
In \( \triangle ABC \), let \( H \) be the orthocenter. Let \( M_A \) be the midpoint of \( AH \) and \( D \) be the foot from \( H \) onto \( BC \), and define \( M_B \), \( M_C \), \( E \) and \( F \) similarly. Suppose \( P \) is a point in the plane distinct from the circumcenter of \( \triangle ABC \), and suppose that \( P_A, P_B \) and \( P_C \) are points such that quadrilaterals \( PABC, P_AAEF, P_BDBF \) and \( P_CDEC \) are similar with vertices in that order. Show that \( M_A P_A, M_B P_B \) and \( M_C P_C \) concur on the circumcircle of \( \triangle DEF \).

We received 5 submissions, all of which were correct. Only the proposer avoided the use of coordinates, so we provide two solutions: the proposer’s together with one example of a solution via coordinates.

Solution 1, by Michel Bataille.

First, a few remarks (see figure):
(1) the circumcircle of $\triangle DEF$ is the nine-point circle whose centre is the midpoint $N$ of $OH$ ($O$ being the circumcentre of $\triangle ABC$). Its radius is half the circumradius of $\triangle ABC$ and it passes through $M_A$. We add the hypothesis that $\triangle ABC$ is not right-angled (to keep the triangle $DEF$ non-degenerate).

(2) The statement of the problem assumes as known that triangles $ABC$ and $AEF$ are oppositely similar, a fact for which a proof is as easily found as a reference: Points $E,F,B,C$ lie on the circle with diameter $BC$ (since $\angle BEC = \angle CFB = 90^\circ$), hence, using directed angles modulo $\pi$, we have

$$\angle(EB,EF) = \angle(EB,EB) + \angle(EB,EF) = \frac{\pi}{2} + \angle(CB,CF)$$

Similarly, $\angle(FA,FE) = \angle(FA,CA)$, whence corresponding angles of triangles $ABC$ and $AEF$ are equal. The similarity $\sigma$ such that $\sigma(A) = A, \sigma(B) = E, \sigma(C) = F$ is indirect (since $\angle(\sigma(CB),\sigma(CA)) = \angle(CA,CB)$); moreover, by definition $\sigma(P) = P_A$.

This said, we embed the problem in the complex plane and suppose without loss of generality that $\triangle ABC$ is inscribed in the unit circle. We denote by $a,b,c,m,a_p,p,m_a,n$ the affixes of $A,B,C,M,A_P,P_A,N$ and we set $m = \frac{b+c}{2}$, the affix of the midpoint of $BC$. Note that $m \neq 0$ since $\triangle ABC$ is not right-angled.

We have

$$h = a + b + c, \quad n = \frac{h}{2} = \frac{a}{2} + m, \quad \text{and} \quad m_a = \frac{a + h}{2} = a + m.$$ 

The equation of the line $AC$ is $z + ac\overline{z} = a + c$ and the equation of the perpendicular to $AC$ through $B$ is

$$\frac{z - b}{c - a} + \frac{\overline{z} - \overline{b}}{\overline{c} - \overline{a}} = 0,$$

that is, $b\overline{z} - abc\overline{z} = b^2 - ac$ (using $\overline{\overline{a}} = \frac{1}{a}$, etc.). From these equations, we deduce the affix $e$ of $E$:

$$e = \frac{hb - ac}{2b}.$$ 

The similarity $\sigma$ transforms the point with affix $z$ into the point with affix $z' = \alpha z + \beta$, where $\alpha, \beta$ satisfy

$$a = \alpha a + \beta \quad \text{and} \quad \frac{hb - ac}{2b} = \alpha b + \beta.$$ 

