The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by April 15, 2020.

MA56. For a given arithmetic series the sum of the first 50 terms is 200, and the sum of the next 50 terms is 2700. What is the first term of the series?

MA57. Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees (see the accompanying figure). Let $C$ be a convex polygon having $s$ sides. Suppose that the interior region of $C$ is the union of $q$ quadrilaterals, none of whose interiors intersect one another. Also suppose that $b$ of these quadrilaterals are boomerangs. Show that $q \geq b + \frac{s-2}{2}$.


If the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are randomly ordered to form a nine-digit number, what is the probability that the number is divisible by 99?

MA59. Find positive integer solutions of

$$x^{x^x} = (19 - y^x)y^y - 74.$$
MA60. Three equilateral triangles with sides of length 1 are shown shaded in a larger equilateral triangle. The total area of the three small triangles is half the area of the large triangle. What is the side-length of the larger equilateral triangle?
Les problèmes proposés dans cette section sont appropriés aux étudiants de l’école secondaire.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 avril 2020.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l’Université de Saint-Boniface, d’avoir traduit les problèmes.

MA56. Pour une certaine série arithmétique, la somme des 50 premiers termes est 200, tandis que la somme des 50 termes suivants est 2700. Déterminer le premier terme de la série.

MA57. On définit un boomerang comme étant un quadrilatère dont les côtés opposés ne se coupent pas et dont un des angles internes est de plus que 180 degrés, tel qu’illustré. Soit maintenant $C$ un polygone convexe ayant $s$ côtés. Supposer que la région interne de $C$ est la réunion de $q$ quadrilatères dont les intérieurs s’intersectent pas. Supposer de plus que $b$ de ces quadrilatères sont des boomerangs. Démontrer que $q \geq b + \frac{s - 2}{2}$.


Les chiffres 1, 2, 3, 4, 5, 6, 7, 8 et 9 sont réarrangés de façon aléatoire pour former un entier à neuf chiffres. Déterminer la probabilité que cet entier soit divisible par 99.

MA59. Déterminer les solutions entières positives à l’équation

$$x^{x^x} = (19 - y^x)y^x - 74.$$
MA60. Trois triangles équilatéraux à côtés de longueur 1 sont indiqués à l’intérieur d’un plus gros triangle équilatéral. La surface totale des trois petits triangles égale la moitié de la surface du gros triangle. Déterminer la longueur du côté du gros triangle.
MATHEMATTIC
SOLUTIONS


MA31. Given that the areas of an equilateral triangle with side length \( t \) and a square with side length \( s \) are equal, determine the value of \( \frac{t}{s} \).

The problem was proposed by John Grant McLoughlin.

We received 4 correct solutions. We present an amalgamation of the submitted solutions.

The formula for the area of the square is \( s^2 \) and the formula for the area of the equilateral triangle is \( \frac{\sqrt{3}}{4} t^2 \). We can deduce that \( s^2 = \frac{\sqrt{3}}{4} t^2 \), which we can rearrange to \( \frac{4}{\sqrt{3}} = \frac{s^2}{t^2} \). Taking the square root of both sides and rationalizing the denominator we get \( t = \frac{2 \sqrt{27}}{3} \).

MA32. Jack and Madeline are playing a dice game. Jack rolls a 6-sided die (numbered 1 to 6) and Madeline rolls an 8-sided die (numbered 1 to 8). The person who rolls the higher number wins the game. If Jack and Madeline roll the same number, the game is replayed. If a tie occurs a second time, then Jack is declared the winner. Which person has the better chance of winning? What are the odds in favour of this person winning the game?

Adapted from NLTA Math League Problem.

We received 2 correct submissions. We present the solution by Digby Smith, modified by the editor.

