes. Have you noticed that the astronomers and mathematicians are much the most cheerful people of the lot? I suppose that perpetually contemplating things on so vast a scale makes them feel either that it doesn't matter a hoot anyway, or that anything so large and elaborate must have some sense in it somewhere.

Dorothy L. Sayers with R. Eustace, "The Documents in the Case", New York: Harper and Row, 1930, p 54.

PROBLEM SOLVING VIGNETTES

No. 9

Shawn Godin

To Assume or Not to Assume

The problem that follows was the September 2019 problem of the month from the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. The problem of the month is a new feature for the CEMC and consists of a problem that is meant to be quite challenging for high school students. One problem appears every month with a hint partway through the month and full solution at the end. You can check out the problem of the month and other CEMC features at `https://www.cemc.uwaterloo.ca`.

> *A point (x, y) in the plane is called a lattice point if it has integer coordinates. Points P, Q, and R are distinct lattice points. Prove that the measure of $\angle PQR$ cannot be 60°.*

This problem seems easy enough on the surface but you may be wondering "where do I begin?" Sometimes a problem is lacking information and we have to make an assumption in order to solve it. For example, if the problem asked: what is the next number in the sequence $1, 2, 4, 8, 16, \ldots$? In this case, since all we are given are the numbers, without context, we can only assume that any pattern we find continues to hold. Thus, if we see that the numbers given satisfy $t_{n+1} = 2t_n$, we can assume this continues to hold and the sequence continues $32, 64, 128, \ldots$. In problems of this sort, we must make an assumption in order to get an answer because the next number could be *anything*. For example, searching *The On-Line Encyclopedia of Integer Sequences* (`https://oeis.org`) yields our sequence (**A000079**: Powers of 2), as well as

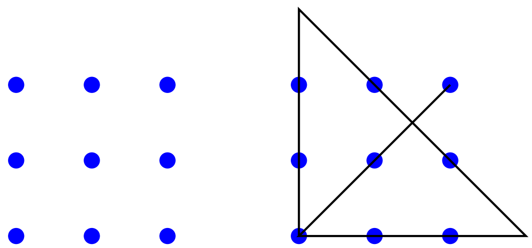
- **A027423**: Number of positive divisors of $n!$ (starting with $n = 1$): 1, 2, 4, 8, 16, 30, ...
- **A000127**: Maximal number of regions obtained by joining n points around a circle by straight lines. Also number of regions in 4-space formed by $n-1$ hyperplanes.: $1, 2, 4, 8, 16, 31, \ldots$.
- **A006261**: The sum of the first six terms of the n-th row in Pascal's triangle, i.e.

$$t_n = \sum_{i=0}^{5} \binom{n}{i}$$

  $1, 2, 4, 8, 16, 32, 63, \ldots$.

If you allow the sequence to start $1, 1, 2, \ldots$ there are a few more.

In other situations people make assumptions that lead to an incorrect solution. For example when asked to draw the smallest number of line segments so that at least one line segment has passed through each dot in the square array pictured below, many people will assume that the line segments have to stay within the square defined by the dots and conclude that five segments are needed. If we didn't make that assumption, we might have eventually found the four segment solution on the right. We can also better the solution to three if "dots" are meant to mean "small circles" and our page is large enough.



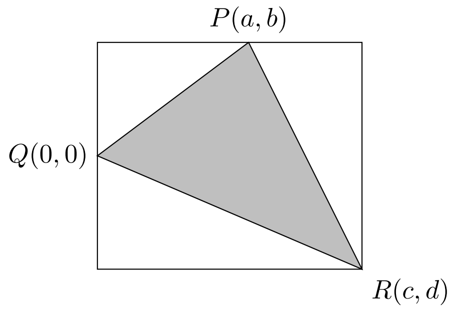
So we have seen assumptions that we need to make in order to solve a problem as well as assumptions that people sometimes make that change the conditions of the original problem and don't give the desired result. There are other "assumptions" that can be made that can simplify the problem, without changing it. Suppose, for example, we had $P(13, 9)$, $Q(9, 7)$ and $R(15, 4)$ and we wanted to measure the angle $\angle PQR$. What if we moved each point four units to the left and 2 units down to get $P'(9, 7)$, $Q'(5, 5)$ and $R'(11, 2)$ – would the angle change?

The angle stays the same because the angle is *invariant* under rigid transformations like translations, rotations and reflections. As a result, moving all the points by the same amount doesn't change the angle. We can say *without loss of generality*, assume that Q is at the origin. What this means is that we will assume that Q is $(0, 0)$. We do this, without loss of generality, because we could have had Q anywhere, then moved it to the origin to make the numbers easier and the problem would be unchanged. We can also argue that rotating our coordinate system by a multiple of 90° changes the location of the points, but not the size of the angle. Finally, notice that if we swap the coordinates of P and R the angle $\angle PQR$ remains unchanged. Thus, before we start solving the problem we will make the following assumption:

Assume, without loss of generality, that Q is at the origin, $P(a, b)$ is in the first quadrant and $R(c, d)$ is in the first or fourth quadrant such that the slope of QP is greater than the slope of QR.

Since we are only interested in the angle $\angle PQR$, then if $\gcd(a, b) = d > 1$, then $P'\left(\frac{a}{d}, \frac{b}{d}\right)$ is on the segment QP and hence $\angle PQR = \angle P'QR$. Similarly with the coordinates of R. So we can further assume, without loss of generality, that $\gcd(a, b) = \gcd(c, d) = 1$ and we are ready to start our proof.

Solution 1: Construct the triangle PQR and inscribe it in a rectangle whose sides are parallel to the coordinate axes.



Thus the side lengths of the rectangle are integer, which means so is its area. The rectangle is partitioned into four triangles. Three of the triangles are right angled with integer legs, which means the area of these triangles are either an integer or half of an integer. This means that twice the area of triangle PQR must be an integer.

But,

$$[PQR] = \frac{1}{2}|PQ||QR|\sin(\angle PQR) = \frac{\sin(\angle PQR)\sqrt{a^2+b^2)}\sqrt{c^2+d^2}}{2},$$

hence, if $\angle PQR = 60^\circ$

$$2[PQR] = \frac{\sqrt{3}}{2}\sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

Clearly the left side of the equation is an integer, so the right side must be as well. Since we have a factor of $\sqrt{3}$ on the right, we need either $3 \mid (a^2+b^2)$ or $3 \mid (c^2+d^2)$ in order to rationalize the $\sqrt{3}$. Since $1^2 \equiv 2^2 \equiv 1 \pmod 3$, the only way to get $3 \mid (a^2+b^2)$ is to have $3 \mid a$ and $3 \mid b$, which, since we assumed $\gcd(a,b) = 1$ is impossible. Similarly, we cannot have $3 \mid (c^2+d^2)$, so we cannot make the angle 60°.

Solution 2: As in solution 1, we assume that Q is at the origin, $P(a,b)$ is in the first quadrant and the slope of QR is smaller than the slope of PQ. Then let θ and ϕ represent the *directed* angles between the positive x-axis and QP and QR, respectively. That is, if R is in the first quadrant $\phi > 0$, and if R is in the fourth quadrant $\phi < 0$.

Thus, $\angle PQR = \theta - \phi$, but $\tan\theta = \frac{b}{a}$ and $\tan\phi = \frac{d}{c}$. Hence

$$\begin{aligned}\tan(\angle PQR) &= \tan(\theta-\phi)\\ &= \frac{\tan(\theta)-\tan(\phi)}{1+\tan(\theta)(\tan(\phi)}\\ &= \frac{\frac{b}{a}-\frac{d}{c}}{1+\frac{bd}{ac}}\end{aligned}$$

which is defined and rational as long as $1 + \frac{bd}{ac} \neq 0$. Given that $\tan 60^\circ = \sqrt{3}$ is not rational, and if $1 + \frac{bd}{ac} = 0$, we would get $\angle Q = 90^\circ$, then we conclude that $\angle Q \neq 60^\circ$.

Judicially use assumptions in your proofs where they simplify the process without altering the problem itself. You may want to check out the official solutions to this problem which are slight variations on mine (or mine is a variation of theirs). For your enjoyment, the current problem of the month, for December, is reproduced below. You may want to check the others out.

> *Let a, b, c, and d be rational numbers and $f(x) = ax^3 + bx^2 + cx + d$. Suppose $f(n)$ is an integer whenever n is an integer and that*
>
> $$\frac{1}{3}n^3 - n - \frac{2}{3} \leq f(n) \leq \frac{1}{3}n^3 + n^2 + 2n + \frac{4}{3}$$
>
> *for every integer n with the possible exception of $n = -2$.*
> *(a) Show that $a = \frac{1}{3}$.*
> *(b) Find $f(10^{2019}) - f(10^{2019} - 1)$.*

TEACHING PROBLEMS

No. 7

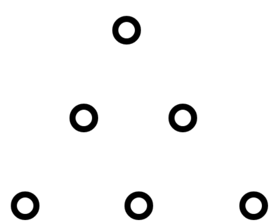
John McLoughlin

Magic Triangles as a Starting Point

One of the challenges in teaching mathematics is allowing time for processing while ensuring that individuals finishing quickly do not diminish opportunities for satisfaction and accomplishment of others. For instance, blurting out an answer diminishes both motivation and satisfaction for others attempting to determine a solution. One avenue to address this concern is to provide problems with multiple solutions. An elementary example of this is posed here with the challenge being to create a *magic triangle*. Magic triangles have equal sums of numbers along each of the three sides.

Problem

Using the digits 1, 2, 3, 4, 5, and 6, fill in the circles so that the numbers along each side of the triangle add up to the same amount.



The Solving Process

Typically people play with the numbers moving them around until a magic triangle appears, or some frustration perhaps emerges. By that time someone in the class will have found a magic triangle. The instinct upon finding a solution is to stop, until either another person gets a triangular arrangement with a different magic sum, or a teacher prompts the class suggesting multiple solutions are possible.