This yields $\alpha = -\frac{a(b+c)}{2} = -am$ and $\beta = a + m$ and therefore,

$$p_a = -am\overline{p} + a + m.$$ 

Now, let $W$ (affix $w$) be the point of intersection distinct from $M_A$ of the line $M_A P_A$ and the nine-point circle: for some real number $\lambda$, we have

$$w = m_a + \lambda(m_a - p_a) = a + m + \lambda am\overline{p}.$$ 

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Expressing that \( w \) must satisfy \(|w - n|^2 = \frac{1}{4}\), we obtain
\[
\lambda^2 mmp + \frac{\lambda}{2}(m\overline{m} + \overline{m}p) = 0.
\]
Since \( w \neq m_n \), we have \( \lambda \neq 0 \), hence \( \lambda = -\frac{1}{2\sqrt{m\overline{m} + \overline{m}p}} \) and we readily deduce that
\[
w = \frac{a + b + c}{2} - \frac{abc\overline{p}}{2p}.
\]
Since the affix \( w \) is invariant under permutations of \( a, b, c \), the lines \( M_B P_B \) and \( M_C P_C \) also pass through \( W \) and the required result follows.

Solution 2 by the proposer, revised by the editor.

We work in the Euclidean plane extended by the line at infinity. We shall be using properties of isogonal conjugation with respect to triangle \( M_A M_B M_C \): Recall that the isogonal conjugate of a point with respect to \( \Delta M_A M_B M_C \) is constructed by reflecting each line joining it to a vertex in the internal angle bisector at the corresponding vertex; the three reflected lines then concur at the isogonal conjugate. This type of conjugation is an involution of the points of the extended plane that are not on a sideline of the triangle; in particular, every point not on an extended side of the triangle is interchanged with its conjugate; moreover, each point other than a vertex on the circumcircle is interchanged with a point at infinity. Details can be found in standard textbooks such as Roger A. Johnson’s Advanced Euclidean Geometry and Nathan Altshiller Court’s College Geometry, as well as in standard internet sources.

Note that \( EF \) is antiparallel to \( BC \) with respect to \( \angle BAC \) (where two lines are said to be antiparallel with respect to an angle if the image of either line under reflection in the angle bisector is parallel to the other line). For a proof, see the second preliminary remark in Solution 1 above. Because \( P_A B C \) is similar to \( P_A A E F \), the similarity that takes \( \Delta ABC \) to \( \Delta AEF \) takes the circumcenter \( O \) of the former to the circumcenter \( M_A \) of the latter, whence the line \( OP \) and its image line \( M_A P_A \) must also be antiparallel with respect to \( \angle BAC \). Since the triangles \( M_A M_B M_C \) and \( ABC \) are homothetic, it follows that \( OP \) and \( M_A P_A \) are antiparallel with respect to \( \angle M_B M_A M_C \). Letting \( P_\infty \) denote the point at infinity of the line \( OP \), we have \( M_A P_\infty \) is parallel to \( OP \), so we may conclude, finally, that \( M_A P_\infty \) and \( M_A P_A \) are isogonal in \( \angle M_B M_A M_C \) (in the sense that \( \angle M_B M_A M_C \) and \( \angle P_A M_A P_\infty \) have the same angle bisectors).

Analogously, \( M_B P_\infty \) and \( M_B P_B \) are isogonal in \( \angle M_C M_B M_A \), and \( M_C P_\infty \) and \( M_C P_C \) are isogonal in \( \angle M_A M_C M_B \). Because these three lines (namely \( M_A P_\infty \), \( M_B P_\infty \) and \( M_C P_\infty \)) meet on the line at infinity, their isogonal conjugates with respect to \( \Delta M_A M_B M_C \), namely the lines \( M_A P_A \), \( M_B P_B \) and \( M_C P_C \), must concur at the isogonal conjugate of \( P_\infty \), which must lie on the circumcircle of \( \Delta M_A M_B M_C \). But that circle is the nine-point circle of \( \Delta ABC \), which coincides with the circumcircle of \( \Delta DEF \); in other words, the three lines concur at a point of the circumcircle of \( \Delta DEF \) as claimed.
\textit{Editor’s comments.} The proposer observed that the concurrence point in question is the Poncelet point of the isogonal conjugate of \( P \) with respect to \( \triangle ABC \). He also stated that by starting with \( P \) at the orthocenter, the result becomes,

The Euler lines of \( \triangle AEF, \triangle BFD \) and \( \triangle CDE \) concur on the nine-point circle.

4472. \textit{Proposed by Liam Keliher.}

Let \( n \) be a positive integer. Prove that \( n \) divides

\[
\prod_{i=0}^{n-1} (2^n - 2^i).
\]

We received 22 submissions, all correct. Several solvers proved a stronger result that the given product is actually divisible by \( n! \). We present the solution of Prithwijit De.

