BONUS PROBLEMS

These problems appear as a bonus. Their solutions will not be considered for publication.


Let $ABCD$ be a cyclic quadrilateral. Let $P$ be a point on the arc $BC$ and let $L$ and $Q$ be the feet of perpendiculars dropped from $P$ on the sides $AD$ and $BC$, respectively. Let $M$ and $N$ be the feet of perpendiculars dropped from $P$ on the lines $AB$ and $DC$, respectively. Prove that

$$
\frac{AM}{MB} \cdot \frac{BQ}{QC} \cdot \frac{CN}{ND} \cdot \frac{DL}{LA} = 1.
$$


Find the real numbers $x, y, z$ and $t$ such that

$$
xt - yz = -1 \quad \text{and} \quad x^2 + y^2 + z^2 + t^2 - xz - yt = \sqrt{3}.
$$


Let $a, b$ and $c$ be positive real numbers such that $ab + bc + ca = 3$. Prove the inequality

$$
\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \leq 1.
$$


Let $ABC$ be a triangle with no angle exceeding $120^\circ$ with $BC = a, AC = b$ and $AB = c$. Let $T$ be its Fermat-Torricelli point, that is the point such that the total distance from the three vertices of $ABC$ to $T$ is minimum possible. Prove that

$$(b + c)|TA| + (c + a)|TB| + (a + b)|TC| \geq \sqrt{3}(|TA| + |TB| + |TC|)^2 - 4\text{Area}(ABC).$$


Let $n$ be an integer such that $n \geq 4$. Consider real numbers $a_k, 1 \leq k \leq n$ such that $2 \geq a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n$ and $\sum_{k=1}^{n} a_k = n$. Prove that

a) $\sum_{k=1}^{n} a_k^2 \leq n + 2$.

b) $\sum_{1 \leq i < j \leq n} a_i a_j \geq \frac{(n-2)(n+1)}{2}$.
Let $ABC$ be a triangle such that $\angle BAC \geq \frac{2\pi}{3}$. Prove that
\[
\frac{r}{R} \leq \frac{2\sqrt{3} - 3}{2},
\]
where $r$ is the inradius and $R$ is the circumradius of $ABC$.

Let $a, b, c$ and $d$ be real numbers such that $2 \geq a \geq 1 \geq b \geq c \geq d \geq 0$ and $a + b + c + d = 4$. Prove that
\[
\frac{2}{a^3 + b^3 + c^3 + d^3} + \frac{9}{ab + bc + cd + da + ac + bd} \leq 2.
\]

Let $ABCD$ be a rectangle with center $O$. Let $M$ and $P$ be two points in the plane (not necessarily distinct) such that $O$ lies on the line $MP$ and $OM = 3 \cdot OP$. Prove that
\[
MA + MB + MC + MD \geq PA + PB + PC + PD.
\]

Let $a, b, c \geq 1$ and $0 \leq d, e, f \leq 1$ such that $a + b + c + d + e + f = 6$. Prove that
\[
6 \leq a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \leq 18.
\]

Let $ABC$ be a nonobtuse triangle with smallest angle $A$. Prove that
\[
\cos (B - C) \geq \cos B + \cos C
\]
and determine when equality holds.

Prove that if $a, b, c$ and $d$ are non-negative real numbers such that $a + b + c + d = 4$, then
\[
ab + bc + cd + da + ac + bd \geq 3\sqrt{(a^2 + b^2 + c^2 + d^2)abcd}.
\]

Let $x$, $y$ and $z$ be positive real numbers such that $xyz = 512$. Prove that
\[
\frac{1}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 + y}} + \frac{1}{\sqrt{1 + z}} \geq 1.
\]
**B13.** Proposed by Leonard Giugiuc.

