OC451. Determine the least natural number $a$ such that

$$a \geq \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)$$

for any nonzero natural number $n$ and for any positive real numbers $a_1, a_2, \ldots, a_n$ whose sum is at most $\pi$.

We received 1 submission. We present the solution by Oliver Geupel.

We show that

$$\sup \left\{ \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) : n \geq 1; a_1, \ldots, a_k > 0; \sum_{k=1}^{n} a_k \leq \pi \right\} = 1,$$

which implies that the least value of $a$ is 1. For $1 \leq k \leq n$, let

$$x_k = \sum_{j=1}^{k} a_j.$$

Then, $0 = x_0 < x_1 < x_2 < \cdots < x_n \leq \pi$, and the sum

$$\sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cos x_k \quad (1)$$

is a right Riemann sum which underestimates the integral

$$I(x_n) = \int_{0}^{x_n} \cos x \, dx$$

of the decreasing function $\cos x$ on the interval $[0, x_n]$. Since $I(x_n) = \sin x_n \leq 1$ for every $x_n \leq \pi$, we obtain that $a \leq 1$. However, if $a_k = \pi/(2n)$ and $x_k = k\pi/(2n)$ for any $1 \leq k \leq n$ then the sequence defined by the sums (1) converges towards $I(\pi/2) = 1$ as $n \to \infty$. Hence $a = 1$.

Editor’s Comment. The restrictions on $a_n$’s can be changed. For example, if

$$a_1 + \cdots + a_n \leq \pi/2,$$

then the value of the upper bound $a$ is 1, as before. However, if

$$a_1 + \cdots + a_n = \pi,$$

then $a = 0$. 
OC452. Let $ABCD$ be a square. Consider the points $E \in AB$, $N \in CD$ and $F, M \in BC$ such that triangles $AMN$ and $DEF$ are equilateral. Prove that $PQ = FM$, where $\{P\} = AN \cap DE$ and $\{Q\} = AM \cap EF$.

We received 11 correct submissions. We present two solutions.

Solution 1, by Miguel Amengual Covas.

The right-angled triangles $ABM$ and $ADN$ have equal hypotenuses $AM$ and $AN$, and the legs $AB$ and $AD$ are respectively equal. Thus $\triangle ABM$ and $\triangle ADN$ are congruent with $\angle MAB = \angle NAD$.

Now, $\angle DAB = \angle MAB + \angle NAM + \angle NAD$. Next, since $\angle DAB = 90^\circ$ and $\angle NAM = 60^\circ$, it follows that $\angle MAB$ and $\angle NAD$ are each $15^\circ$. Analogously, $\angle CDF = \angle ADE = 15^\circ$.

Clearly, then, $\triangle ABM$, $\triangle CDF$, $\triangle DAE$, $\triangle ADN$ are congruent (these are right-angled triangles which have equal legs $AB$, $CD$ and $DA$ and contain another pair of equal angles) with $AE = BM = FC = ND$.

Consequently, $AEND$ is a rectangle, so that the segments $AN$ and $DE$ bisect each other. Thus $P$ is the midpoint of segment $DE$. Then, we have in equilateral triangle $DEF$ that $\angle FPE$ is a right angle and $\angle PFE = 30^\circ$.

Subtracting $AE = FC$ from both sides of $AB = BC$ gives $AB - AE = BC - FC$. This makes $EB = BF$ and consequently $\triangle EBF$ is an isosceles right-triangle with $\angle EFB = 45^\circ$. Therefore,

\[
\angle PFM = \angle PFB = \angle PFE + \angle EFB = 30^\circ + 45^\circ = 75^\circ = 90^\circ - 15^\circ = 90^\circ - \angle MAB = \angle AMB,
\]

implying that $PF$ is parallel to $AM$, that is, $PF \parallel QM$.

Next, $PQ$ subtends $60^\circ$ angles at $A$ and $E$, making $PAEQ$ cyclic and on chord $EQ$ we have

$$\angle QPE = \angle QAE = \angle MAB = 15^\circ$$

and

$$\angle FPQ = \angle FPE - \angle QPE = 90^\circ - 15^\circ = 75^\circ.$$ 

That is to say, the exterior angle $QMB$ in quadrilateral $PQMF$ is equal to the interior and opposite angle $P$. Thus $PQMF$ is cyclic. Since $PF \parallel QM$, $PQMF$ is an isosceles trapezium. The conclusion follows.

