2019 CMO Qualifying Repêchage

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

Official Solutions

1. A function f is called injective if when $f(n) = f(m)$, then $n = m$. Suppose that f is injective and

$$\frac{1}{f(n)} + \frac{1}{f(m)} = \frac{4}{f(n) + f(m)}.$$

Prove $m = n$.

Solution: Clearing denominators gives:

$$\begin{aligned}
(f(n) + f(m))(f(n) + f(m)) &= 4f(n)f(m) \\
f(n)^2 + 2f(m)f(n) + f(m)^2 &= 4f(n)f(m) \\
f(n)^2 - 2f(m)f(n) + f(m)^2 &= 0 \\
(f(n) - f(m))^2 &= 0
\end{aligned}$$

Thus, $f(n) = f(m)$, and since f is injective, that means $m = n$.

2. Rosemonde is stacking spheres to make pyramids. She constructs two types of pyramids S_n and T_n. The pyramid S_n has n layers, where the top layer is a single sphere and the i^{th} layer is an $i \times i$ square grid of spheres for each $2 \leq i \leq n$. Similarly, the pyramid T_n has n layers where the top layer is a single sphere and the i^{th} layer is $\frac{i(i+1)}{2}$ spheres arranged into an equilateral triangle for each $2 \leq i \leq n$.

   If all the spheres have radius 2, determine the smallest n so that the difference between the height of S_n and the height of T_n is greater than 2019.

   Solution: We can determine formuale for the heights of S_n and T_n as:

$$H(S_n) = 4 + 2\sqrt{2}(n-1)$$
$$H(T_n) = 4 + \frac{4}{3}\sqrt{6}(n-1)$$

   Setting $|H(S_n) - H(T_n)| > 2019$, we find that:

$$n > \frac{2019}{\frac{4}{3}\sqrt{6} - 2\sqrt{2}} + 1.$$

   And the smallest integer value for n is 4616.

3. Let $f(x) = x^3 + 3x^2 - 1$ have roots a, b, c.

   (a) Find the value of $a^3 + b^3 + c^3$.

   (b) Find all possible values of $a^2b + b^2c + c^2a$.

Solution:

(a) Vieta's formula gives $abc = 1, ab + ac + bc = 0,$ and $a + b + c = -3$. Let $p = a^2b + b^2c + c^2a$, and let $q = a^c + c^2b + b^2a$. First, we noet that $p + q = (ab + ac + bc)(a + b + c) - 3abc = -3$. So,

$$\begin{aligned}
a^3 + b^3 + c^3 &= (a + b + c)^3 - 3(p + q) - 6abc \\
&= (-3)^3 - 3(-3) - 6(1) \\
&= -24
\end{aligned}$$

(b) Now,

$$\begin{aligned}
pq &= (a^4bc + b^4ca + c^4ab) + 3a^2b^2c^2 + (a^3b^3 + b^3c^3 + c^3a^3) \\
&= abc(a^3 + b^3 + c^3) + 3(abc)^2 \\
&+ ((a^2b^2 + a^2c^2 + b^2c^2)(ab + ac + bc) - abc(p + q)) \\
&= 1(-24) + 3(1)^2 + ((a^2b^2 + a^2c^2 + b^2c^2)(0) - 1(-3)) \\
&= -18
\end{aligned}$$

Thus, $\{p, q\}$ are the roots of $y^2 + 3y - 18 = 0$, hence $\{p, q\} = \{-6, 3\}$. Since switching b and c switches p and q, we see that both values are possible, hence $\{-6, 3\}$ is the set of possible values for $a^2b + b^2c + c^2a$.

4. Let n be a positive integer. For a positive integer m, we partition the set $\{1, 2, 3, \ldots, m\}$ into n subsets, so that the product of two different elements in the same subset is never a perfect square. In terms of n, find the largest positive integer m for which such a partition exists.

   Solution: Suppose that $m \geq (n+1)^2$. Then each of the $n+1$ numbers $1^2, 2^2, \ldots, (n+1)^2$ must lie in a different subset. But there are only n subsets, so such a partition is not possible.

   Conversely, assume that $m \leq (n+1)^2 - 1 = n^2 + 2n$. For $1 \leq i \leq m$, we can write i uniquely in the form
$$i = a_i^2 \cdot b_i,$$
   where a_i and b_i are positive integers, and b_i is not divisible by the square of a prime. Since $i \leq m < (n+1)^2, a_i \leq n$ for all $1 \leq i \leq m$.

   For $1 \leq j \leq n$, let
$$S_j = \{i : 1 \leq i \leq m, a_i = j\}.$$
   We see that $S_1, S_2, S_3, \ldots, S_n$ form a partition of $\{1, 2, 3, \ldots, m\}$. We claim that the product of two different elements in the same subset S_j is never a perfect square.