Observe that \(|\text{GL}_n(\mathbb{Z}_2)| = \prod_{i=0}^{n-1} (2^n - 2^i)\) and the group of permutation matrices of order \( n \) (itself a group of order \( n! \)) is a subgroup of \( \text{GL}_n(\mathbb{Z}_2) \). Therefore

\[
n! \prod_{i=0}^{n-1} (2^n - 2^i).
\]

\textit{Editor’s Comment.} Bataille pointed out that this result is well-known as it was problem 5 of the 4th National Mathematical Olympiad of Turkey, appearing in \textit{Crux} before: see [2000 : 390] and [2002 : 503].

4473. \textit{Proposed by Nguyen Viet Hung.}

Let \( \lfloor a \rfloor \) denote the greatest integer not exceeding \( a \). For every positive integer \( n \),

(a) find the last digit of \( \lfloor (2 + \sqrt{3})^n \rfloor \),

(b) find \( \gcd(\lfloor (2 + \sqrt{3})^{n+1} \rfloor + 1, \lfloor (2 + \sqrt{3})^n \rfloor + 1) \), where \( \gcd(a, b) \) denotes the greatest common divisor of \( a \) and \( b \).

We received 15 correct solutions. Most followed the tack of the general solution presented below.

(a) Let \( u = 2 + \sqrt{3} \) and \( v_n = u^n + u^{-n} \) for \( n \geq 1 \). Then \( 0 < u^{-1} = 2 - \sqrt{3} < 1 \) and

\[
v_n - 1 < u^n < v_n
\]

for each positive integer \( n \). Since \( u \) and \( u^{-1} \) are the roots of the quadratic equation \( x^2 = 4x - 1 \), the sequence \( \{v_n\} \) satisfies the recursion

\[
v_{n+2} = 4v_{n+1} - v_n
\]

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with initial conditions $v_1 = 4$ and $v_2 = 14$. Thus $\{v_n\}$ is a sequence of even integers and $\lfloor u^n \rfloor = v_n - 1$.

Modulo 10, $\{v_n\}$ has the period 3 cycle $\{4, 4, 2\}$, so that $\lfloor u^n \rfloor$ ends in the digit 3 when $n \equiv \pm 1 \pmod{3}$ and in the digit 1 when $n \equiv 0 \pmod{3}$.

(b) Since $\gcd(v_1, v_2) = 2$ and $v_{n+2} = 4v_{n+1} - v_n$ for $n \geq 1$, an induction argument establishes that

$$\gcd(\lfloor (2 + \sqrt{3})^{n+1} \rfloor + 1, \lfloor (2 + \sqrt{3})^n \rfloor + 1) = \gcd(v_{n+1}, v_n) = 2.$$

**4474. Proposed by Kadir Altintas and Leonard Giugiuc.**

Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \frac{\pi}{2}$, $\angle ADB = \frac{\pi}{12}$, $\angle BDC = \frac{\pi}{6}$ and $\angle DBC = \frac{\pi}{8}$. Prove that $BD$ passes through the midpoint of $AC$.

We received 23 submissions, all correct, and from the rich variety of solutions we have chosen two that were relatively light on the use of trigonometric identities.

**Solution 1, by Cristobal Sánchez-Rubio.**

Let $M$ be the point of intersection of the lines $AC$ and $BD$. Since $\triangle ABC$ is a right triangle, it is enough to prove that the triangle $MCB$ is isosceles; specifically, since we have $\angle DBC = \angle MBC = 22.5^\circ$, we must show that $\angle MCB = 22.5^\circ$. By the sine law applied to triangle $ABD$,

$$\frac{AB}{\sin 15^\circ} = \frac{BD}{\sin 97.5^\circ} = \frac{BD}{\cos 7.5^\circ},$$

and therefore,

$$AB = \frac{BD \sin 15^\circ}{\cos 7.5^\circ}.$$

By the sine law for triangle $BCD$,

$$\frac{BC}{\sin 30^\circ} = \frac{BD}{\sin 127.5^\circ},$$

and therefore

$$BC = \frac{BD \sin 30^\circ}{\cos 37.5^\circ}.$$
Then
\[ \frac{AB}{BC} = \frac{\sin 15^\circ \cdot \cos 7.5^\circ}{\sin 30^\circ \cos 37.5^\circ} = \frac{2 \sin 7.5^\circ \cdot \cos 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ \cos 37.5^\circ} = \frac{2 \sin 7.5^\circ \cdot \cos 7.5^\circ}{\sin 30^\circ}. \]