Solution 2, by Miguel Amengual Covas.

As in Solution 1, we conclude that $\angle MAB = \angle NAD = \angle CDF = \angle ADE = 15^\circ$.

Suppose (wlog) the unity of measurement equal to the length of the side of the given square. Then

$$DN = AE = BM = CF = \tan 15^\circ.$$  \hspace{1cm} (1)  

Hence,

$$FM = BC - CF - MB = 1 - 2 \tan 15^\circ.$$  \hspace{1cm} (2)  

Moreover, since $AE \parallel DN$, $AEND$ is a rectangle, so that segments $AN$ and $DE$ bisect each other. Therefore, $P$ is the midpoint of $DE$ and we have

$$PE = \frac{1}{2} DE = \frac{1}{2 \cos 15^\circ}.$$  \hspace{1cm} (3)  

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Observing that the equal segments \(FB\) and \(BE\) make \(EBF\) an isosceles right-angled triangle, the exterior angle theorem, applied to \(\triangle AEQ\) at \(E\), yields

\[
\angle EQA = \angle QEB - \angle QAE = \angle FEB - \angle MAB = 45^\circ - 15^\circ = 30^\circ.
\]

Now, the law of sines asserts that

\[
\frac{QE}{\sin 15^\circ} = \frac{AE}{\sin 30^\circ},
\]

and therefore

\[
\frac{QE}{\sin 15^\circ} = 2 \cdot AE,
\]

yielding (by (2))

\[
QE = \frac{2 \sin^2 15^\circ}{\cos 15^\circ}.
\] (4)

Applying the law of cosines to \(\triangle PEQ\) we get

\[
PQ^2 = PE^2 + QE^2 - 2 \cdot PE \cdot QE \cdot \cos 60^\circ.
\]

Substituting for \(PE\) and \(QE\) from (4) and (5),

\[
PQ^2 = \frac{1}{4 \cos^2 15^\circ} + \frac{4 \sin^4 15^\circ}{\cos^2 15^\circ} - \tan^2 15^\circ.
\] (5)

Now, we write the identity \(2 \sin 30^\circ = 1\) in the equivalent form \(4 \sin 15^\circ \cos 15^\circ = 1\), multiply it by \(\tan 15^\circ\) and square, obtaining \(16 \sin^4 15^\circ = \tan^2 15^\circ\). Hence we can rewrite (6) as

\[
PQ^2 = \frac{1 + \tan^2 15^\circ}{4 \cos^2 15^\circ} - \tan^2 15^\circ,
\]
or, equivalently,

\[
PQ^2 = \frac{1}{4 \cos^4 15^\circ} - \tan^2 15^\circ.
\]

This, in turn, is equivalent to

\[
PQ^2 = \frac{1 - 4 \sin^2 15^\circ \cos^2 15^\circ}{4 \cos^4 15^\circ} = \frac{1 - \sin^2 30^\circ}{4 \cos^4 15^\circ} = \frac{3}{16 \cos^4 15^\circ}.
\]

Thus

\[
PQ = \frac{\sqrt{3}}{4 \cos^2 15^\circ} = \sqrt{3} \tan 15^\circ.
\] (6)

Taking into account that \(\tan 15^\circ = 2 - \sqrt{3}\), from (3) and (7) we conclude that

\[
FM = 2\sqrt{3} - 3 = PQ.
\]

_Crux Mathematicorum_, Vol. 46(4), April 2020
**OC453.** Let $n \geq 2$ be an integer and let $A, B \in M_n(\mathbb{C})$. If $(AB)^3 = O_n$, is it true that $(BA)^3 = O_n$? Justify your answer.

We received 4 correct submissions. We present two solutions.

**Solution 1, by Oliver Geupel.**

We prove that the deduction is correct if and only if $n \leq 3$.