Suppose that the first person mentions getting a magic triangle with sums of 11 along each side, and then the next person gets a magic sum of 9. As a teacher, my inclination would be to tell all of the students that it is possible to find a magic triangle with a sum of 9 along each side while mentioning that there are more sums that work. Those of you looking for a first solution can work with the total being 9, whereas, others can try to find more solutions.

The pedagogical value of the problem is enhanced as students engage at a level suited to their own experience with the problem to that point. Further, the class has a common goal of trying to identify the various solutions. We will reach a place where magic triangles with sums of 9, 10, 11 and 12 have been found. This represents all of the possible solutions.

The Underlying Mathematics

How do we know that there are no other solutions? Would it be practical for us to find these solutions without being so random in our approach? How might the mathematics figure into this problem?

Let us begin to consider the possible sums for the magic triangle using the digits from 1 through 6. First, we note that the sum of these numbers is 21. Secondly, note that each of the numbers will appear in one sum, and those in the vertices will appear a second time. Hence, the largest possible total of the three sums would be $21 + (4 + 5 + 6)$ and likewise, the smallest would be $21 + (1 + 2 + 3)$. The case with the larger total of 36 corresponds to a magic triangle with a sum of 12 along each side, whereas, the smaller total would require sums of 9 along the sides. Hence, the only possible magic sums are 9, 10, 11, and 12.

Consider one of the extreme cases as an example. Suppose we want the magic sum to be 9. Recall that necessitates the placement of 1, 2, and 3 in the vertices. The middle numbers are placed accordingly to produce sums of 9, as shown.

```
    1
  6   5
2   4   3
```

Suppose that we had wanted to know whether it would be possible to have a magic sum of 12. (Of course, we now know that it is.) How could we proceed?

Observe that $3 \times 12 = 36$ and $1 + 2 + 3 + 4 + 5 + 6 = 21$. Hence, the difference of 15 must be the sum of the numbers in the vertices. This can only be if we place the digits 4, 5, and 6 in the corners. (Note that with a triangle the ordering of these does not matter.)

```
    5
  1   3
6   2   4
```

It turns out that the two extreme cases are easier to analyze as they only give us one choice for the set of numbers that must appear in the vertices. Suppose that we wanted to consider whether it would be possible to make the magic sum 11. Proceeding as above, we require $3 \times 11 - 21 = 12$ as the sum of the numbers in the vertices. This gives three possibilities: $(1, 5, 6)$, $(2, 4, 6)$, and $(3, 4, 5)$. Mixed results will emerge with efforts to complete the triangles. For example, $5 + 6 = 11$

already without adding a third number. The $(3, 4, 5)$ arrangement is unworkable since the side with 3 and 4 will require a second 4 to make 11. Hence, only one magic triangle results with a sum of 11. It is shown below.

$$\begin{array}{ccccc} & & 2 & & \\ & 3 & & 5 & \\ 6 & & 1 & & 4 \end{array}$$

Analysis of the case with the magic sum of 10 is included in the concluding set of questions to consider.

Concluding comments

In summary, a problem that is accessible to students in elementary school can serve as a starting point for deeper investigation. The simplicity of the concept allows for valuable problem solving qualities such as playfulness, conjecturing and pattern seeking at the forefront. The relative absence of any mathematical complication allows for elaboration and development of the balance between finding a solution and analyzing a problem. The simplicity of the task brings attention to the process and the mathematics.

For those who want to delve deeper into the idea here, it is worth noting that any discussion becomes considerably more advanced by having four numbers along each side and using the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9. Suggestions are offered for consideration beginning with #3 in the closing set of ideas for consideration.

Ideas to Consider

1. Complete the analysis of the elementary magic triangle problem by considering all possible cases for the vertices in a magic triangle with a sum of 10 along each side.

2. The smallest and largest digits from our example are 1 and 6. Their sum is 7. Revisit the solutions for each of the magic sums (9, 10, 11, and 12) that worked. For each solution, create a complementary solution by replacing each number in the solution with its positive difference from 7. That is, for each of $x = 1, 2, 3, 4, 5$, and 6, replace x with $(7 - x)$. What do you notice?

3. Extend the ideas from the elementary example discussed to the next level using the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9. Each magic triangle will have four digits on each side in this case. Begin by determining the range of possible magic sums. What are the smallest and largest possible magic sums?

4. Select either the largest or smallest possible magic sums from Question 3, and create all possible magic triangles for that sum. Note that simply interchanging the middle two numbers (those not in the vertices) along a side will not be considered to be creating a new magic triangle.

5. Choose any other possible magic sum between the extreme values. Complete the analysis for that case and determine how many (if any) magic triangles with that sum are possible.

6. Observe that the extreme digits of 1 and 9 add to 10. Apply the ideas of Question 2 above to any of the magic triangles you found in questions 4 and/or 5. Comment on your findings.

The author welcomes feedback via email `johngm@unb.ca`. Comments from teachers or students would be particularly helpful as we move into the second year of this feature, *Teaching Problems*.

OLYMPIAD CORNER

No. 378

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **February 15, 2020**.

OC456. Solve the system of equations

$$\begin{aligned}(x^2+1)(x-1)^2 &= 2017yz\\(y^2+1)(y-1)^2 &= 2017zx\\(z^2+1)(z-1)^2 &= 2017xy,\end{aligned}$$

where $x \geq 1$, $y \geq 1$, $z \geq 1$.

OC457. On a blackboard are written the numbers $1!, 2!, 3!, \ldots, 2017!$. What is the smallest among these numbers that should be deleted so that the product of all the remaining numbers is a perfect square?

OC458. Let A be the product of eight consecutive positive integers and let k be the largest positive integer for which $k^4 \leq A$. Find the number k knowing that it is represented in the form $2p^m$, where p is a prime number and m is a positive integer.

OC459. Points P and Q lie respectively on sides AB and AC of a triangle ABC such that $BP = CQ$. Segments BQ and CP intersect at R. The circumcircles of triangles BPR and CQR intersect again at point S different from R. Prove that point S lies on the angle bisector $\angle BAC$.

OC460. Prove that the set of positive integers $\mathbb{Z}^+$ can be represented as a union of five pairwise disjoint subsets with the following property: each 5-tuple of numbers of the form $(n, 2n, 3n, 4n, 5n)$, where $n \in \mathbb{Z}^+$, contains exactly one number from each of these five subsets.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 février 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC456. Résoudre le système d'équations

$$\begin{aligned}(x^2+1)(x-1)^2 &= 2017yz\\(y^2+1)(y-1)^2 &= 2017zx\\(z^2+1)(z-1)^2 &= 2017xy,\end{aligned}$$

où $x \geq 1$, $y \geq 1$, $z \geq 1$.

OC457. Soient les nombres $1!, 2!, 3!, \ldots, 2017!$. Lequel parmi ces nombres pourrait être supprimé, de façon à ce que le produit des nombres restants soit un carré parfait ?

OC458. Soit A le produit de huit entiers positifs consécutifs et soit k le plus gros entier positif tel que $k^4 \leq A$. Déterminer le nombre k, prenant pour acquis qu'il est de la forme $2p^m$, où p est un nombre premier et m est un entier positif.

OC459. Les points P et Q se situent, respectivement, sur les côtés AB et AC d'un triangle ABC, de façon à ce que $BP = CQ$. Les segments BQ et CP intersectent en R. Enfin, les cercles circonscrits des triangles BPR et CQR intersectent de nouveau au point S, distinct de R. Démontrer que le point S se trouve sur la bissectrice de l'angle $\angle BAC$.

OC460. Démontrer que l'ensemble des entiers positifs $\mathbb{Z}^+$ peut être partitionné comme réunion de cinq ensembles disjoints avec la propriété suivante : chaque 5-tuple de nombres de la forme $(n, 2n, 3n, 4n, 5n)$, avec $n \in \mathbb{Z}^+$, contient exactement un nombre de chacun des cinq sous ensembles.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(5), p. 245–246.

OC431. All natural numbers greater than 1 are coloured with blue or red so that the sum of every two blue numbers (not necessarily distinct) is blue, and the product of every two red ones (not necessarily distinct) is red. It is known that the number 1024 is blue. What colour can the number 2017 be?

Originally from Moscow Math Olympiad, Problem 2, Grade 10, Final Round 2017.

We received 7 *submissions. We present* 2 *solutions.*

Solution 1, by Kathleen E. Lewis, modified by the editor.

The number 2017 can be red. We know that 1024 is blue, so 2 must be blue, because if 2 were red, every power of 2 would have to be red. Since 2 is blue, and the sum of two blues is blue, then every even number must be blue. But we cannot determine the colours of the odds. Since two colours are used, then one odd number is red, say a. Assume that $a \geq 5$. Since $a - (a-2) = 2$, then also $a-2$ must be red and applying the same reasoning to $a-2$ and so on, we get that all the odd numbers less than a are red. Now, all the powers a^n with $k = 1, 2, \dots$ are red and there is a power $a^k > 2017$ which is coloured with red. Using the same reasoning, we conclude that all the odd numbers less than a^k are red, therefore 2017 is red.

Solution 2, by the Missouri State University Problem Solving Group.

Assuming that both colours must be used (otherwise, every number could be coloured blue), we will show that 2017 must be red.

We will prove, more generally, that (assuming both colours must be used) the only valid colourings are ones in which all the multiples of a given prime number are blue and the rest of the numbers are red. We need a lemma.

Lemma. If $\gcd(a,b) = 1$ and a is blue, then b is red.