   For the sake of contradiction, suppose that there exist distinct $k \in S_j$ and $l \in S_j$ so that kl is a perfect square. Since
$$kl = (a_k^2 \cdot b_k)(a_l^2 \cdot b_l) = (j^2 \cdot b_k)(j^2 \cdot b_l) = j^4 b_k b_,$$
   the product $b_k b_l$ must also be a perfect square.

   Since $k = j^2 \cdot b_k$ and $l = j^2 \cdot b_l$ are distinct, b_k and b_l must also be distinct. This means that there must be some prime p that appears in the prime factorization of one of b_k and b_l, but not the other. Without loss of generality, assume that p appears in the prime factorization of b_k but not b_l. Since b_k is not divisible by the square of a prime, b_k has exactly one factor of p. But then $b_k b_l$ also has exactly one factor of p, so it cannot be a perfect square, contradiction. Thus, the partition has exactly the property we seek. We conclude that the largest such possible value of m is $n^2 + 2n$.

5. Let (m, n, N) be a triple of positive integers. Bruce and Duncan play a game on an $m \times n$ array, where the entries are all initially zeroes. The game has the following rules.

   - The players alternate turns, with Bruce going first.
   - On Bruce's turn, he picks a row and either adds 1 to all of the entries in the row or subtracts 1 from all the entries in the row.
   - On Duncan's turn, he picks a column and either adds 1 to all of the entries in the column or subtracts 1 from all of the entries in the column.
   - Bruce wins if at some point there is an entry x with $|x| \geq N$.

   Find all triples (m, n, N) such that no matter how Duncan plays, Bruce has a winning strategy.

   Solution: Bruce can win in all cases except $m = n = 1, N > 1$. Indeed, if $m = n = 1$, then Bruce obviously wins with $N = 1$, and if $N > 1$ then Duncan can just undo Bruce's move every time, resetting the array back to 0.

   We now break into two cases: $m > 1$ and $n > 1$. If $n > 1$, then Bruce will follow the strategy of adding 1 to the first row every time. Let the sum of the elements of the first row be S_r after r turns. Then $S_0 = 0$, on Bruce's turn he increases the sum by n, and on Duncan's turn he must add either 1 or -1 to the sum, by adding that amount to the element in the first row of whichever column he chooses. In particular, after any 2 turns, the sum of the elements of the first row goes up by at least $n - 1 \geq 1$. Therefore $S_{2r} \geq r$, and therefore $S_{2nN} \geq nN$. This is a sum of n integers, therefore, at least one of them is greater than or equal to $\frac{nN}{n} = N$, and Bruce has won.

   Now assume $m > 1$. After r turns, let the element in the top left of the board be a_r, and the element below it be b_r. Define $d_r = a_r - b_r$. Bruce's strategy is gain to add 1 to the first row at every turn. Thus if the $r + 1^{\text{th}}$ turn is Bruce's, then $d_r + 1 = d_r + 1$. However if it were Duncan's turn,then he changes both a_r and b_r by the same value (either -1, 0, or 1), which does not change d_r. Hence we have that $d_{2r} \geq r$. In particular, $d_{4N-3} \geq 2N - 1$. If Bruce has not won yet, then $-N + 1 \leq a_{4N-3}, b_{4N-3} \leq N - 1$ and so $d_{4N-3} \leq (N - 1) - (-N + 1) = 2N - 2$, which is a contradiction, so Bruce wins.

6. Pentagon $ABCDE$ is given in the plane. Let the perpendicular from A to line CD be F, the perpendicular from B to DE be G, from C to EA be H, from D to AB be I, and from E to BC be J. Given that lines AF, BG, CH, and DI concur, show that they also concur with line EJ.

   Solution: We start with a lemma. Given four points W, X, Y, Z on the plane, lines WX and YZ are perpendicular if and only if $WY^2 - WZ^2 = XY^2 - XZ^2$.

   Proof: Let WX be on the x-axis, with coordinates $(w, 0)$ and $(x, 0)$. Let $Y = (y_1, y_2)$ and $Z = (z_1, z_2)$ be the coordinates of points Y and Z. Then, we have that $WY^2 - XY^2 = (w - y_1)^2 - (x - y_1)^2 = w^2 - x^2 - 2(w - x)y_1$. Similarly, $WZ^2 - XZ^2 = w^2 - x^2 - 2(w - x)z_1$, so that these two quantities are equal precisely when $y_1 = z_1$; that is, YZ and WX are perpendicular.

   Let the common intersection of lines AF, BG, CH, and DI be point P. We know that because PA and CD are perpendicular, we have $PC^2 - PD^2 = AC^2 - AD^2$. Similarly, we have the equations $PD^2 - PE^2 = BD^2 - BE^2$, $PE^2 - PA^2 = CE^2 - CA^2$, and $PA^2 - PB^2 = DA^2 - DB^2$. Adding all of these equations up, we get that $PB^2 - PC^2 = EB^2 - EC^2$, and hence that lines EP and BC are perpendicular.