First, let $n \leq 3$ and let $\lambda$ be an eigenvalue of the matrix $C = BA$ with eigenvector $v$. Then, $\lambda v = Cv$ and

$$
\lambda^4 v = C \cdot \lambda^3 v = C^2 \cdot \lambda^2 v = C^3 \cdot \lambda v = C^4 v = B(AB)^3 Av = BO_n Av = O_n.
$$

Hence $\lambda = 0$. So 0 is the only eigenvalue of $C$. The characteristic polynomial of $C$ is then $\lambda^n$. By the Cayley-Hamilton theorem, the matrix $C$ satisfies its own characteristic equation, so that $C^n = O_n$ and therefore $(BA)^3 = C^3 = O_n$.

We now turn to the case where $n \geq 4$. Let

$$
A_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B_4 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Consider the $n$-by-$n$ block matrices

$$
A = \begin{bmatrix}
A_4 & O_{4 \times (n-4)} \\
O_{(n-4) \times 4} & O_{(n-4) \times (n-4)}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_4 & O_{4 \times (n-4)} \\
O_{(n-4) \times 4} & O_{(n-4) \times (n-4)}
\end{bmatrix}.
$$

Straightforward computations yield $(A_4 B_4)^3 = O_4$ and

$$(B_4 A_4)^3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Thus,

$$(AB)^3 = \begin{bmatrix}
(A_4 B_4)^3 & O_{4 \times (n-4)} \\
O_{(n-4) \times 4} & O_{(n-4) \times (n-4)}
\end{bmatrix} = O_n
$$

and

$$(BA)^3 = \begin{bmatrix}
(B_4 A_4)^3 & O_{4 \times (n-4)} \\
O_{(n-4) \times 4} & O_{(n-4) \times (n-4)}
\end{bmatrix} \neq O_n.
$$

This completes the proof.
Solution 2, by Missouri State University Problem Solving Group.

We prove a more general result. Fix \( k \geq 1 \). Then

1. If \( n \leq k \) then \((AB)^k = 0\) implies \((BA)^k = 0\).

2. If \( n \geq k + 1 \) then there exist \( n \times n \) matrices \( A \) and \( B \) such that \((AB)^k = 0\) but \((BA)^k \neq 0\).

Suppose \( n \leq k \) and \((AB)^k = 0\). Every eigenvalue of \( AB \) is 0 and the characteristic polynomial of \( AB \) is \( x^n \). But \( AB \) and \( BA \) have the same characteristic polynomial, hence \((BA)^k = 0\). This proves (1).

2. Let \( n = k + 1 \), let \( A \) be the \( n \times n \) matrix whose \((i,j)\) entry is 1 if and only if \( i = j > 1 \) and 0 otherwise, and let \( B \) be the \( n \times n \) matrix whose \((i,j)\) entry is 1 if and only if \( j = i + 1 \) and 0 otherwise. Then \( BA = B \) and \( AB = C \) where the \((i,j)\) entry of \( C \) is 1 if and only if \( j = i + 1 > 2 \) and 0 otherwise.

A direct calculation yields \((AB)^{n-1} = 0\), but \((BA)^{n-1}\) is non-zero. By taking the direct sum of \( A \) and \( B \) with the zero matrix, we can get counterexamples for all larger \( n \).

For the original problem with \( k = 3 \), we take \( n = 4 \) and get

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then

\[
AB = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
(AB)^3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

but

\[
BA = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
(BA)^3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**OC454.** Find all the functions \( f : \mathbb{N} \to \mathbb{N} \) having the following property for each natural number \( m \): if \( d_1, d_2, \ldots, d_n \) are all the divisors of the number \( m \), then

\[
f(d_1)f(d_2) \cdots f(d_n) = m.
\]

We received 8 submissions. We present the solution by Oliver Geupel.

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It is readily checked that the following function is a solution to the problem:

\[ f(m) = \begin{cases} 
  p & \text{if } m = p^k \text{ where } p \text{ is a prime number and } k \geq 1 \\
  1 & \text{otherwise.}
\end{cases} \]

We show that there are no other solutions. Suppose \( f \) is any solution.

Putting \( m = 1 \) in the given condition, we obtain \( f(1) = 1 \). Setting \( m = p \) where \( p \) is a prime number in the given condition, we get \( f(p) = f(1)f(p) = p \). A straightforward induction shows that \( f(p^k) = p \) for \( k \geq 1 \).