Proof. Suppose b were blue. Then every number of the form $ka + \ell b$ with integers $k, \ell \geq 1$ must be blue. It is well known that since $\gcd(a,b) = 1$, every integer greater than or equal to $N = (a-1)(b-1)$ is of this form. Therefore every integer greater than or equal to N must be blue. Now there must be some number c that is red (since we are assuming both colours are used) so c^m is red for all positive m. But for m sufficiently large, $c^m > N$ and so must be blue. This gives a contradiction, so b must be red.

Suppose the number n is blue. Then at least one of the primes dividing it must be blue (if all of the primes dividing n were red, n would be red). Denote this prime by p. All the multiples of p must be blue and any non-multiple is relatively prime to p so must be red by the lemma.

In the case at hand, since 1024 is blue, 2 must be blue, so the even numbers are blue and the odd numbers are red hence 2017 is red.

Editor's Comment. The original problem included the hypothesis that *it is known that when colouring the numbers both colours were used*. Unfortunately, this was missed in the translation by the Editor. We apologize for that.

OC432. Find the smallest natural number that is a multiple of 80 such that you can rearrange two of its distinct digits and the resulting number will also be a multiple of 80.

Originally from Moscow Math Olympiad, Problem 1, Grade 11, Final Round.

We received 3 *submissions, of which* 1 *was correct and complete. We present the solution by Oliver Geupel.*

A number with the desired property is 1520 because it is a multiple of 80, as well as the number 5120. We prove that 1520 is the smallest number with the required property.

The multiples of 80 that are less than 1000 are 80, 160, 240, 320, 400, 480, 560, 640, 720, 800, 880, and 960. By inspection, they all do not have the property. Hence, the least number with the desired property is a four digit number $1bc0$.

We cannot rearrange the leading 1 with the digit c, because a number that ends in 10 is not divisible by 80. If we can rearrange the leading 1 with the digit b, then the number $b1c0 - 1bc0$ is a multiple of 80. Hence, $80 \mid 900(b-1)$, that is, $4 \mid b-1$, so that $b \geq 5$. By inspection, $(b,c) = (5,0)$ fails, and $(b,c) = (5,2)$ yields the solution 1520.

Finally, if we can rearrange the digits b and c, then b and c are even digits such that $b < c$, and the number $1cb0 - 1bc0$ is a multiple of 80, that is $80 \mid 90(c-b)$. Hence, $c-b$ is divisible by 8, which leads to $b = 0$ and $c = 8$. But this is impossible, because the number 1080 is not divisible by 80.

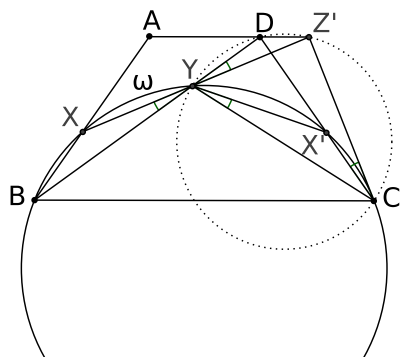
The proof is complete.

OC433. Consider an isosceles trapezoid $ABCD$ with bases AD and BC. A circle ω passing through B and C intersects the side AB and the diagonal BD at points X and Y, respectively. The tangent to ω at C intersects the line AD at Z. Prove that the points X, Y, and Z are collinear.

Originally from Moscow Math Olympiad, Problem 2, Grade 9, Final Round 2017.

We received 4 *submissions. We present the solution by Oliver Geupel.*

The line XY intersects the line AD at a point Z' (see figure on the next page). It is enough to show that $Z = Z'$.



We have

$$\begin{aligned} \angle CYZ' &= 180^\circ - \angle XYC && \text{because } X,\, Y,\, Z' \text{ are collinear} \\ &= \angle CBX && \text{because } B,\, C,\, Y,\, X \text{ are concyclic} \\ &= 180^\circ - \angle ADC && \text{because } ABCD \text{ is an isosceles trapezoid} \\ &= \angle CDZ' && \text{because } A,\, D,\, Z' \text{ are collinear.} \end{aligned}$$

Hence, the points C, Z', D, and Y are concyclic.

The circle ω intersects the side CD at a point X'. It holds

$$\begin{aligned} \angle Z'CD &= \angle Z'YD && \text{because } C,\, Z',\, D,\, Y \text{ are concyclic} \\ &= \angle XYB && \text{because } Y \text{ is intersection of } BD \text{ and } XZ' \\ &= \angle CYX' && \text{because arcs } XB \text{ and } CX' \text{ have equal lengths} \\ &= \angle ZCD && \text{angle between tangent and chord.} \end{aligned}$$

Therefore, $Z = Z'$.

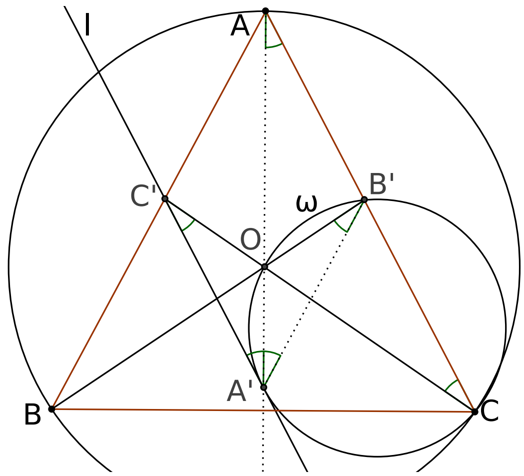
OC434. The acute isosceles triangle ABC ($AB = AC$) is inscribed in a circle with center O. The rays BO and CO intersect the sides AC and AB at the points B' and C', respectively. A line l parallel to the line AC passes through point C'. Prove that the line l is tangent to the circumcircle ω of the triangle $B'OC$.

Originally from Moscow Math Olympiad, Problem 2, Grade 10, Final Round 2017.

We received 4 *submissions. We present* 2 *solutions.*

Solution 1*, by Oliver Geupel.*

Let A' denote the point of intersection of the lines l and AO. Let φ denote the size of the angle $\angle OAC$.



Since the triangle AOC is isosceles, we have $\angle ACO = \varphi$. Also, $\angle OA'C' = \angle A'C'O = \varphi$, because the triangle $A'OC'$ is homothetic to the triangle AOC. Moreover, $\angle B'A'O = \angle OB'A' = \varphi$, since the triangle $A'B'O$ is the reflection of the triangle $A'C'O$ in the line AO.

Since $\angle B'A'O$ has the same size as the inscribed angle $\angle B'CO$ in ω, it follows that A' lies on ω.

The angle between the chord $A'O$ and the line l has the size φ, which is identical to the size of the inscribed angle $\angle OB'A'$ and, thus, to the size of the angle between the chord $A'O$ and the tangent to ω in A'.

We conclude that l is the tangent to ω in A'.

Solution 2, by Ivko Dimitrić.

Let $A = \angle BAC$. The line $q = \overleftrightarrow{AO}$ is the axis of symmetry of $\triangle ABC$. Let D be the intersection point of l and q. Because of the axial symmetry of the triangle ABC about q in which the lines AB, AC and BB', CC' each correspond to the other in the pair, it follows that $B'D \parallel AB$ just as $C'D \parallel AC$. That implies

$$\angle B'DO = \angle B'DA = \angle DAB = \frac{A}{2}$$

and $\angle DB'C = \angle BAC = A$. Since O is the circumcenter of the triangle ABC then

$$\angle DOC = \frac{1}{2}\angle BOC = \angle BAC = A.$$

Because $\angle DB'C = \angle DOC = A$, we conclude that the quadrilateral $DCB'O$ is cyclic and its circumcircle is ω, implying

$$\angle B'CO = \angle B'DO = \angle OAB = \frac{A}{2}.$$

By symmetry, $\angle OBC' = A/2$ and since $B'D \parallel AB$ it follows

$$\angle OCD = \angle OB'D = \angle OBC' = \frac{A}{2}.$$

Thus,

$$\angle B'CD = \angle B'CO + \angle OCD = \frac{A}{2} + \frac{A}{2} = A.$$

Hence, since $\angle B'CD = \angle DB'C$, the triangle DCB' is isosceles with base $B'C$. The common circumcenter S of $\triangle B'OC$ and $\triangle B'DC$ belongs to the perpendicular bisector s of $B'C$ that is also perpendicular to $l \parallel B'C$ and passes through the vertex D. Since the circumcircle ω of triangle $B'OC$ passes through D and the line l is perpendicular to the radius SD of ω at D, it follows that l is tangent to the circumcircle of triangle $B'OC$ at point D.

OC435. There are n positive numbers $a_1, a_2, \ldots, a_n$ written on a blackboard. Under each number a_i, Vasya wants to write a number $b_i \geq a_i$ so that for every pair of numbers chosen from $b_1, b_2, \ldots, b_n$, the ratio of one of them to the other is an integer. Prove that Vasya can write out the required numbers so that

$$b_1 b_2 \cdot \ldots \cdot b_n \leq 2^{(n-1)/2} a_1 a_2 \cdot \ldots \cdot a_n.$$

Originally from Moscow Math Olympiad, 4th Problem, Grade 10, Final Round 2017.