There are \( 6 \times 8 = 48 \) possible outcomes of dice tosses. Let \( P(X) \) denote the probability of \( X \) occurring. We observe the following:

1. \( P(\text{Jack and Madeline roll the same number}) = \frac{1}{8} \)

2. \( P(\text{Madeline wins a toss}) = P(\text{Madeline rolls 8}) + P(\text{Madeline rolls 7}) + P(\text{Madeline rolls 6, Jack rolls less than 6}) + P(\text{Madeline rolls 5, Jack rolls less than 5}) + P(\text{Madeline rolls 4, Jack rolls less than 4}) + P(\text{Madeline rolls 3, Jack rolls less than 3}) + P(\text{Madeline rolls 2, Jack rolls less than 2}) \)

\[
= \frac{1}{8} + \frac{1}{8} + \left( \frac{1}{8} \right) \left( \frac{5}{6} + \frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6} \right) = \frac{9}{16}
\]
3. \( P(\text{Madeline wins on first toss}) = \frac{9}{16} \).

4. \( P(\text{Madeline wins on second toss}) = P(\text{Jack and Madeline roll the same number}) \times P(\text{Madeline wins a toss}) = \frac{1}{8} \left( \frac{9}{16} \right) = \frac{9}{128} \)

It follows that
\[
P(\text{Madeline wins}) = \frac{9}{16} + \frac{9}{128} = \frac{81}{128}
\]
and
\[
P(\text{Jack wins}) = 1 - \frac{81}{128} = \frac{47}{128}.
\]
We conclude that the odds in favour of Madeline are 81:47 and she has a better chance of winning.

**MA33.** Note that \( \sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}} \). Determine conditions for which \( \sqrt{\frac{a^2}{c}} = a\sqrt{\frac{b}{c}} \), where \( a, b, c \) are positive integers.

*The problem was proposed by John Grant McLoughlin.*

We received 7 solutions, with varying conditions. We present the solution of the Missouri State University Problem Solving Group, which went so far as to be able to generate all such \( a, b, \) and \( c \).

Assume that \( a, b, \) and \( c \) satisfy
\[
\sqrt{\frac{b}{c}} = a\sqrt{\frac{b}{c}}
\]
or rather
\[
\sqrt{\frac{ac + b}{c}} = a\sqrt{\frac{b}{c}}
\]

Squaring both sides we find
\[
\frac{ac + b}{c} = \frac{a^2b}{c}
\]
which simplifies to \( ac + b = a^2b \) and then \( ac = b(a^2 - 1) \).

Now, since \( \gcd(a, a^2 - 1) = 1 \) we can see that \( a \mid b(a^2 - 1) \) implies \( a \mid b \). Let \( b = ka \).

Then we find that
\[
ac = ka(a^2 - 1)
\]
or rather \( c = k(a^2 - 1) \).

It is readily verified that for any choice of \( a \geq 2 \) and \( k \geq 1 \), the triple
\[
(a, b, c) = (a, ka, k(a^2 - 1))
\]
will satisfy the condition.

*Crux Mathematicorum, Vol. 46(2), February 2020*
MA34. Try to replace each * with a different digit from 1 to 9 so that the multiplication is correct. (Each digit from 1 to 9 must be used once.)

\[
\begin{array}{cccc}
* & * & * & * \\
\times & * & & \\
* & * & * & * \\
\end{array}
\]

Determine whether a solution is possible. If so, determine whether the solution is unique.

*Originally from “Mathematical Puzzling” by Anthony Gardiner.*

We received 1 correct solution and one incorrect solution. We present an approach for the problem and the conclusion of Doddy Kastanya.

A formal proof of the answers will just devolve into case based work. One approach is to consider the possible values for the single digit in the product. It cannot be 1 as otherwise the four digit numbers would be identical.

If the single digit were \(d\) and the four digit number in the product is \(n\), then the resulting four digit number is \(dn\). We can see \(1234 \leq n\) and \(dn \leq 9876\) so \(n \leq \frac{9876}{d}\). Putting these together, we narrow down the possible values of \(n\) to the range \(1234 \leq n \leq \frac{9876}{d}\). Then we just check for each possible \(d = 2, \cdots, 9\) for possible \(n\) in this range with distinct digits which has \(dn\) with distinct digits, using all 9 nonzero digits once.