Finally, let \( m \) have at least two distinct prime divisors, say \( m = p_1^{k_1}p_2^{k_2} \cdots p_\ell^{k_\ell} \), where \( p_1, \ldots, p_\ell \) are distinct prime divisors (\( \ell \geq 2 \)) and \( k_j \geq 1 \) for \( 1 \leq j \leq \ell \).

Let \( D \) denote the set of those divisors of \( m \) that have at least two distinct prime divisors. Then,

\[
\prod_{d \in D} f(d) = m \prod_{d \in D} f(d).
\]

Hence,

\[
\prod_{d \in D} f(d) = 1.
\]

It follows \( f(d) = 1 \) for all \( d \in D \). Since \( m \in D \), we conclude that \( f(m) = 1 \).

\textbf{OC455.} Let \( D \) be a point on the base \( AB \) of an isosceles triangle \( ABC \). Select a point \( E \) so that \( ADEC \) is a parallelogram. On the line \( ED \), take a point \( F \) such that \( E \in DF \) and \( EB = EF \). Prove that the length of the chord that the line \( BE \) cuts on the circumcircle of triangle \( ABF \) is twice the length of the segment \( AC \).

We received 4 submissions. We present the solution by Ivko Dimitrić.

Without loss of generality, we may assume that the vertices are labeled counterclockwise and that \( |DB| \leq |AD| \). Let \( \omega \) be the circumcircle of \( \triangle ABF \), \( O \) its center and \( K \) the point where the line \( BE \) meets \( \omega \) again. Let \( M \) be the midpoint of \( BK \), \( S \) the midpoint of \( BF \) and \( N = \overrightarrow{CE} \cap \overrightarrow{BF} \). Further, set

\[
\alpha = \angle CAB = \angle CBA = \angle ECB, \quad \theta = \angle FBE = \angle EFB \quad \text{and} \quad \varphi = \angle ECM.
\]

To prove the claim, it suffices to show that \( \triangle CBM \) is isosceles.

From parallelogram \( ADEC \) we have \( \angle CED = \alpha \) and from isosceles \( \triangle BFE \) we get \( \angle DEB = 2\theta \), so that \( \angle CEB = \alpha + 2\theta \). The points \( O, E \) and \( S \) are collinear, because \( BE = EF \) and \( E, O \) belong to the perpendicular bisector of \( BF \) at \( S \). Since \( |DB| \leq |AD| \) the foot of the perpendicular from \( B \) to \( CN \) belongs to the segment \( \overrightarrow{CE} \) just as the foot of the perpendicular from \( C \) to \( AB \) belongs to \( \overrightarrow{AD} \).
That implies that $\angle BEN \geq 90^\circ$ and $\angle MEO = \angle BES < 90^\circ$. Hence, $S$ is between $B$ and $N$ and $M$ is between $E$ and $K$.

Since $OM \perp BK$ and $O$ and $C$ belong to the perpendicular bisector of $AB$ we have $\angle ECO = \angle OME = 90^\circ$ and the quadrilateral $CEMO$ is cyclic so that

$$\angle ECM = \angle EOM = \varphi$$ and $$\angle OEC = \angle OMC.$$ 

Since the triangles $ESN$ and $OEM$ are right-angled we have

$$\angle CNB = \angle ENS = 90^\circ - \angle SEN$$
$$= 90^\circ - \angle OEC$$
$$= 90^\circ - \angle OMC$$
$$= \angle CME = \angle CMB.$$ 

Therefore, quadrilateral $BCMN$ is cyclic so that $\varphi = \angle NCM = \angle NBM = \theta$. 

Now, we have

$$\angle BCM = \angle BCE + \angle ECM = \alpha + \varphi$$

and from $\triangle CME$ we get

$$\angle CMB = \angle CEB - \angle ECM = (\alpha + 2\theta) - \varphi = \alpha + 2\varphi - \varphi = \alpha + \varphi.$$ 

Therefore, we conclude that $\angle BCM = \angle CMB$, so that

$$AC = CB = MB = \frac{1}{2} BK,$$

proving the claim.