But
\[ 2 \sin 7.5^\circ \cdot \cos 37.5^\circ = \sin (7.5^\circ + 37.5^\circ) + \sin (7.5^\circ - 37.5^\circ) = \sin 45^\circ - \sin 30^\circ = \sqrt{2} - \frac{1}{2}, \]
whence, finally,
\[ \frac{AB}{BC} = \frac{2 \sin 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ} = \sqrt{2} - 1 = \tan 22.5^\circ. \]

In other words, \( \angle MCB = 22.5^\circ \), so that \( MB = MC \), as desired.

**Solution 2, by Ivko Dimitrić.**

Let \( M \) be the point of intersection of \( AC \) and \( BD \), and let \( P \) and \( Q \) be the feet of the perpendiculars from \( A \) and \( C \) to \( BD \), respectively. Then, from \( \triangle ABP \) and \( \triangle APD \) we have
\[ \frac{BD}{AP} = \frac{BP + PD}{AP} = \cot \frac{3\pi}{8} + \cot \frac{\pi}{12} = \tan \frac{\pi}{8} + \cot \frac{\pi}{12}, \]
and from \( \triangle BCQ \) and \( \triangle CDQ \),
\[ \frac{BD}{CQ} = \frac{BQ + QD}{CQ} = \cot \frac{\pi}{8} + \cot \frac{\pi}{6}. \]

Using the relevant half-angle formulas we have
\[ \tan \frac{\pi}{8} = \sqrt{\frac{1 - \cos \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}}} = \sqrt{\frac{(2 - \sqrt{2})^2}{(2 + \sqrt{2})(2 - \sqrt{2})}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1 \]
and \( \cot \frac{\pi}{8} = \sqrt{2} + 1 \), whereas
\[ \cot \frac{\pi}{12} = \sqrt{\frac{1 + \cos \frac{\pi}{6}}{1 - \cos \frac{\pi}{6}}} = \sqrt{\frac{2 + \sqrt{3}}{2 - \sqrt{3}}} = 2 + \sqrt{3}. \]

Then,
\[ \tan \frac{\pi}{8} + \cot \frac{\pi}{12} = (\sqrt{2} - 1) + (2 + \sqrt{3}) = (\sqrt{2} + 1) + \sqrt{3} = \cot \frac{\pi}{8} + \cot \frac{\pi}{6}. \]

Hence, \( \frac{BD}{AP} = \frac{BD}{CQ} \), implying \( AP = CQ \). As a consequence, the right triangles \( APM \) and \( CQM \) are congruent, implying \( AM = CM \); that is, \( BD \) passes through the midpoint of \( AC \).

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4475. Proposed by Michel Bataille.

Let \( a, b \) be real numbers with \( a, b, a + b, a - b \neq 0 \). Prove the inequality

\[
\frac{\sinh(2(a + b))}{a + b} + \frac{\sinh(2(a - b))}{a - b} \geq 4 \left( \frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b} \right).
\]

There were 4 correct and one incomplete solution. The correct solutions all used an integration argument along the lines of the following.

We first establish that

\[(\cosh x)(\cosh y) \geq \sinh x + \sinh y\]

for real \( x \) and \( y \). This can be done either by using the identity \( \cosh^2 t = 1 + \sinh^2 t \) and squaring, or by making the substitutions

\[(\sinh x, \cosh x) = (\tan u, \sec u) \quad \text{and} \quad (\sinh y, \cosh y) = (\tan v, \sec v)\]

with \( \pi/2 < u, v < \pi/2 \), and noting that the inequality is equivalent to

\[1 \geq \sin(u + v).\]

Let \((x, y) = (2ta, 2tb)\). For \(0 \leq t \leq 1\),

\[
\frac{1}{2} [\cosh(2t(a + b)) + \cosh(2t(a - b))] = \cosh(2ta) \cosh(2tb) \geq \sinh(2ta) + \sinh(2tb).
\]

Integrate this inequality from 0 to 1 with respect to \( t \) to obtain

\[
\frac{\sinh(2(a + b))}{4(a + b)} + \frac{\sinh(2(a - b))}{4(a - b)} \geq \frac{\cosh(2a) - 1}{2a} + \frac{\cosh(2b) - 1}{2b}
\]

\[
= \left( \frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b} \right),
\]

as desired.