We received only 1 *submission. We present the solution by Oliver Geupel.*

There are n positive numbers $a_1, a_2, \ldots, a_n$ written on a blackboard. Under each number a_i, Vasya wants to write a number $b_i \geq a_i$ so that for every pair of numbers chosen from $b_1, b_2, \ldots, b_n$, the ratio of one of them to the other is an integer. Prove that Vasya can write out the required numbers so that

$$b_1 b_2 \cdot \ldots \cdot b_n \leq 2^{(n-1)/2} a_1 a_2 \cdot \ldots \cdot a_n.$$

For $i, j \in \{1, 2, \ldots, n\}$, let $c_{i,j} = \lceil \log_2 a_i - \log_2 a_j \rceil$. We prove that one of the following n sequences satisfies the conditions of the problem:

$$\begin{array}{cccccc}
2^{c_{1,1}} \cdot a_1, & 2^{c_{2,1}} \cdot a_1, & 2^{c_{3,1}} \cdot a_1, & \ldots, & 2^{c_{n-1,1}} \cdot a_1, & 2^{c_{n,1}} \cdot a_1, \\
2^{c_{1,2}} \cdot a_2, & 2^{c_{2,2}} \cdot a_2, & 2^{c_{3,2}} \cdot a_2, & \ldots, & 2^{c_{n-1,2}} \cdot a_2, & 2^{c_{n,2}} \cdot a_2, \\
2^{c_{1,3}} \cdot a_3, & 2^{c_{2,3}} \cdot a_3, & 2^{c_{3,3}} \cdot a_3, & \ldots, & 2^{c_{n-1,3}} \cdot a_3, & 2^{c_{n,3}} \cdot a_3, \\
\multicolumn{6}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
2^{c_{1,n}} \cdot a_n, & 2^{c_{2,n}} \cdot a_n, & 2^{c_{3,n}} \cdot a_n, & \ldots, & 2^{c_{n-1,n}} \cdot a_n, & 2^{c_{n,n}} \cdot a_n.
\end{array}$$

In the j-th sequence, we have

$$b_i = 2^{c_{i,j}} \cdot a_j = 2^{\lceil \log_2 a_i - \log_2 a_j \rceil} \cdot a_j \geq 2^{\log_2 a_i - \log_2 a_j} \cdot a_j = a_i.$$

Moreover, the ratio of every two members of the sequence is a power of 2 with an integer exponent. Hence, the ratio of one of them to the other is an integer.

Note that for real numbers x and y it holds $\lceil x \rceil + \lceil y \rceil \leq \lceil x + y \rceil + 1$. Thus, for $i, j \in \{1, 2, \ldots, n\}$, it holds

$$c_{i,j} + c_{j,i} = \lceil \log_2 a_i - \log_2 a_j \rceil + \lceil \log_2 a_j - \log_2 a_i \rceil \leq \lceil 0 \rceil + 1 = 1.$$

As a consequence, the product of all of the n^2 members of the n sequences is

$$\begin{aligned} 2^{\left(\sum_{i,j=1}^{n} c_{i,j}\right)} \cdot a_1^n a_2^n \cdots a_n^n &= 2^{\left(\sum_{1 \leq i < j \leq n} c_{i,j} + c_{j,i}\right)} \cdot a_1^n a_2^n \cdots a_n^n \\ &\leq 2^{(n-1)n/2} \cdot a_1^n a_2^n \cdots a_n^n. \end{aligned}$$

We conclude that the product of the members of at least one of the n sequences is not greater than $2^{(n-1)/2} \cdot a_1 a_2 \cdots a_n$. This completes the proof.

How To Write A *Crux* Article... Revisited!

Robert Dawson

About four years ago (November 2015, to be precise) I wrote a note for *Crux* detailing what a *Crux* article should be – and what it should not be. Since then, much has changed. Most noticeably, we're now an electronic-only journal. While the purple paper problem poser was familiar to many, there are advantages to the new regime. Not only is the new e-*Crux* free to all the day that it comes out, but we have no printing costs. While that doesn't mean that we can publish everything people send us, it does give us a little more flexibility. Here's what that means to us – and to you.

My first point in 2015 was that *Crux* **is accessible**. "*Crux* is read by university professors and graduate students. It's also read by undergraduates, school teachers, school students, and amateurs whose day jobs have nothing to do with mathematics. We ask prospective writers to write for a very clever high school student. Assume high intelligence but not a lot of specialized knowledge." Still true! We stop about where the Putnam does – if your article needs much more than standard sophomore math, it's not for us.

My second point was that *Crux* **is for problem solvers**. "The sort of thing that might appear on a regional or national math contest." That's still the case, and we still want articles about tricks for problem solving. "Assume that most *Crux* readers know the standard tricks of the trade. Don't stop and explain mathematical induction or double counting unless you're explaining something new or unusual about those topics."

Not everything to do with problem solving makes a good *Crux* article.

> A new problem, with or without solution: send it to *Crux* as a problem (not an article)
>
> A solution to a *Crux* problem: submit it to *Crux* as a solution (not an article)
>
> One problem, many solutions: unlikely to be of much interest.
>
> One technique that works for multiple problems: that's a *Crux* article.

I wrote that "*Crux* **is short.** Our articles don't usually run more than five or six pages, and we're more likely to run it if it's three or four. That's the right length for the sort of thing we publish. We do like well-done illustrations, nice examples, and interesting asides – but please keep it all brief." This is still true – we have a bit more flexibility but our readers' free time is still the same.

Crux is **still not a research journal**. If your research paper is high level, send it to the *CMS Bulletin* or one of the many other great math journals out there. If you're an undergraduate student, try the *PUMP Journal of Undergraduate Research*. If it's elementary, try the *College Math Journal*, *Mathematics Magazine*, or the *American Mathematical Monthly*. If it's accessible and quirky, try the *Mathematical Intelligencer*. But don't send it to us, that's not what we do. And in particular, we don't want "research" that other journals won't print. "You cannot trisect the angle, square the circle, or duplicate the cube using classical tools. If you understand Wantzel's classic 1837 proof, you won't try. If you don't understand it, you have not done your preparatory work and you have no business trying."

However, with our newfound freedom we have the luxury of running a few **general-interest articles**. I'm thinking about light and entertaining survey articles, the sort of thing that Martin Gardner used to write. The level will have to be just right – I anticipate rejecting some articles for being too difficult, others for being too elementary, more for being too specialized and still others for being common knowledge. I suspect that the successful writer will know a lot about the subject, and write much less than they know.

This raises the question of originality. We don't expect original research – indeed, like Wikipedia, we feel it's probably not right for us. But we don't want articles that just rephrase material that's already available from one source, with or without attribution. Your article should combine material from several sources, and say something that none of them says in quite the same way. (Young writers: please run it past a teacher or professor, *explaining all your sources*, before submitting.)

A note on **format**. Normally, we'd rather that you use LaTeX. Make sure that you know how to use things like theorem environments, \label, \cite, and \ref. If you're a mathematician you already know LaTeX; if you're going to be one, you'll have to learn some day, so might as well do it now. If your article has very few equations in it, you may submit it as a .doc or .docx file, or even flat text. (PDF is okay but be prepared to submit something editable on request.) Please don't use a theorem-proof-lemma form. It's not our style.

We love diagrams and pictures. Please make them neat; you can create them with Maple, Cinderella, Geogebra, or other software. Now we're paperless, colour and video are both fine, where appropriate. Okay? Now you know what we're looking for. Write it – get somebody to check it over to make sure the math and English (or French) is correct – and send it to us.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2020**.

4491. *Proposed by Lorian Saceanu.*

Let a, b, c be the side lengths of acute-angled triangle ABC lying opposite of angles $\angle A, \angle B, \angle C$, respectively. Let r be the inradius of ABC and let R be its circumradius. Prove that

$$\frac{a\angle A + b\angle B + c\angle C}{a+b+c} \leq \arccos \frac{r}{R}.$$

4492. *Proposed by George Stoica.*

Find the number of classes $\hat{u}$ in $\mathbb{Z}_n$ $(n \geq 2)$ with the property that both $\hat{u}$ and $\hat{u} - \hat{1}$ have multiplicative inverses in $\mathbb{Z}_n$.

4493. *Proposed by Nguyen Viet Hung.*

Find all real numbers x, y such that

$$\left\{\frac{x+2y+1}{x^2+y^2+7}\right\} = \frac{1}{2}$$

where $\{a\}$ denotes the fractional part of a.

4494. *Proposed by Michel Bataille.*

Let O be the circumcentre of a triangle ABC such that $\angle BAC \neq 90^\circ$ and let γ be the circumcircle of ΔBOC and Ω its centre. If P is a point of the side BC, let Q denote the point of intersection other than O of the line OP and γ. For which P do the lines OA and ΩQ intersect at M such that $MA = MQ$?

4495. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Prove Mihaileanu's theorem: Given a point P inside triangle ABC set $x = [PBC]$, $y = [PCA]$, and $z = [PAB]$, where square brackets denote area. If M is a point on side AB and N is a point on side AC, then line MN contains P if and only if

$$y \cdot \frac{BM}{MA} + z \cdot \frac{CN}{NA} = x.$$

[*Ed.*: See Solution 2 of 4445.]

4496. *Proposed by Leonard Giugiuc.*

Let a and b be two fixed numbers such that $0 < a < b$. We consider the function

$$f : [a, b] \times [a, b] \times [a, b] \to \mathbb{R}, \quad f(x, y, z) = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Find the maximum value of the function.

4497. *Proposed by Hoang Le Nhat Tung.*

Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

4498. *Proposed by Sergey Sadov.*

Consider the function $f(x) = 1/(x^2 + 1)$ for $x > 0$. Prove that there exists n such that the nth derivative $f^{(n)}(x)$ does not have constant sign for $x > 2019$.

4499. *Proposed by H. A. ShahAli.*

Prove that the following system of Diophantine equations has infinitely many unproportional solutions in positive integers:

$$\begin{cases} a + b + c + d = e + f + g, \\ a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2. \end{cases}$$

4500. *Proposed by Chudamani R. Pranesachar.*

Let AB be an arc of a circle with radius r and centre O, its angle subtended at the center, denoted θ, being less than π. Let M be the mid-point of the shorter arc AB. Points P on radius OA, S on radius OB, Q and R on arc AB are taken such that $PQRS$ is a rectangle. Prove that when the area of $PQRS$ is maximum, the line segments OQ, OM, OR divide angle AOB into four equal parts of common value $\frac{\theta}{4}$). Determine this maximum area in terms of r and θ.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 février 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4491. *Proposée par Lorian Saceanu.*

Dans un triangle acutangle ABC, soient a, b, c les longueurs $\angle A$, $\angle B$, $\angle C$, respectivement. Soient r et R les rayons du cercle inscrit et du cercle circonscrit, repectivement. Démontrer que

$$\frac{a\angle A + b\angle B + c\angle C}{a+b+c} \leq \arccos \frac{r}{R}.$$

4492. *Proposée par George Stoica.*

Déterminer le nombre de classes $\hat{u}$ dans $\mathbb{Z}_n$ $(n \geq 2)$ telles que $\hat{u}$ et $\hat{u} - \hat{1}$ ont toutes deux une inverse multiplicative dans $\mathbb{Z}_n$.

