2001
SOLUTIONS

Several solutions are edited versions of solutions submitted by the contestants whose names appear in *italics*..

1. *(Daniel Brox)*
   Let R be Rachel's age, and let J be Jimmy's age. Rachel's quadratic is

   $$a(x-R)(x-J) = ax^2 - a(R+J)x + aRJ.$$

   for some number a. We are given that the coefficient a is an integer. The sum of the coefficients is

   $$a - a(R+J) + aRJ = a(R-1)(J-1).$$

   Since this is a prime number, two of the three integers $a, R-1, J-1$ multiply to 1. We are given that $R > J > 0$, so we must have that $a = 1$, $J = 2$, $R-1$ is prime, and the quadratic is

   $$(x-R)(x-2).$$

   We are told that this quadratic takes the value $-55 = -5 \cdot 11$ for some positive integer x. Since $R > 2$, the first factor, $(x-R)$, must be the negative one. We have four cases:
   $x - R = -55$ and $x - 2 = 1$, which implies $x = 3$, $R = 58$.
   $x - R = -11$ and $x - 2 = 5$, which implies $x = 7$, $R = 18$.
   $x - R = -5$ and $x - 2 = 11$, which implies $x = 13$, $R = 18$.
   $x - R = -1$ and $x - 2 = 53$, which implies $x = 57$, $R = 58$.

   Since $R-1$ is prime, the first and last cases are rejected, so $R = 18$ and $J = 2$.

2. *(Lino Demasi)*
   After ten coin flips, the token finishes on the square numbered $2k-10$, where k is the number of heads obtained. Of the $2^{10} = 1024$ possible results of ten coin flips, there are exactly $\binom{10}{k}$ ways to obtain exactly k heads, so the probability of finishing on the square labeled $2k-10$ equals $\binom{10}{k}/1024$.

   The probability of landing on a red square equals $c/1024$ where c is the sum of a selection of the numbers from the list

   $$\binom{10}{0}, \binom{10}{1}, \binom{10}{2}, \ldots, \binom{10}{10} = 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1. \tag{1}$$

   We are given that for some integers a, b satisfying $a + b = 2001$,

   $$a/b = c/1024.$$

   If we assume (as most contestants did!) that a and b are relatively prime, then the solution proceeds as follows. Since $0 \le a/b \le 1$ and $a + b = 2001$, we have $1001 \le b \le 2001$. Also b divides 1024, so we have $b = 1024$. Thus $a = c = 2001 - 1024 = 977$. There is only one way to select terms from (1) so that the sum equals 977.

   $$977 = 10 + 10 + 45 + 120 + 120 + 210 + 210 + 252. \tag{2}$$

   (This is easy to check, since the remaining terms in (1) must add to $1024 - 977 = 47$, and $47 = 45 + 1 + 1$ is the only possibility for this.)

In order to maximize n, we must colour the strip as follows. Odd numbered squares are red if positive, and white if negative. Since $252 = \binom{10}{5}$ is in the sum, the square labeled $2 \cdot 5 - 10 = 0$ is red. For $k = 0, 1, 2, 3, 4$, if $\binom{10}{k}$ appears twice in the sum (2), then both $2k - 10$ and $10 - 2k$ are coloured red. If $\binom{10}{k}$ does not appear in the sum, then both $2k - 10$ and $10 - 2k$ are coloured white. If $\binom{10}{k}$ appears once in the sum, then $10 - 2k$ is red and $2k - 10$ is white. Thus the maximum value of n is obtained when the red squares are those numbered $\{1, 3, 5, 7, 9, -8, 8, -4, 4, -2, 2, 0, 6\}$ giving $n = 31$.

(Alternatively) If we do not assume a and b are relatively prime, then there are several more possibilities to consider. The greatest common divisor of a and b divides $a + b = 2001$, so $\gcd(a, b)$ is one of

$$1,\ 3,\ 23,\ 29,\ 3 \cdot 23,\ 3 \cdot 29,\ 23 \cdot 29,\ 3 \cdot 23 \cdot 29.$$

Since $a/b = c/1024$, dividing b by $\gcd(a, b)$ results in a power of 2. Thus the prime factorization of b is one of the following, for some integer k.

$$2^k,\ 3 \cdot 2^k,\ 23 \cdot 2^k,\ 29 \cdot 2^k,\ 69 \cdot 2^k,\ 87 \cdot 2^k,\ 667 \cdot 2^k,\ 2001.$$

Again we have $1001 \le b \le 2001$, so b must be one of the following numbers.

$$1024,\ 3 \cdot 512 = 1536,\ 23 \cdot 64 = 1472,\ 29 \cdot 64 = 1856,\ 69 \cdot 16 = 1104,\ 87 \cdot 16 = 1392,\ 667 \cdot 2 = 1334,\ 2001.$$