Prove that for any real numbers \( a, b \) and \( c \), we have

\[3\sqrt{6}(ab(a - b) + bc(b - c) + ca(c - a)) \leq ((a - b)^2 + (a - c)^2 + (b - c)^2)^{3/2}.
\]

We received 10 submissions, all correct. We present a composite of nearly the same solutions by Michel Bataille and Marie-Nicole Gras.

Let

\[p = \frac{1}{2} [(a - b)^2 + (b - c)^2 + (c - a)^2] = a^2 + b^2 + c^2 - ab - bc - ca,
\]
\[q = ab(a - b) + bc(b - c) + ca(c - a) = a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2.
\]
We need to prove that
\[ 3\sqrt[3]{6} q \leq (2p)^{\frac{3}{2}}. \quad (1) \]
If \( q < 0 \), then (1) clearly holds. If \( q \geq 0 \), then
\[ (1) \iff 54q^2 \leq (2p)^3 \iff 27q^2 \leq 4p^3. \quad (2) \]
Consider the polynomial
\[
P(x) = (x - a + b)(x - b + c)(x - c + a) \\
= x^3 + [(a - b)(b - c) + (b - c)(c - a) + (c - a)(a - b)]x - (a - b)(b - c)(c - a) \\
= x^3 + (-a^2 - b^2 - c^2 + ab + bc + ca)x - (-a^2b - b^2c - c^2a + ab^2 + bc^2 + ca^2) \\
= x^3 - px + q,
\]
of discriminant
\[ \Delta = -4p^3 + 27q^2. \]
Since \( P(x) \) has 3 real roots by its definition, we must have \( \Delta \leq 0 \), so \( 4p^3 - 27q^2 \geq 0 \), from which (2) follows, completing the proof.

**4477. Proposed by Warut SukSompong.**

Given a positive integer \( n \), let \( a_1 \geq \cdots \geq a_n \geq 0 \) and \( b_1 \geq \cdots \geq b_n \geq 0 \) be integers such that
1. \( a_1 + \cdots + a_i \geq b_1 + \cdots + b_i \) for all \( i = 1, \ldots, n - 1 \);
2. \( a_1 + \cdots + a_n = b_1 + \cdots + b_n \).

Assume that there are \( n \) boxes, with box \( i \) containing \( a_i \) balls. In each move, Alice is allowed to take two boxes with an unequal number of balls, and move one ball from the box with more balls to the other box. Prove that Alice can perform a finite number of moves after which each box \( i \) contains \( b_i \) balls.

*There were 4 correct solutions, all along the lines of the following.*

The proof is by induction, the result being trivial for \( n = 1 \).

Assume it holds for at most \( n - 1 \) boxes, with \( n \geq 2 \). If \( a_j = b_j \) for some \( j \) with \( 1 \leq j \leq n \), we can remove box \( j \) from consideration. The conditions hold for the remaining \( n - 1 \) boxes and we can invoke the induction hypothesis to rearrange the balls among them.

Henceforth, let \( a_i \neq b_i \) for each \( i \).

Since \( a_1 > b_1 \) and
\[ a_n = b_n - [(a_1 + \cdots + a_{n-1}) - (b_1 + \cdots + b_{n-1})] \leq b_n, \]
then \( a_n < b_n \) and there is a positive integer \( k \leq n - 1 \) for which \( a_k > b_k \) and \( a_{k+1} < b_{k+1} \). Start removing balls one at a time from box \( k \) and placing them in box \( k + 1 \).