4493. *Proposée par Nguyen Viet Hung.*

Déterminer tous nombres réels x, y tels que

$$\left\{ \frac{x+2y+1}{x^2+y^2+7} \right\} = \frac{1}{2}$$

où $\{a\}$ dénote la partie fractionnelle de a.

4494. *Proposée par Michel Bataille.*

Soit ABC un triangle tel que $\angle BAC \neq 90^\circ$ et soit O le centre de son cercle circonscrit ; soit γ le cercle circonscrit de ΔBOC, avec Ω comme centre. Soit P un point sur le côté BC, la ligne OP intersectant γ en O et Q. Pour quels P les lignes OA et ΩQ intersectent-elles en M de façon à ce que $MA = MQ$?

4495. *Proposée par Leonard Giugiuc et Dan Stefan Marinescu.*

Démontrer le théorème de Mihaileanu : Pour un point P à l'intérieur du triangle ABC, soient

$$x = [PBC], \quad y = [PCA], \quad \text{et} \quad z = [PAB],$$

où les crochets représentent les surfaces. Si M est un point sur le côté AB et N est un point sur le côté AC, alors la ligne MN contient P si et seulement si

$$y \cdot \frac{BM}{MA} + z \cdot \frac{CN}{NA} = x.$$

[*Éd.*: Voir la solution à 4445.]

4496. *Proposée par Leonard Giugiuc.*

Soient a et b deux nombres réels tels que $0 < a < b$; posons la fonction

$$f : [a,b] \times [a,b] \times [a,b] \to \mathbb{R}, \quad f(x,y,z) = (x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Déterminer la valeur maximale de cette fonction.

4497. *Proposée par Hoang Le Nhat Tung.*

Soient a, b, c des nombres réels positifs. Démontrer que

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

4498. *Proposée par Sergey Sadov.*

Soit la fonction $f(x) = 1/(x^2+1)$ pour $x > 0$. Démontrer qu'il existe n tel que la $n$ième dérivée $f^{(n)}(x)$ n'est pas de signe constant pour $x > 2019$.

4499. *Proposée par H. A. ShahAli.*

Démontrer qu'il existe infiniment de solutions entières positives, non proportionnelles, au système Diophantin

$$\begin{cases} a+b+c+d = e+f+g, \\ a^2+b^2+c^2+d^2 = e^2+f^2+g^2. \end{cases}$$

4500. *Proposée par Chudamani R. Pranesachar.*

Soit AB un arc d'un cercle de rayon r et centre O, son angle soutendu au centre, dénoté θ, étant inférieur à π, epuissoit M le centre de l'arc AB. Les points P sur le rayon OA, Q et R sur l'arc AB et S sur le rayon OB sont choisis pour que $PQRS$ soit un rectangle. Démontrer que lorsque la surface de $PQRS$ est maximale, les segments OQ, OM et OR divisent $\angle AOB$ en quatre parties égales, de valeur commune $\frac{\theta}{4}$. Déterminer la surface maximale en termes de r et θ.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 44(5), p. 265–269.

4441. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle, with circumcenter O and orthocenter H. Let A', B' and C' be the intersection of AH, BH, CH with BC, AC, AB, respectively. Let A_1, B_1 and C_1 be the intersection of AO, BO, CO with BC, AC, AB, respectively. If A'', B'' and C'' are midpoints of AA_1, BB_1 and CC_1, show that $A'A'', B'B''$ and $C'C''$ have a common intersection point.

We received 7 submissions, all of which were correct; we feature a solution by Ivko Dimitrić supplemented by a small, but critical, contribution from the solution by Marie-Nicole Gras.

The requirement that the triangle be acute is unnecessary; we shall see that for any ΔABC the lines $A'A''$, $B'B''$, and $C'C''$all pass through the nine-point center.

When $AB = AC$, the points A' and A_1 coincide, whence the line $A'A''$ is an altitude and passes through the nine-point center.

Let us therefore assume that $AB \neq AC$, and let M be the midpoint of $\overline{BC}$, K the midpoint of $\overline{HA}$, and N the intersection point of $A'A''$ and KM. Since $AH = 2OM$ and $AH, OM \perp BC$, we have $AK = OM$ and $AK \parallel OM$, so the quadrilateral $AKMO$ is a parallelogram and hence $KM \parallel AA_1$. Consequently, triangles NMA' and $A''A_1A'$ are similar as are triangles $A'NK$ and $A'A''A$, and we have

$$\frac{NM}{A''A_1} = \frac{A'N}{A'A''} = \frac{NK}{A''A}.$$

Because $A''A_1 = A''A$ we have $NM = NK$. Hence, N is the midpoint of the hypotenuse KM of $\triangle A'MK$, and therefore the center of its circumcircle.

Since a circle is uniquely determined by any three of its points, and the nine-point circle passes through the feet of the altitudes, the midpoints of the sides and the midpoints of the segments from the orthocenter to the vertices, it follows that the circumcircle of $\triangle A'MK$ is the nine-point circle of $\triangle ABC$ and its center is $N = A'A'' \cap KM$. Hence, the line $A'A''$ passes through the nine-point center N.

In the same manner, the lines $B'B''$ and $C'C''$ also pass through the nine-point center, so that the three lines in question have the common intersection point N.

4442. *Proposed by Nguyen Viet Hung.*

Find the following limit

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\right).$$

We received 21 submissions, of which all but one were correct. We present several solutions.

Solution 1, by Florentin Viescu.

We show that the limit is $1/\sqrt{2}$ as an application of Cesaro-Stolz Lemma. Let

$$a_n=\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}.$$

Then

$$a_{n+1}-a_n=1/(\sqrt{2n+1}+\sqrt{2n+2}).$$

Let $b_n=\sqrt{n}$. Then $b_{n+1}-b_n=\sqrt{n+1}-\sqrt{n}>0$, for all $n\geq 1$. So $(b_n)_{n\geq 1}$ is strictly increasing and approaches $+\infty$.

Moreover, the following limit exists

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}&=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{2n+1}+\sqrt{2n+2}}}{\sqrt{n+1}-\sqrt{n}}\\&=\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{2n+1}+\sqrt{2n+2}}\\&=\lim_{n\to\infty}\frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}{\sqrt{n}\left(\sqrt{2+\frac{1}{n}}+\sqrt{2+\frac{2}{n}}\right)}\\&=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}.\end{aligned}$$

Then according to Cesàro-Stolz Lemma $\lim_{n\to\infty}a_n/b_n=1/\sqrt{2}$.

Solution 2, by Nguyen Viet Hung.

We show that the limit is $1/\sqrt{2}$ as an application of the Squeeze Theorem. We have clearly

$$\begin{aligned}S_n&=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\\&<\frac{1}{\sqrt{0}+\sqrt{1}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-2}+\sqrt{2n-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}2S_n &< 1+\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\\ &= 1+\frac{\sqrt{1}-\sqrt{2}}{-1}+\frac{\sqrt{2}-\sqrt{3}}{-1}+\cdots+\frac{\sqrt{2n-1}-\sqrt{2n}}{-1}\\ &= \sqrt{2n}. \qquad (1)\end{aligned}$$

On the other hand,

$$\begin{aligned}S_n &= \frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\\ &> \frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\cdots+\frac{1}{\sqrt{2n}+\sqrt{2n+1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}2S_n &> \frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}+\frac{1}{\sqrt{2n}+\sqrt{2n+1}}\\ &= \frac{\sqrt{1}-\sqrt{2}}{-1}+\frac{\sqrt{2}-\sqrt{3}}{-1}+\cdots+\frac{\sqrt{2n}-\sqrt{2n+1}}{-1}\\ &= \sqrt{2n+1}-1. \qquad (2)\end{aligned}$$

From (1) and (2), we find that

$$\frac{\sqrt{2n+1}-1}{2} < S_n < \frac{\sqrt{2n}}{2}.$$

We apply Squeeze Theorem to find

$$\lim_{n\to\infty}\frac{S_n}{\sqrt{n}}=\frac{1}{\sqrt{2}}.$$

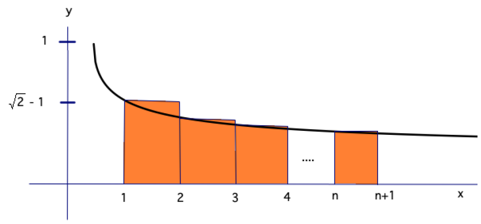
Solution 3, by Rob Downes.

