

1. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$ be a function, and define $h : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^+$ by $h(x, y) = \gcd(f(x), f(y))$. If $h(x, y)$ is a two-variable polynomial in x and y , prove that it must be constant.
2. Alphonse and Beryl play a game involving n safes. Each safe can be opened by a unique key and each key opens a unique safe. Beryl randomly shuffles the n keys, and after placing one key inside each safe, she locks all of the safes with her master key. Alphonse then selects m of the safes (where $m < n$), and Beryl uses her master key to open just the safes that Alphonse selected. Alphonse collects all of the keys inside these m safes and tries to use these keys to open up the other $n - m$ safes. If he can open a safe with one of the m keys, he can then use the key in that safe to try to open any of the remaining safes, repeating the process until Alphonse successfully opens all of the safes, or cannot open any more. Let $P_m(n)$ be the probability that Alphonse can eventually open all n safes starting from his initial selection of m keys.
 - (a) Show that $P_2(3) = \frac{2}{3}$.
 - (b) Prove that $P_1(n) = \frac{1}{n}$.
 - (c) For all integers $n \geq 2$, prove that

$$P_2(n) = \frac{2}{n} \cdot P_1(n - 1) + \frac{n - 2}{n} \cdot P_2(n - 1).$$

- (d) Determine a formula for $P_2(n)$.
3. Let $1000 \leq n = ABCD_{10} \leq 9999$ be a positive integer whose digits $ABCD$ satisfy the divisibility condition:

$$1111 \mid (ABCD + AB \times CD).$$

Determine the smallest possible value of n .

4. In $\triangle ABC$, the interior sides of which are mirrors, a laser is placed at point A_1 on side BC . A laser beam exits the point A_1 , hits side AC at point B_1 , and then reflects off the side. (Because this is a laser beam, every time it hits a side, the angle of incidence is equal to the angle of reflection). It then hits side AB at point C_1 , then side BC at point A_2 , then side AC again at point B_2 , then side AB again at point C_2 , then side BC again at point A_3 , and finally, side AC again at point B_3 .
 - (a) Prove that $\angle B_3A_3C = \angle B_1A_1C$.
 - (b) Prove that such a laser exists if and only if all the angles in $\triangle ABC$ are less than 90° .
5. Let $f(x) = x^4 + 2x^3 - x - 1$.
 - (a) Prove that $f(x)$ cannot be written as the product of two non-constant polynomials with integer coefficients.
 - (b) Find the exact values of the 4 roots of $f(x)$.

6. Given a triangle A, B, C , X is on side AB , Y is on side AC , and P and Q are on side BC such that $AX = AY$, $BX = BP$ and $CY = CQ$. Let XP and YQ intersect at T . Prove that AT passes through the midpoint of PQ .
7. A bug is standing at each of the vertices of a regular hexagon $ABCDEF$. At the same time each bug picks one of the vertices of the hexagon, which it is not currently in, and immediately starts moving towards that vertex. Each bug travels in a straight line from the vertex it was in originally to the vertex it picked. All bugs travel at the same speed and are of negligible size. Once a bug arrives at a vertex it picked, it stays there. In how many ways can the bugs move to the vertices so that no two bugs are ever in the same spot at the same time?
8. For any given non-negative integer m , let $f(m)$ be the number of 1's in the base 2 representation of m . Let n be a positive integer. Prove that the integer

$$\sum_{m=0}^{2^n-1} \left((-1)^{f(m)} \cdot 2^m \right)$$

contains at least $n!$ positive divisors.