Thus $a/b = (2001 - b)/b$ is one of the following fractions.

$$\frac{977}{1024}, \frac{465}{1536}, \frac{529}{1472}, \frac{145}{1856}, \frac{897}{1104}, \frac{609}{1392}, \frac{667}{1334}, \frac{0}{2001}.$$

Thus $c = 1024a/b$ is one of the following integers.

$$977,\ 310,\ 368,\ 80,\ 832,\ 448,\ 512,\ 0.$$

After some (rather tedious) checking, one finds that only the following sums with terms from (1) can add to a possible value of c.

$$\begin{aligned}
977 &= 10 + 10 + 45 + 120 + 120 + 210 + 210 + 252 \\
310 &= 10 + 45 + 45 + 210 \\
512 &= 10 + 10 + 120 + 120 + 252 \\
512 &= 1 + 1 + 45 + 45 + 210 + 210 \\
0 &= 0.
\end{aligned}$$

Again only those terms appearing exactly once in a sum can affect maximum value of n. We make the following table.

c	Terms appearing once in sum	Corresponding red squares
977	$\{45, 252\}$	$\{6, 0\}$
310	$\{10, 210\}$	$\{8, 2\}$
512	$\{252\}$	$\{0\}$
512	$\emptyset$	$\emptyset$
0	$\emptyset$	$\emptyset$

Evidently, the maximum possible value of n is obtained when $c = 310 = \binom{10}{2} + \binom{10}{6} + \binom{10}{8} + \binom{10}{9}$, the red squares are $\{1, 3, 5, 7, 9, -6, 6, 2, 8\}$, the probability of landing on a red square is $a/b = 465/1536 = 310/1024 = 155/512$, and $n = 35$.

3. *Solution 1: (Daniel Brox)*
Set O be the centre of the circumcircle of $\triangle ABC$. Let the angle bisector of $\angle BAC$ meet this circumcircle at R. We have

$$\angle BOR = 2\angle BAR = 2\angle CAR = \angle COR$$

Thus $BR = CR$ and R lies on the perpendicular bisector of BC. Thus $R = P$ and $ABCP$ are concyclic. The points X, Y, M are the bases of the three perpendiculars dropped from P onto the sides of $\triangle ABC$. Thus by Simson's rule, X, Y, M are collinear. Thus we have $M = Z$ and $BZ/ZC = BM/MC = 1$.

Note: XYZ is called a *Simson line*, *Wallace line* or *pedal line* for $\triangle ABC$. To prove Simson's rule, we note that $BMPX$ are concyclic, as are $AYPX$, thus

$$\angle BXM = \angle BPM = 90 - \angle PBC = 90 - \angle PAC = \angle APY = \angle AXY$$

Solution 2: (Kenneth Ho)
Since $\angle PAX = \angle PAY$ and $\angle PXA = \angle PYA = 90$, triangles $\triangle PAX$ and $\triangle PAY$ are congruent, so $AX = AY$ and $PX = PY$. As P is on the perpendicular bisector of BC, we have $PC = PB$. Thus $\triangle PYC$ and $\triangle PXB$ are congruent right triangles, which implies $CY = BX$. Since X, Y and Z are collinear, we have by Menelaus' Theorem

$$\frac{AY}{YC}\,\frac{CZ}{ZB}\,\frac{BX}{XA} = -1.$$

Applying $AX = AY$ and $CY = BX$, this is equivalent to $BZ/ZC = 1$.

4. We shall see that the only solution is $n = 2$. First we show that if $n \neq 2$, then the table $T_0 = \begin{bmatrix} 1 \\ n-1 \end{bmatrix}$ can not be changed into a table containing two zeros. For $n = 1$, this is very easy to see. Suppose $n \geq 3$. For any table $T = \begin{bmatrix} a \\ b \end{bmatrix}$, let $d(T)$ be the quantity $b - a \pmod{n-1}$. We shall show that neither of the two permitted moves can change the value of $d(T)$. If we subtract n from both elements in T, then $b - a$ does not change. If we multiply the first row by n, then the element a changes to na, for a difference of $(n-1)a$, which is congruent to $0 \pmod{n-1}$. Similarly, multiplying the second row by n does not change $d(T)$. Since $d(T_0) = (n-1) - 1 \equiv -1 \pmod{n-1}$, we can never obtain the table with two zeros by starting with T_0, because $0 - 0$ is not congruent to -1 modulo $n - 1$.

For $n = 2$, and any table of positive integers, the following procedure will always result in a table of zeros. We shall begin by converting the first column into a column of zeros as follows.