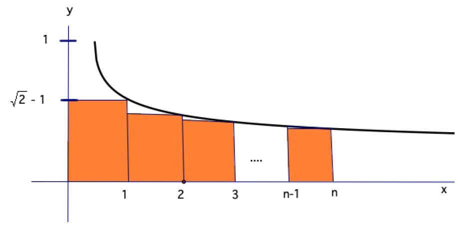
We show that the limit is $1/\sqrt{2}$ as an application of the Squeeze Theorem. Let

$$\begin{aligned}S_n &= \frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}\\ &= (\sqrt{2}-1)+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{2n}-\sqrt{2n-1}).\end{aligned}$$

Consider $f(x)=\sqrt{2x}-\sqrt{2x-1}$. Note that $f(x)$ is a strictly decreasing function for $x\geq 1$. Therefore, as illustrated in the left figure below, we have:

$$S_n \geq \int_1^{n+1}(\sqrt{2x}-\sqrt{2x-1})\,dx$$





Additionally, as illustrated in the right figure we have:

$$S_n \leq \sqrt{2} - 1 + \int_1^n (\sqrt{2x} - \sqrt{2x-1})\, dx.$$

Evaluating the integrals, dividing by $\sqrt{n}$, and simplifying yields the compound inequality:

$$\begin{aligned} &\frac{(2n+2)^{\frac{3}{2}} - (2n+1)^{\frac{3}{2}} - (2^{\frac{3}{2}} - 1)}{3\sqrt{n}} \leq \frac{S_n}{\sqrt{n}} \\ &\leq \frac{(2n)^{\frac{3}{2}} - (2n-1)^{\frac{3}{2}} - (2^{\frac{3}{2}} - 1) + 3(\sqrt{2} - 1)}{3\sqrt{n}}. \end{aligned} \tag{1}$$

Next, we use the Squeeze Theorem to evaluate the given limit. For the expression on the left in (1), we have:

$$\begin{aligned} &\lim_{n\to\infty} \frac{(2n+2)^{\frac{3}{2}} - (2n+1)^{\frac{3}{2}} - (2^{\frac{3}{2}} - 1)}{3\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{(2n+2)^{\frac{3}{2}} - (2n+1)^{\frac{3}{2}}}{3\sqrt{n}} - \lim_{n\to\infty} \frac{(2^{\frac{3}{2}} - 1)}{3\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{12n^2 + 18n + 7}{3(\sqrt{n(2n+2)^3} + \sqrt{n(2n+1)^3})} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Dividing the numerator and denominator by n^2 and taking the limit yields the result $1/\sqrt{2}$. Similarly, for the expression on the right in (1), we have:

$$\begin{aligned} &\lim_{n\to\infty} \frac{(2n)^{\frac{3}{2}} - (2n-1)^{\frac{3}{2}} - (2^{\frac{3}{2}} - 1) + 3(\sqrt{2} - 1)}{3\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{12n^2 - 6n + 1}{3(\sqrt{8n^4} + \sqrt{n(2n-1)^3})} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Since the limits of the two outer expressions in (1) are equal, we have by the Squeeze Theorem

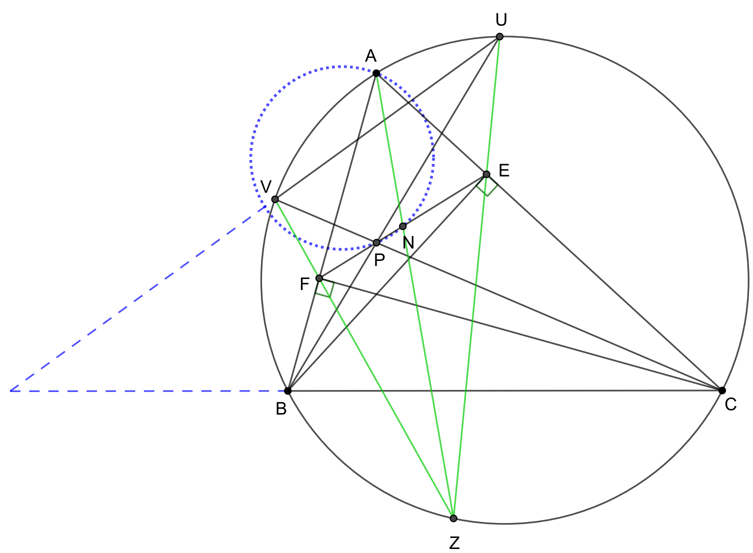
$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}} = \frac{1}{\sqrt{2}}.$$

Editor's comments. The solutions received used either Cesàro-Stolz Lemma or Squeeze Theorem to calculate the limit. The authors used various inequalities or approximations to be able to apply these theorems. Many solutions were similar and we chose to feature the representative ones.

4443. *Proposed by Andrew Wu.*

Acute scalene $\triangle ABC$ has circumcircle Ω and altitudes $\overline{BE}$ and $\overline{CF}$. Point N is the midpoint of $\overline{EF}$ and line $\overline{AN}$ meets Ω again at Z. Let lines $\overline{ZF}$ and $\overline{ZE}$ meet Ω again at V and U, respectively, and let lines $\overline{CV}$ and $\overline{BU}$ meet at P. Prove that $\overline{UV}$ and $\overline{BC}$ meet on the tangent from P to the circumcircle of $\triangle APN$.

We received 3 solutions. We present the solution by Andrea Fanchini.



Use barycentric coordinates with reference to $\triangle ABC$. We have $A(1:0:0)$, $B(0:1:0)$ and $C(0:0:1)$. Denote the side lengths of the triangle by a, b and c. We use Conway's notation, in particular S for twice the area of $\triangle ABC$, and the shorthand S_A, S_B and S_C, where $S_\alpha = S\cot(\alpha)$ for an angle α.

The equation of the circumcircle Ω is $a^2yz + b^2zx + c^2xy = 0$. As the feet of the altitudes from B and C respectively, E and F have coordinates $E(S_C : 0 : S_A)$ and $F(S_B : S_A : 0)$.

Point $N(c^2S_C + b^2S_B : b^2S_A : c^2S_A)$ is the midpoint of EF. The line through A and N has equation $c^2y - b^2z = 0$, and meets Ω again at $Z(-a^2 : 2b^2 : 2c^2)$.

The line through Z and F has equation

$$2c^2S_Ax - 2c^2S_By + (a^2S_A + 2b^2S_B)z = 0$$

and meets Ω again at $V(2S_B(a^2S_A + 2b^2S_B) : S_A(a^2S_A + 2b^2S_B) : -2c^2S_AS_B)$. Similarly, the line through Z and E has equation

$$2b^2S_Ax + (a^2S_A + 2c^2S_C)y - 2b^2S_Cz = 0$$

and meets Ω again at $U(2S_C(a^2S_A + 2c^2S_C) : -2b^2S_AS_C : S_A(a^2S_A + 2c^2S_C))$.

The lines $CV : S_Ax - 2S_By = 0$ and $BU : S_Ax - 2S_Cz = 0$ meet at the point $P(2S_BS_C : S_AS_C : S_AS_B)$.

The circumcircle of $\triangle APN$ thus has equation

$$a^2yz + b^2zx + c^2xy - \frac{4S^2 - a^2S_A}{4S^2 - 2a^2S_A}(S_By + S_Cz)(x + y + z) = 0.$$

The tangent to this circumcircle at P has equation

$$t : -2S_AS^2x + S_B(a^2S_A + 2c^2S_C)y + S_C(a^2S_A + 2b^2S_B)z = 0.$$

The lines $UV : -a^2S_A^2x + 2S_B(a^2S_A + 2c^2S_C)y + 2S_C(a^2S_A + 2b^2S_B)z = 0$ and $BC : x = 0$ meet at the point $X(0 : S_C(a^2S_A + 2b^2S_B) : -S_B(a^2S_A + 2c^2S_C))$, which we can easily verify is on the tangent t, concluding the proof.

4444. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate in closed form

$$\sum_{k=0}^{n-1}\left(\tan^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{4}\right) + \cot^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{4}\right)\right).$$

We received 9 correct solutions. We present the solution by Angel Plaza.

Since

$$\tan^2 x + \cot^2 x = 4\cot^2(2x) + 2 = 4\tan^2(\pi/2 - 2x) + 2,$$

the sum can be written as

$$4\sum_{k=0}^{n-1}\cot^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{2}\right) + 2n = 4\sum_{k=1}^{n}\tan^2\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right) + 2n.$$

We will prove that

$$\sum_{k=0}^{n}\tan^2(2k\pi/2(2n+1)) = n(2n+1).$$

Observe that, for $1 \le k \le n$,

$$1 = (-1)^{2k} = \left(\cos\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right) + i\sin\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)\right)^{2n+1}$$
$$= \sum_{j=0}^{2n+1}\binom{2n+1}{j}\left(\cos\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)\right)^j\cdot\left(i\sin\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)\right)^{2n+1-j}.$$

Taking the imaginary parts of both sides and dividing by

$$\cos^{2n+1}\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)\cdot\tan\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)$$

yields

$$0=\sum_{j=0}^{n}\binom{2n+1}{2j}\left(i\tan\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)\right)^{2n-2j}.$$

Thus, $\tan^2(2k/(2n+1))$ with $1\le k\le n$ are the zeros of the polynomial

$$\sum_{j=0}^{n}\binom{2n+1}{2j}(-z)^{n-j}.$$

The sum of these zeros is

$$\frac{\binom{2n+1}{2}}{\binom{2n+1}{0}}=n(2n+1).$$

Therefore,

$$\sum_{k=1}^{n}\tan^2\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)=n(2n+1)$$

and the proposed sum is

$$4n(2n+1)+2n=2n(4n+3).$$

Editor's comments. Brian Bradie and Oliver Geupel, independently, rendered the sum as

$$\begin{aligned}\sum_{k=0}^{2n}\tan^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{4}\right)-1&=\sum_{k=1}^{4n+1}\tan^2\left(\frac{k\pi}{8n+4}\right)-\sum_{k=1}^{2n}\tan^2\left(\frac{k\pi}{4n+2}\right)-1\\&=\frac{(8n+3)(4n+1)}{3}-\frac{2n(4n+1)}{3}-1\\&=(2n+1)(4n+1)-1=8n^2+6n.\end{aligned}$$

Eight of the solutions relied on the value of a trigonometric sum of the type in the foregoing solutions. Three of the solvers justified it by a polynomial argument like Plaza's, and one used a partial fractions representation. Two people gave a reference to the literature, while the remaining two regarded it as "well-known" and "classical". The ninth solution appealed to the series development

$$\frac{\pi^2}{\sin^2(\pi x)}=\sum_{m=-\infty}^{+\infty}\frac{1}{(m+x)^2},$$

and through some delicate bookkeeping got the result from the sum of the odd square reciprocals.