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If \( a_k - b_k \leq b_{k+1} - a_{k+1} \), then transfer a total of \( a_k - b_k \) balls from box \( k+1 \), leaving \( b_k \) balls in box \( k \) and

\[
a_{k+1} + a_k - b_k \leq a_{k+1} + b_k - a_{k+1} = b_{k+1}
\]
balls in box \( k+1 \).

Since

\[
(a_1 + \cdots + a_{k-1}) + b_k + (a_{k+1} + a_k - b_k) = a_1 + \cdots + a_{k+1} \geq b_1 + \cdots + b_{k+1},
\]
we see that the consequent arrangement of balls in the boxes satisfies the two conditions.

If \( a_k - b_k > b_{k+1} - a_{k+1} \), transfer \( b_{k+1} - a_{k+1} \) balls from box \( k \) to box \( k+1 \), leaving

\[
a_k - (b_{k+1} - a_{k+1}) > a_k - (a_k - b_k) = b_k
\]
balls in box \( k \) and \( b_{k+1} \) balls in box \( k+1 \).

Since

\[
(a_1 + \cdots + a_{k-1}) + [a_k - (b_{k+1} - a_{k+1})] > (b_1 + \cdots + b_k) + b_k
\]
and

\[
(a_1 + \cdots + a_{k-1}) + [a_k - (b_{k+1} - a_{k+1})] + b_{k+1} = a_1 + \cdots + a_{k-1} + a_k + a_{k+1}
\]
\[\geq b_1 + \cdots + b_{k+1},\]
the consequent arrangement of balls in the boxes satisfies the conditions.

In either case, we have a rearrangement of balls for which the number of balls in one of the \( k \)th and \((k+1)\)th boxes is equal to the corresponding value of \( b \), so we can apply the induction step.

4478. Proposed by Florin Stanescu.

Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
f(a^2) - f(b^2) \leq (f(a) + b)(a - f(b))
\]
for all \( a, b \in \mathbb{R} \).

There were 11 correct and 3 incorrect submitted solutions.

The only solutions are \( f(x) = x \) and \( f(x) = -x \).

Make the substitutions \((a, b) = (x, y)\) and \((a, b) = (y, x)\) to obtain

\[
f(x^2) - f(y^2) \leq xy - f(x)f(y) + xf(x) - yf(y)
\]
and

\[
f(y^2) - f(x^2) \leq xy - f(x)f(y) - xf(x) + yf(y).
\]
Adding these inequalities leads to \( f(x)f(y) \leq xy \) for all real \( x \) and \( y \). In particular, \( f(0)^2 \leq 0 \), so that \( f(0) = 0 \).

Substituting \((a, b) = (x, 0)\) and \((a, b) = (0, x)\) yields
\[
\begin{align*}
  f(x^2) \leq x f(x) \quad \text{and} \quad -f(x^2) \leq -x f(x),
\end{align*}
\]
whence \( f(x^2) = x f(x) \) for all real \( x \). In particular, \( f(1) = -f(-1) \).

Since \( f(1) f(x) \leq x \) and \( -f(1) f(x) = f(-1) f(x) \leq -x \),
then \( f(1) f(x) = x \) and \( f(1)^2 = 1 \). When \( f(1) = 1 \), then \( f(x) \equiv x \), and when \( f(1) = -1 \), then \( f(x) \equiv -x \).

**4479. Proposed by George Apostolopoulos.**

Let \( ABC \) be a triangle with \( \angle A = 90^\circ \) and let \( H \) be the foot of the altitude from \( A \). Prove that
\[
\frac{6}{(AB + AC)^2} - \frac{1}{2 \cdot AH^2} \leq \frac{1}{BC^2}.
\]

We received 32 solutions, including two from the featured solver. We present the solution by Miguel Amengual Covas.