We repeatedly subtract 2 from all entries in the first column until at least one of the entries equals either 1 or 2. Now we repeat the following sequence of three steps:

(a) multiply by 2 all rows with 1 in the first column
(b) now multiply by 2 every row having a 2 in the first column (there is at least one such row)
(c) subtract 2 from all entries in the first column.

Each iteration of the three steps decreases the sum of those entries in the first column which are greater than 2. Thus the first column eventually consists entirely of ones and twos, at which time we apply (a) and (c) once again to obtain a column of zeros. We now repeat the above procedure for each successive column of the table. The procedure does not affect any column which has already been set to zero, so we eventually obtain a table with all entries zero.

5. *(Daniel Brox)*
Let $\angle P_1P_3P_2 = 2\alpha$. As $\triangle P_1P_2P_3$ is isosceles, we have that

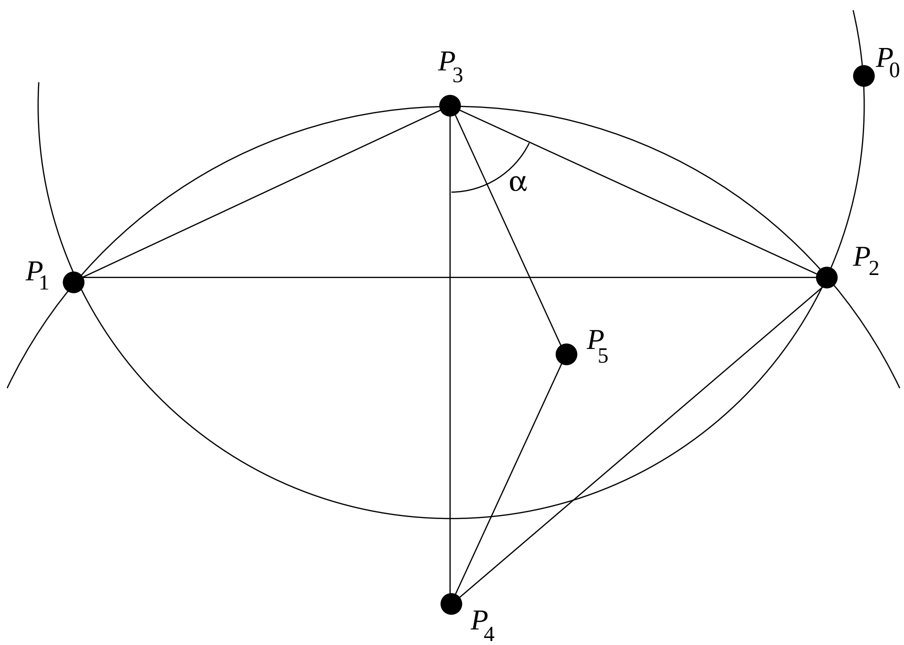
$$t = P_1P_2 = 2\sin\alpha.$$

The line P_3P_4 is the perpendicular bisector of P_1P_2. Since $\triangle P_2P_3P_4$ is isosceles, we calculate its length,

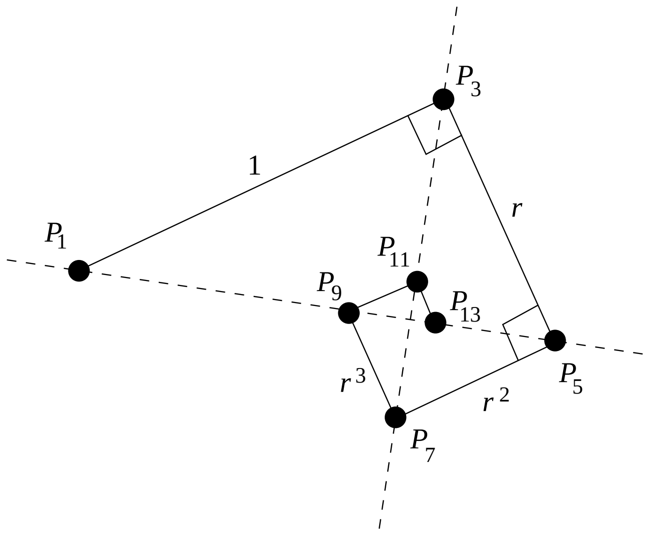
$$P_3P_4 = \frac{P_2P_3/2}{\cos\alpha} = \frac{1}{2\cos\alpha}.$$