4445. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let ABC be a triangle with $AC > BC > AB$, incenter I and centroid G .

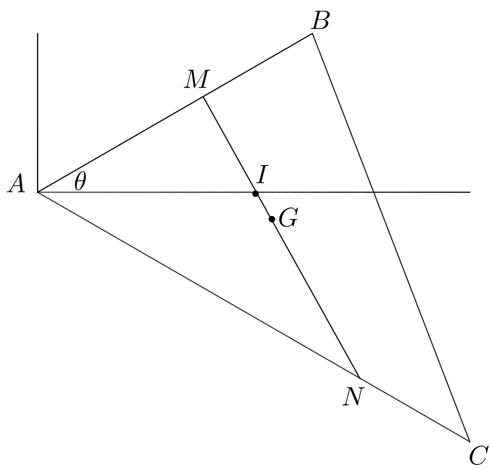
1. Prove that point A lies in one half-plane of the line GI, while points B and C lie in the other half-plane.

2. The line GI intersects the sides AB and AC at M and N, respectively. Prove that $BM = CN$ if and only if $\angle BAC = 60°$.

We received 5 submissions, all correct. Four of the solutions relied on coordinates; we have chosen one example of such a solution to feature together with the solution that avoided coordinates.

Solution 1, by Marie-Nicole Gras.

For triangle ABC we denote by a, b, c the lengths of the sides BC, CA and AB; by assumption, $b > a > c$. We put $\theta = \dfrac{\angle BAC}{2}\cdot$



The bisectors at vertex A define an orthonormal system with origin A; the cartesian coordinates of A, B, C are

$$A(0,0),\ B(c\cos\theta, c\sin\theta),\ C(b\cos\theta, -b\sin\theta);$$

using well-known formulas, we get those of G and I :

$$G\left(\frac{b+c}{3}\cos\theta, \frac{c-b}{3}\sin\theta\right), \quad I\left(\frac{2bc}{a+b+c}\cos\theta,\ 0\right)\cdot$$

The point M is the intersection of the lines GI and the line AB; the equation of the line GI is

$$\frac{c-b}{3}x\sin\theta - \left(\frac{b+c}{3} - \frac{2bc}{a+b+c}\right)y\cos\theta = \frac{2bc}{a+b+c}\frac{c-b}{3}\sin\theta\cos\theta;$$

since the equation of AB is $y = x\tan\theta$, we obtain

$$\left(\frac{c-b}{3} - \frac{b+c}{3} + \frac{2bc}{a+b+c}\right) x\sin\theta = \frac{2bc}{a+b+c}\,\frac{c-b}{3}\sin\theta\cos\theta;$$

and we arrive at the coordinates of M :

$$M\left(\frac{c(b-c)}{a+b-2c}\cos\theta, \frac{c(b-c)}{a+b-2c}\sin\theta\right)\cdot$$

Exchanging b and c, and the sign of θ, we obtain

$$N\left(\frac{b(c-b)}{a+c-2b}\cos\theta, \frac{b(c-b)}{a+c-2b}\sin(-\theta)\right)\cdot$$

Since $b > a > c$ we have

$$\frac{c(b-c)}{a+b-2c} = \frac{c(b-c)}{(a-c)+(b-c)} > 0, \qquad \frac{b(c-b)}{a+c-2b} = \frac{b(b-c)}{(b-a)+(b-c)} > 0,$$

and then

$$AM = \frac{c(b-c)}{a+b-2c}, \qquad AN = \frac{b(b-c)}{2b-a-c}\cdot$$

1. Because

$$MB = AB - AM = c - \frac{c(b-c)}{a+b-2c} = \frac{c(a-c)}{a+b-2c} > 0,$$

we deduce that M is between A and B, and

$$BM = \frac{c(a-c)}{a+b-2c}\cdot$$

Exchanging b and c, we get that N is between A and C, and

$$CN = \frac{b(a-b)}{a+c-2b} = \frac{b(b-a)}{2b-a-c}\cdot$$

As points G and I are inside ABC, they belong to the line segment MN, and then point A lies in one half-plane of the line GI, while points B and C lie in the other half-plane. We remark that A is the vertex of ABC such that $\angle BAC$ is between $\angle ABC$ and $\angle BCA$.

2. Finally, we have

$$\begin{aligned} BM = CN \quad &\iff \quad \frac{c(a-c)}{a+b-2c} = \frac{b(b-a)}{2b-a-c} \\ &\iff \quad c(a-c)(b-a+b-c) = b(b-a)(a-c+b-c) \\ &\iff \quad c(a-c)(b-c) - b(b-a)(b-c) = b(b-a)(a-c) - c(a-c)(b-a) \\ &\iff \quad (b-c)(ac-c^2-b^2+ab) = (b-c)(ab-bc-a^2+ac) \\ &\iff \quad (b-c)(b^2+c^2-a^2-bc) = 0. \end{aligned}$$

By assumption, $b \neq c$; then $BM = CN$ is equivalent to

$$a^2 = b^2 + c^2 - bc \iff \cos(\angle BAC) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2} \iff \angle BAC = 60^\circ.$$

Remark. If $\angle BAC = 60^\circ$ and $b \neq c$, then $a^2 - c^2 = b(b - c) \neq 0$ and

$$CN = BM = \frac{c(a-c)(a+c)}{((a-c)+(b-c))(a+c)} = \frac{cb(b-c)}{b(b-c)+(b-c)(a+c)} = \frac{bc}{a+b+c}\cdot$$

Let r be the inradius of ABC; then the area of ABC is

$$\frac{a+b+c}{2}r = \frac{1}{2}bc\sin(\angle BAC) = \frac{\sqrt{3}}{4}bc;$$

we find, finally, that

$$BM = CN = \frac{2r\sqrt{3}}{3} = \frac{2}{3}r\tan 60^\circ\cdot$$

Solution 2, by the proposers.

The argument is based on a two-part lemma.

Lemma. Let M be a point on side AB of ΔABC and N a point on side AC; then

a) the line MN passes through the centroid G if and only if $\frac{BM}{MA} + \frac{CN}{NA} = 1$, and,

b) the line MN passes through the incenter I if and only if $b \cdot \frac{BM}{MA} + c \cdot \frac{CN}{NA} = a$.

Editor's comment: Both parts follow immediately from a theorem of Nicolae Mihaileanu that deserves to be better known outside Romania. Because its proof would make a nice challenge for *Crux* readers, instead of including it here, we have turned it into one of this month's problem proposals, problem number 4495.

Back to the problem. We assume that M,G,I, and N are collinear and set $x = \dfrac{BM}{MA}$, $y = \dfrac{CN}{NA}$. Our lemma tells us that x and y satisfy the simultaneous equations,

$$\begin{cases} x + y = 1 \\ bx + cy = a \end{cases},$$

which, because $b > a > c$, gives us

$$x = \frac{a-c}{b-c} > 0 \quad \text{and} \quad y = \frac{b-a}{b-c} > 0.$$

Because there is a unique point, namely M, that divides segment BA in the ratio $a - c : b - c$, and a unique point, namely N, that divides segment CA in the ratio

$b-a : b-c$, our lemma tells us that with this choice of the points M and N we have both $\dfrac{BM}{MA} + \dfrac{CN}{NA} = 1$, so that MN passes through G, and $b \cdot \dfrac{BM}{MA} + c \cdot \dfrac{CN}{NA} = a$, so that MN passes through I. We conclude that the line GI separates A from both B and C. Furthermore, because

$$\frac{BM}{MA} = \frac{a-c}{b-c},$$

we have

$$\frac{BM}{CM + MA} = \frac{a-c}{(a-c)+(b-c)},$$

whence

$$BM = \frac{c(a-c)}{a+b-2c}.$$

similarly,

$$CN = \frac{b(b-a)}{2b-a-c}.$$

The argument concludes as in part 2 of the first solution.

Editor's comments. For a compilation of many other properties of triangles with an angle of 60°, see Chris Fisher's article "Recurring *Crux* Configurations 3: Triangles Whose Angles Satisfy $2B = C + A$" [**37**:7 (November 2011) pages 449-453]. Problem 4445 is evidently the first *Crux* problem involving the Nagel line GI of such a triangle.

4446. *Proposed by Florin Stanescu.*

Let n be a prime number greater than 4 and let $A \in M_{n-1}(\mathbb{Q})$ be such that $A^n = I_{n-1}$. Evaluate $\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})$ in terms of n.

We received 6 submissions, 5 of which were correct and complete solutions. We present the solution by Brian Bradie, slightly edited.

The condition $A^n = I_{n-1}$ implies

$$(A - I_{n-1})(A^{n-1} + A^{n-2} + A^{n-3} + \cdots + A + I_{n-1}) = 0.$$

Because n is prime, the polynomial

$$x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1$$

is irreducible over $\mathbb{Q}$. Thus, either $A = I_{n-1}$ or A is similar to the $(n-1)\times(n-1)$ matrix

$$C_{n-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & -1 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix},$$

which is the companion matrix associated with the characteristic polynomial

$$x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1.$$

We now consider two cases.

Case 1: If $A = I_{n-1}$, then

$$\begin{aligned}&\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})\\ &\qquad = \det\left(\frac{(n-1)n}{2} I_{n-1}\right) = \left[\frac{(n-1)n}{2}\right]^{n-1}.\end{aligned}$$

Case 2: If A is similar to C_{n-1}, then

$$\begin{aligned}&\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})\\ =&\det(C_{n-1}^{n-2} + 2C_{n-1}^{n-3} + 3C_{n-1}^{n-4} + \cdots + (n-2)C_{n-1} + (n-1)I_{n-1}).\end{aligned}$$

Now note that the companion matrix has the property that

$$C_{n-1}^k = (v_{k+1}|v_{k+2}|\cdots|v_{k+n-1}),$$

where $v_i = v_{n+i} = e_i$, $(i = 1, \ldots, n-1)$ is the i'th canonical base vector and $v_n = (-1, -1, \ldots, -1)^T$.