7. There are n passengers in a line, waiting to board a plane with n seats. For $1 \leq k \leq n$, the k^{th} passenger in line has a ticket for the k^{th} seat. However, the first passenger ignores his ticket, and decides to sit in a seat at random. Thereafter, each passenger sits as follows: If his/her assigned is empty, then he/she sits in it. Otherwise, he/she sits in an empty seat at random. How many different ways can all n passengers be seated?

   Solution: For each possible sequence of seats taken, we construct a sequence as follows: $a_0 = 1$, and for $i \geq 1, a_i$ is the number of the seat that passenger $a_i - 1$ sits in. The last term of the sequence is the number of the passenger sitting in the first seat. For example, if $n = 5$, and passengers 4, 1, 3, 2, 5 are sitting in seats 1, 2, 3, 4, 5, respectively, then the sequence is 1, 2, 4.

   We consider how the passengers take their seats, starting with the first passenger. By construction, the first passenger sits in seat a_1. Then passengers $2, 3, \ldots, a_1 - 1$ take their assigned seats, until passenger a_1, who discovers that his seat is already taken. If passenger a_1 sits in the first seat, then the sequence terminates (with the term a_1). Also, all subsequent passengers take their assigned seats. Otherwise, passenger a_1 sits in some seat numbered a_2, where $a_2 > a_1$.

   Then passengers $a_1 + 1, a_1 + 2, \ldots, a_2 - 1$ take their assigned seats, until passenger a_2, who discovers that his seat is already taken. If passenger a_2 sits in the first seat, then the sequence terminates (with the term a_2). Also, all subsequent passengers take their assigned seats. Otherwise, passenger a_2 sits in some seat numbered a_3, where $a_3 > a_2$, and so on.

   This process continues, until a passenger (say passenger a_k) sits in the first seat, and the remaining passengers take their assigned seats. From our work above, we see that

   - The sequence $a_0 = 1, a_1, a_2, \ldots, a_k$ is increasing, and
   - If the number j does not appear in the sequence above, where $2 \leq j \leq n$, then passenger j sits in his assigned seat.

   Thus, each possible seating corresponds to a subset $\{a_1, a_2, \ldots, a_k\}$ of $\{2, 3, \ldots, n\}$. There are 2^{n-1} such subsets, so there are 2^{n-1} different possible seatings.

8. For $t \geq 2$, define $S(t)$ as the number of times t divides into $t!$. We say that a positive integer t is a *peak* if $S(t) > S(u)$ for all values of $u < t$.

   Prove or disprove the following statement:

   For every prime p, there is an integer k for which p divides k and k is a peak.

   Solution: First, for a prime p and number n, the number of factors of p that divide $n!$ is equal to

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Also, note that since $S(n)$ can be arbitrarily large, there must be infinitely many peaks. Let $T(n) = S(n)/n$. Now, consider a peak n_0 greater than

$$\prod_{q \text{ prime}} q^{\lfloor \frac{p-1}{q-1} \rfloor},$$

which is a finite sum because there is no term when $q > p$. Then, there must be some prime q so that

$$q^{\lfloor \frac{p-1}{q-1} \rfloor + 1} | n_0.$$

Assume for sake of contradiction that p does not such an n. Then, we claim that $S(pn) \geq pS(n)$. We have that for every $q|p$, the number of times that q divides $(pn)!$ is at least p times the number of times that q divides $n!$. Finally, we claim that for our chosen prime q with the above inequality, $S(pn)$ has more factors of p than it does q^k, where k is the number of times q divides n. Indeed, it remains to show that

$$\frac{1}{k} \sum_{i=1}^{\infty} \left\lfloor \frac{n}{q^i} \right\rfloor \leq \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

When $q > p$, this is clearly true. Assume $q < p$. But we may bound

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{q^i} \right\rfloor \leq \frac{n}{q-1} + \log_q n + 2,$$

because we may bound the first $\log_q n$ terms of the fractional part by 1 and the remainder by themselves. Similarly, we have

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \geq \frac{n}{p-1},$$

simply by removing the floors, so we just require

$$k \geq \frac{p-1}{q-1} + \frac{(p-1)(\log_q n + 2)}{n}.$$

However, the second term is clearly smaller than $\frac{1}{q-1}$, because we have $n > (p-1)(q-1)(\log_q n + 2)$ by our above bound on n. In particular, we have that $S(pn_0) \geq pS(n_0)$, so that if pn_0 is not a peak, then there must be some peak $n_1 \in [n_0, pn_0)$ with $S(n_1) > pS(n_0)$. In particular, $T(n_1) > T(n_0)$. Since $n_1 > n_0$, the conditions above are satisfied again, and we can construct an increasing sequence of peaks $n_0, n_1, \ldots$ where the T-values are also increasing. However, for any fixed ϵ, there are only finitely many n with $T(n) > \epsilon$; for instance, the maximum exponent on any prime q must be at most $\frac{1}{(q-1)\epsilon}$. This provides a contradiction.