As P_5 is the circumcentre of $\triangle P_2P_3P_4$, we have $\angle P_3P_5P_4 = 2\angle P_3P_2P_4 = 2\angle P_2P_3P_4 = 2\alpha$. The isosceles triangle $\triangle P_3P_4P_5$ is therefore similar to $\triangle P_1P_2P_3$. As $P_3P_4 \perp P_1P_2$, we have $\angle P_1P_3P_5 = 90$. Furthermore, the ratio $P_3P_5 : P_1P_3$ equals r where

$$r = \frac{P_3P_4}{P_1P_2} = \frac{1}{(2\sin\alpha)(2\cos\alpha)} = \frac{1}{2\sin(2\alpha)}.$$



By the same argument, we see that each $\angle P_iP_{i+2}P_{i+4}$ is a right angle with $P_{i+2}P_{i+4} : P_iP_{i+2} = r$. Thus the points $P_1, P_3, P_5, \ldots$ lie on a logarithmic spiral of ratio r and period four as shown below. It follows that $P_1, P_5, P_9, \ldots$ are collinear, proving part (a).



By the self-similarity of the spiral, we have that $P_1P_{1001} = r^{500}P_{1001}P_{2001}$, so

$$\sqrt[500]{x/y} = 1/r = 2\sin(2\alpha).$$

This is an integer when $\sin(2\alpha) \in \{0, \pm 1/2, \pm 1\}$. Since $0 < \alpha < 90$, this is equivalent to $\alpha \in \{15, 45, 75\}$. Thus $\sqrt[500]{x/y}$ is an integer exactly when t belongs to the set $\{2\sin 15,\ 2\sin 45,\ 2\sin 75\}$. This answers part (b).

GRADERS' REPORT

Eighty four of the eighty five eligible students submitted an examination paper. Each paper contained proposed solutions the some or all of the five examination questions. Each correct and well presented solution was awarded seven marks for a maximum total score of 35. The mean score was 10.8/35. The top three scores were 28, 27, and 22, thus special scrutiny was required to separate the top two papers.

Each solution was independently marked by two graders. If the two marks differed, then the solution was reconsidered until the difference was resolved. The top twenty papers were then carefully regraded by the chair to ensure that nothing was amiss.

The grade distribution and average mark for each question appears in the following table. For example, 13.1% of students were awarded 3 marks for question #1.

Marks	#1	#2	#3	#4	#5
0	10.7	11.9	45.2	60.7	90.5
1	8.3	8.3	26.2	10.7	3.6
2	8.3	6.0	4.8	8.3	1.2
3	13.1	9.5	3.6	9.5	1.2
4	10.7	17.9	0.0	4.8	1.2
5	9.5	32.1	3.6	2.4	2.4
6	20.2	13.1	0.0	1.2	0.0
7	19.0	1.2	16.7	2.4	0.0
Ave. Mark	4.05	3.64	1.79	1.09	.26

PROBLEM 1 Ninety five percent of students found the correct solution, although a surprising number arrived at a solution through trial and error or by guessing and verifying a solution. Many assumed without proof that the leading coefficient equals one, which resulted in a two-point penalty. Another common error was not to consider all four possibilities for the pair $(x - R), (x - 2)$.

PROBLEM 2 There was a flaw in question 2. The proposers intended that the integers a, b be relatively prime. This was made explicit in an early draft, but somehow was lost with the ambiguous phrase "of the form a/b." Without this assumption, the problem is much more tedious to solve. Remarkably, one student (Lino Demasi) considered more (but not all) possible values for $\gcd(a, b)$ and obtained the correct solution $n = 35$. All other students assumed implicitly (and in two cases, explicitly) that $\gcd(a, b) = 1$. Solutions to both problems are presented in this publication.

PROBLEM 3 Most students either completely solved or were baffled by this basic geometry problem. There were at least four types of solutions: one trigonometric, one using basic geometry, and two which refer to standard theorems relating to the triangle. The first two tended to be lengthy or cumbersome, and the last two are presented here. There were complaints from some participants regarding the inaccurate angles appearing in the diagram supplied with the question. The inaccuracy was intensional, since the key observation $M = Z$ would have otherwise been given away. Unfortunately, this caused some students to doubt their own proofs that $BZ : ZC = 1$; as the ratio appears to be closer to 2 in the misleading diagram!

PROBLEM 4 This problem was left unanswered by about 60% of students. Several solutions consisted only of a proof that $n = 1$ is not possible. About 25% described a procedure which works when $n = 2$. Indeed the procedure for $n = 2$ seems to be unique. About 10% proved that for no other value of n was possible, and all of the proofs explicitly or implicitly involved considering residues modulo $n - 1$.

PROBLEM 5 This problem proved to be very difficult. Only two students completely answered part (a), and no students correctly answered part (b). Of the students receiving more than 0 marks, only two were were not among the top 15 This suggests that the question effectively resolved the ranking of the strongest participants, which is arguably the purpose of Problem 5.