We obtain

$$\begin{aligned}&C_{n-1}^{n-2} + 2C_{n-1}^{n-3} + 3C_{n-1}^{n-4} + \cdots + (n-2)C_{n-1} + (n-1)I_{n-1}\\ &\qquad = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ n-2 & n-2 & -2 & \cdots & -2 & -2 \\ n-3 & n-3 & n-3 & \cdots & -3 & -3 \\ & & & \ddots & & \\ 2 & 2 & 2 & \cdots & 2 & 2-n \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.\end{aligned}$$

Adding j times row $n-1$ to row j for $j = 1, 2, 3, \ldots, n-2$ yields

$$\begin{bmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ n & n & 0 & \cdots & 0 & 0 \\ n & n & n & \cdots & 0 & 0 \\ & & & \ddots & & \\ n & n & n & \cdots & n & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Thus,

$$\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1}) = n^{n-2}.$$

4447. *Proposed by Lorian Saceanu.*

Let ABC be a scalene triangle. Prove that

$$2+\frac{\sin A\sin B\sin C}{\sin A+\sin B+\sin C}\geq \sin^2 A+\sin^2 B+\sin^2 C.$$

We received 19 submissions, all of which are correct. Most of the solutions used several known identities and are very similar to one another. We present the proof by Ioan Viorel Codreanu.

Let r, R, and s denote the inradius, circumradius, and semiperimeter of ΔABC, respectively. The following identities are all well known:

$$\prod_{cyc}\sin A=\frac{sr}{2R^2},$$

$$\sum_{cyc}\sin A=\frac{s}{R},$$

and

$$\sum_{cyc}\sin^2 A=\frac{s^2-r(4R+r)}{2R^2}. \qquad (1)$$

Using (1), the proposed inequality is equivalent to

$$2+\frac{r}{2R}\geq\frac{s^2-r(4R+r)}{2R^2}, \qquad \text{or}$$

$$s^2\leq 4R^2+5Rr+r^2, \qquad \text{or}$$

$$s^2\leq(4R^2+4Rr+3r^2)+r(R-2r),$$

which is true by Gerretsen's Inequality [*Editor's comment:* See for reference, for example, Item 5.8 on p. 50 of *Geometric Inequalities* by O. Bottema et al], and $R\geq 2r$ by Euler's Inequality.

4448. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be non-zero complex numbers such that $|a|=|b|=|c|=|d|$ and $\mathrm{Arg}(a)<\mathrm{Arg}(b)<\mathrm{Arg}(c)<\mathrm{Arg}(d)$. Prove that

$$|(a-b)(c-d)|=|(a-d)(b-c)| \quad\Longleftrightarrow\quad (a-b)(c-d)=(a-d)(b-c).$$

There were 6 correct solutions. Three took the approach of Solution 1 and the other three of Solution 2.

Solution 1, by Oliver Geupel.

The reverse implication is clear. Suppose that $|(a-b)(c-d)|=|(a-d)(b-c)|$.

Then for some r,

$$\left|\frac{a-b}{c-b}\right| = r = \left|\frac{a-d}{c-d}\right|,$$

so that

$$\frac{a-b}{c-b} = re^{i\phi} \qquad \text{and} \qquad \frac{a-d}{c-d} = re^{i\psi}$$

for some angles ϕ and ψ; note that these angles have opposite directions. Since the points a, b, c, d in the complex plane form a concyclic quadrilateral $\phi + (-\psi) = \pi$. Therefore

$$\frac{a-d}{c-d} = re^{i\psi} = re^{i(\phi-\pi)} = -re^{i\phi} = -\frac{a-b}{c-b},$$

whence $(a-b)(c-d) = (a-d)(b-c)$.

Solution 2, by Florentin Visescu.

Let $a = r(\cos m + i \sin m)$, $b = r(\cos n + i \sin n)$, $c = r(\cos p + i \sin p)$, $d = r(\cos q + i \sin q)$. Then, after standard trigonometric manipulations, we find that

$$(a-b)(c-d) = -4r^2 \sin\frac{n-m}{2} \sin\frac{q-p}{2}(\cos s + i \sin s)$$

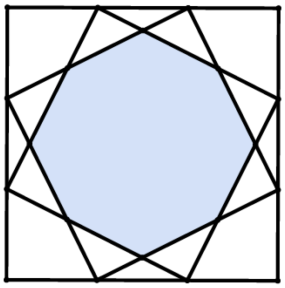
and

$$(a-d)(b-c) = -4r^2 \sin\frac{q-m}{2} \sin\frac{p-n}{2}(\cos s + i \sin s),$$

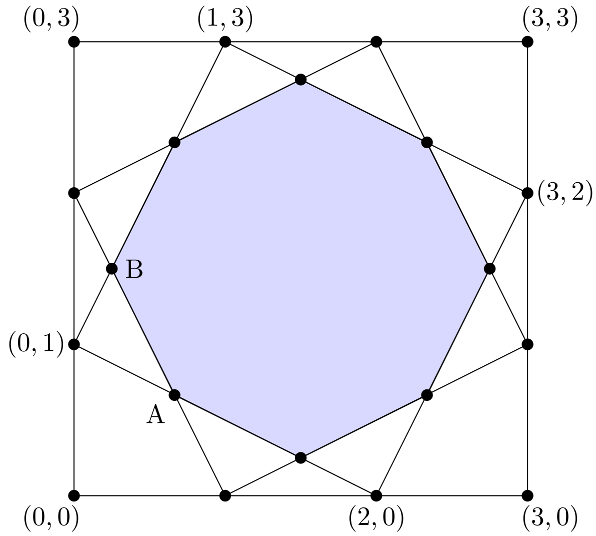
where $2s = m + n + p + q$. From this, the desired result follows.

4449. *Proposed by Arsalan Wares.*

The figure shows two congruent overlapping squares inside a larger square. The vertices of the overlapping smaller squares divide each of the four sides of the largest square into three equal parts. If the area of the shaded region is 50, find the area of the largest square.



We received 18 submissions of which 17 were correct and complete. We present the solution by Brian Beasley.



We model the diagram by placing the vertices of the largest square at $(0, 0)$, $(0, 3)$, $(3, 3)$, and $(3, 0)$. Then the shaded region (which we denote by R) is inside the smaller square with vertices at $(0, 1)$, $(1, 3)$, $(3, 2)$, and $(2, 0)$, yielding an area of 5 for the smaller square. Since the smaller square may be partitioned into R and four remaining congruent triangles, we calculate the area of R by subtracting the areas of the four congruent triangles from 5. By solving for the intersection of lines, it is easy to obtain the coordinates of A and B as $(2/3, 2/3)$ and $(1/4, 3/2)$ respectively. Thus the right-triangle formed by $(0, 1)$, A, and B has area

$$\frac{1}{2} \cdot \frac{\sqrt{5}}{4} \cdot \frac{\sqrt{5}}{3} = \frac{5}{24}.$$

Hence in our model, the area of R is $5 - 5/6 = 25/6$, so in general the ratio of the areas of R and the largest square is $(25/6)/9 = 25/54$. Thus we conclude that an area of 50 for R must correspond to an area of 108 for the largest square.

4450. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let $n \geq 3$ be an integer and consider positive real numbers $a_1, a_1, \dots, a_n$ such that $a_n \geq a_1 + a_2 + \cdots + a_{n-1}$. Prove that

$$(a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq 2((n-1)^2 + 1).$$

We received 18 solutions, all correct. We present the proof by Angel Plaza, modified slightly by the editor.

Let $S_{n-1} = \sum_{k=1}^{n-1} a_k$ and $T_{n-1} = \sum_{k=1}^{n-1} \frac{1}{a_k}$. Then the proposed inequality reads as

$$(S_{n-1} + a_n) \left(T_{n-1} + \frac{1}{a_n} \right) \geq 2((n-1)^2 + 1). \qquad (1)$$

Now, by the AM-HM inequality, we have

$$S_{n-1}T_{n-1} = \left(\sum_{k=1}^{n-1} a_k\right)\left(\sum_{k=1}^{n-1} \frac{1}{a_k}\right) \geq (n-1)^2,$$

so

$$(S_{n-1} + a_n)\left(T_{n-1} + \frac{1}{a_n}\right) \geq (n-1)^2 + 1 + a_nT_{n-1} + \frac{S_{n-1}}{a_n}. \qquad (2)$$

Since $a_n \geq S_{n-1}$, we have $a_n = S_{n-1} + d$ for some $d \geq 0$. Then

$$\begin{aligned} a_nT_{n-1} + \frac{S_{n-1}}{a_n} &= (S_{n-1} + d)\,T_{n-1} + \frac{S_{n-1}}{S_{n-1} + d} \\ &\geq (n-1)^2 + 1 + dT_{n-1} + \left(\frac{S_{n-1}}{S_{n-1} + d} - 1\right) \\ &= (n-1)^2 + 1 + \frac{dT_{n-1}S_{n-1} + d^2T_{n-1} - d}{S_{n-1} + d} \\ &\geq (n-1)^2 + 1 + \frac{d(n-1)^2 + d^2T_{n-1} - d}{S_{n-1} + d} \geq (n-1)^2 + 1. \qquad (3) \end{aligned}$$

(Therefore $d(n-1)^2 - d = d(n^2 - 2n) \geq 0$.)

Finally, (1) follows by substituting (3) into (2).

Editor's comment. Note that equality holds if and only if $a_1 = a_2 = \cdots = a_{n-1} = c$ and $a_n = (n-1)c$ for some $c > 0$.