Denote the length of the hypotenuse of the given triangle by \( a \) and the legs by \( b \) and \( c \). Then the area of \( \triangle ABC \) may be expressed as \( bc/2 \), and also as \( a \cdot AH/2 \). Equating these and solving for \( AH \), we get
\[
AH = \frac{bc}{a}.
\]

When this is substituted into the proposed inequality, the proposed inequality becomes
\[
\frac{6}{(b + c)^2} - \frac{a^2}{2b^2 c^2} \leq \frac{1}{a^2}.
\]
(1)

We substitute \( b^2 + c^2 \) for \( a^2 \) in (1), obtaining
\[
\frac{6}{(b + c)^2} - \frac{b^2 + c^2}{2b^2 c^2} \leq \frac{1}{a^2},
\]
or, equivalently,
\[
b^6 + 2b^5 c - 7b^4 c^2 + 8b^3 c^3 - 7b^2 c^4 + 2bc^5 + c^6 \geq 0.
\]

This in turn is equivalent to
\[
(b - c)^2 (b^4 + 4b^3 c + 4bc^3 + c^4) \geq 0,
\]
whose validity is obvious. Equality occurs if and only if \( b = c \).

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Find all the solutions to the system
\[
\begin{align*}
    a + b + c + d &= 4, \\
    a^2 + b^2 + c^2 + d^2 &= 6, \\
    a^3 + b^3 + c^3 + d^3 &= \frac{94}{9},
\end{align*}
\]
in \([0, 2]^4\).

We received 11 submissions, all correct. We present the solution by Digby Smith.

Note first that
\[
ab + ac + ad + bc + bd + cd = \frac{1}{2} \left( (a + b + c + d)^2 - (a^2 + b^2 + c^2 + d^2) \right)
= \frac{1}{2} (4^2 - 6) = 5, \quad (1)
\]
and
\[
(a + b + c + d)^3 = a^3 + b^3 + c^3 + d^3 + 3a^2(b + c + d) + 3b^2(c + d + a)
+ 3c^2(d + a + b) + 3d^2(a + b + c) + 6(abc + abd + acd + bcd)
= 3(a + b + c + d)(a^2 + b^2 + c^2 + d^2) - 2(a^3 + b^3 + c^3 + d^3)
+ 6(abc + abd + acd + bcd)
\Rightarrow 64 = 3(4)(6) - 2 \frac{94}{9} + 6(abc + abd + acd + bcd)
\Rightarrow abc + abd + acd + bcd = \frac{1}{6} \left( 64 + \frac{188}{9} - 72 \right) = \frac{58}{27}. \quad (2)
\]

Next, let \(k = abcd\) and \(p(x)\) be the polynomial function defined by
\[p(x) = (x - a)(x - b)(x - c)(x - d).\]

Then by (1) and (2) we have
\[
p(x) = x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2
- (abc + bcd + cda + dab)x + abcd
= x^4 - 4x^3 + 5x^2 - \frac{58}{27} x + k.
\]

Since
\[
p'(x) = 4x^3 - 12x^2 + 10x - \frac{58}{27}
= \frac{1}{27} (108x^3 - 324x^2 + 270x - 58)
= \frac{2}{27} (3x - 1)(18x^2 - 48x + 29),
\]
solving $p'(x) = 0$ yields

$$x = \frac{1}{3}, \frac{4}{3} \pm \frac{\sqrt{6}}{2}.$$  

Since $p(x)$ is a 4th degree polynomial with positive leading coefficient and $p'(x)$ has 3 distinct real roots in $(0,2)$, it follows that in order for $a, b, c, d$ to be solutions of the given equations, where $0 \leq a, b, c, d \leq 2$, we must have

$$p(0) \geq 0, \ p\left(\frac{1}{3}\right) \leq 0, \ p\left(\frac{4}{3} - \frac{\sqrt{6}}{2}\right) \geq 0, \ p\left(\frac{4}{3} + \frac{\sqrt{6}}{2}\right) \leq 0, \ p(2) \geq 0.$$  

Evaluating, we find $p\left(\frac{1}{3}\right) = p(2) = k - \frac{8}{27}$. Hence, $k = \frac{8}{27}$, from which we obtain

$$p(x) = x^4 - 4x^3 + 5x^2 - \frac{58}{27}x + \frac{8}{27}$$

$$= \frac{1}{27}(27x^4 - 108x^3 + 135x^2 - 58x + 8)$$

$$= \frac{1}{27}(3x - 1)^2(3x - 4)(x - 2).$$  

Therefore, the solutions in $[0,2]^4$ are the 12 permutations of $\left(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}, 2\right)$. 

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