<table>
<thead>
<tr>
<th>(d)</th>
<th>range of possible values for (n)</th>
<th>(n) that fit the description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(1234 \leq n \leq 4938)</td>
<td>none</td>
</tr>
<tr>
<td>3</td>
<td>(1234 \leq n \leq 3292)</td>
<td>none</td>
</tr>
<tr>
<td>4</td>
<td>(1234 \leq n \leq 2469)</td>
<td>(1738, 1963)</td>
</tr>
<tr>
<td>5</td>
<td>(1234 \leq n \leq 1975)</td>
<td>none</td>
</tr>
<tr>
<td>6</td>
<td>(1234 \leq n \leq 1646)</td>
<td>none</td>
</tr>
<tr>
<td>7</td>
<td>(1234 \leq n \leq 1410)</td>
<td>none</td>
</tr>
<tr>
<td>8</td>
<td>(1234 \leq n \leq 1234)</td>
<td>none</td>
</tr>
<tr>
<td>9</td>
<td>(1234 \leq n \leq 1097)</td>
<td>impossible</td>
</tr>
</tbody>
</table>

It takes some effort, or better yet a computer program, but we find exactly two solutions:

\[
\begin{array}{cccc}
1 & 7 & 3 & 8 \\
\times & 4 \\
\hline
6 & 9 & 5 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 9 & 6 & 3 \\
\times & 4 \\
\hline
7 & 8 & 5 & 2 \\
\end{array}
\]

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MA35. A polygon has angles that are all equal. If the sides of this polygon are not all equal, show that the polygon must have an even number of sides.

Originally from “Mathematical Puzzling” by Anthony Gardiner.

We received two submissions (including the author), one proving that the claim of the author is false. We present the solution by the Missouri State University Problem Solving Group showing that the claim is false.

The claim is false. Given any regular $n$-gon, $n > 3$, with vertices $A_1, \ldots, A_n$, let $B_2$ be a point in the interior of $\overrightarrow{A_1A_2}$ and $B_3$ be a point in the interior of $\overrightarrow{A_3A_4}$ such that $B_2A_2 = A_3B_3$. Since $\overrightarrow{B_2B_3}$ is parallel to $\overrightarrow{A_2A_3}$, $\angle A_1B_2B_3 \cong \angle A_1A_2A_3 \cong \angle A_2A_3A_4 \cong \angle B_2B_3A_4$. Therefore the polygon with vertices $A_1, B_2, B_3, A_4, \ldots, A_n$ also has all angles congruent. However, $B_2B_3 \geq A_2A_3$ and $A_1B_2 < A_1A_2 = A_2A_3$, so all the sides are not congruent.
Sometimes a simple idea can be very powerful. In mathematics this happens all the time. In this issue, we revisit an idea, the pigeonhole principle, that I used to solve a problem from the course C&O 380 that I took as an undergraduate from professor Ross Honsberger [2018: 44(4), p. 157-159].

**Pigeonhole principle (a.k.a. Dirichlet box principle):**
If you have \( n \) pigeonholes and \( m > n \) pigeons, then there must be at least one pigeonhole that contains at least two pigeons.

To see this in action, think about the following statement: if 8 people are gathered together in a room, at least two of them were born on the same day of the week. Why does this work? Let’s think of the worst case scenario. Imagine the people arrive at the room one at a time, and we only let people in that were born on a different day of the week than everyone else that is already in the room. We can do this up to a point. Once we reach seven people, every day has been accounted for which means that if we allow an eighth person to enter, this person must have been born on the same day as somebody already present. The pigeonhole principle is an example of an **existence** theorem in mathematics. It tells us something must exist, but it doesn’t really give us any idea how to find it.