Let $n$ be an integer with $n \geq 4$. Prove or disprove that for any positive real numbers $a_i$, $i = 1, 2, \ldots, n$ that sum up to 1, we have:

$$\sqrt{(1-a_1^2)(1-a_2^2)\cdots(1-a_n^2)} \geq (n^n-1)a_1a_2\cdots a_n.$$ 

**B14.** Proposed by Leonard Giugiuc.

Let $k$ be a real number with $k > \frac{7+3\sqrt{5}}{2}$. Prove or disprove that for any non-negative real numbers $x, y, z$ no two of which are zero, we have:

$$\sqrt{\frac{x}{ky+z}} + \sqrt{\frac{y}{kz+x}} + \sqrt{\frac{z}{kx+y}} \geq \frac{3}{\sqrt{k+1}}.$$ 

**B15.** Proposed by Leonard Giugiuc.

Let $a, b$ and $c$ be real numbers such that $a \geq b \geq c \geq 0$ and $a + b + c = 3$.

a) Show that $2 \leq ab + bc + ca \leq 3$.

b) Prove that $a^3 + b^3 + c^3 + \frac{45}{a^2+b^2+c^2} \leq 18$ and study the equality cases.

**B16.** Proposed by Dao Thanh Oai and Leonard Giugiuc.

Let $ABCD$ be a cyclic quadrilateral. Prove that the following two statements are equivalent:

a) $AC \geq BD$,

b) $AB \cdot AD + CB \cdot CD \geq BA \cdot BC + DA \cdot DC$.

**B17.** Proposed by Dao Thanh Oai and Leonard Giugiuc.

Let $ABCD$ be a cyclic quadrilateral. Prove that

$$AB + AC + AD + BC + BD + CD \leq 4R(\sqrt{2}+1),$$

where $R$ is the circumradius of $ABCD$.

**B18.** Proposed by Leonard Giugiuc and Dorin Marghidanu.

Let $n \geq 2$ be a natural number, and $a_k$ be real numbers such that $0 < a_k < 2$ for all $k = 1, 2, \ldots, n$ with $\prod_{k=1}^n a_k = 1$. Prove that

$$\sum_{k=1}^n \frac{1}{\sqrt{1+a_k}} \leq \frac{n}{\sqrt{2}}.$$ 

Prove further that the condition $a_k < 2$ can be dropped when $n = 2$ or $n = 3$. 

Copyright © Canadian Mathematical Society, 2020
Find the maximum value \( k \) such that
\[
a^2 + b^2 + c^2 + k(ab + bc + ca) \geq 3 + k(a + b + c)
\]
for any positive numbers \( a, b \) and \( c \) such that \( abc = 1 \).

B20\*. Proposed by Leonard Giugiuc.
Let \( x, y \in (0, 3/2) \) be real numbers that satisfy \((x - 2)(y - 2) = 1\). Prove or disprove that
\[
x^3 + y^3 \geq 2.
\]

Consider an arbitrary triangle \( ABC \) with medians \( m_a, m_b, m_c \), circumradius \( R \),
inradius \( r \) and exradii \( r_a, r_b, r_c \). Show that
\[
m_a + m_b + m_c \leq \sqrt{16R^2 + 4rR + 9r^2} \leq r_a + r_b + r_c.
\]

Let \( a, b, c, d, e, f \) be non-negative real numbers such that \( a + b + c + d + e + f = 4 \). If \( a \geq b \geq c \geq 1 \geq d \geq e \geq f \geq 0 \), prove that
\[
a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 180abcdef \leq 10.
\]

Given a triangle \( ABC \), let the tangent to its circumcircle at \( A \) intersect the line \( BC \) at \( D \), and let the circle through \( A \) that is tangent to \( BC \) at \( D \) intersect the circumcircle again at \( E \). Prove that
\[
\frac{EB}{EC} = \left(\frac{AB}{AC}\right)^3.
\]

Let \( ABCD \) be a square. Let \( \omega \) be the circle centered at \( A \) with radius \( AB \). A point \( M \) lies inside the square on \( \omega \); the line \( BM \) intersects the side \( CD \) at \( N \). Prove that \( CM = 2MN \) if and only if \( CM \) and \( BN \) are perpendicular.

Let \( ABC \) be a triangle with semiperimeter \( s \). The A-excircle of the triangle touches the side \( BC \) at \( Q \) and the lines \( AB \) and \( AC \) at \( M \) and \( N \), respectively. Suppose that \( AQ \) intersects \( MN \) at \( P \). Prove that
\[
AP = \frac{s \sqrt{a(s - a)(as + (b - c)^2)}}{b(s - c) + c(s - b)}.
\]