The key idea for using the pigeonhole principle is to define our groups in such a way that we are forced to have an overlap. In some problems, like the example above, it is obvious. In others we have to work a bit. Let’s look at problem #7 from C&O 380 [2018: 44(10), p. 419]:

> **Of 5 points inside a square of unit side, show that some pair is less than \( \sqrt{2} \) units apart.**

Since we have 5 points and we are trying to force a pair of them to have a condition, it gives us the hint that we want to work with 4 categories. It seems natural to cut the square into four congruent squares as shown below.

```
  3 | 4
  ---+---
  1 | 2
```

Thus the pigeonhole principle guarantees that at least two points will be in the same region. How far apart can they be? The furthest apart would be if they were
at opposite vertices of the square, which the Pythagorean theorem tells us are $\sqrt{2}$ units apart. Since we are picking points from inside the square, we are guaranteed that two must be less than $\frac{\sqrt{2}}{2}$ units apart, so we are done.

Care is needed since we can break the square up into any four disjoint regions and the pigeonhole principle will ensure that at least two are in the same region. If we don’t create our regions carefully, we may not get what we are after. For example, we could have created the four regions as shown below.

```
1 2 3 4
```

We still know for sure that at least two of the points will fall in the same rectangle, but now they can be as far apart as (almost) $\sqrt{17}$ which is larger than $\sqrt{2}$. We cannot conclude that the problem is impossible just because our configuration gives a larger number. So build your groups carefully.

Next, let’s consider problem #11 from C&O 380 [2019: 45(4), p. 176]: Prove that no matter how the points of a closed unit square are coloured red or blue, either some two red points or some two blue points are at least $\frac{\sqrt{5}}{2}$ units apart.

Consider any three corners of the square. Since we have only two colours, at least two of the points have the same colour. If two of the same coloured corners are diagonally opposite of each other, they are $\sqrt{2}$ units apart which is larger than $\frac{\sqrt{5}}{2}$ and we are done.

Suppose the two points that are the same colour are not diagonally opposite. Then we can assume, without loss of generality, that two vertices on the same edge are red and the other vertex is blue, as in the diagram below. Consider the midpoint of the edge nearest one of the red vertices, but not between one of the three coloured vertices, as indicated by a “?” in the diagram. This point, taken with the further red vertex and the blue vertex, forms an isosceles triangle. Once again, we can conclude that at least two vertices will be the same colour. Since the original two vertices are coloured oppositely, the new point is the same colour as one of them. The Pythagorean theorem again yields that these points are $\frac{\sqrt{5}}{2}$ units apart, and we are done.

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```

Notice that in the first case we were told we had 5 points which gave us the hint that we are looking at four categories. The second problem was a little different. We had to focus in on the two colours, which would be our categories. That meant
we had to look at 3 points to force two of them to be the same colour and piece our proof together from there. Let’s look at problems #21–25 from C&O 380 and choose one to focus on.

#21. Suppose each point of the plane is coloured red or blue. Show that some rectangle has its vertices all the same colour.

#22. In chess, is it possible for a knight to go from the lower left corner square of the board to the upper right corner square and in the process to land exactly once on each other square?

#23. Prove that the number of people at the opera next Thursday night who will shake hands an odd number of times is an even number of people.

#24. Prove that no matter what three points of a square lattice are joined, an equilateral triangle will never occur.

#25. Prove that

\[
\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100} < \frac{1}{10}
\]

Let’s looks closely at #21. Again we have two colours, so again we know that for any three points we pick on the plane, at least two will be the same colour. We need four points, all the same colour, forming a rectangle. Let’s investigate all the possibilities of three coloured points. Taking into account the order of the points, there are \(2 \times 2 \times 2 = 8\) possibilities, pictured below.

Notice that, if I have two identically coloured arrangements of three points, correctly arranged, I will have satisfied the conditions of the problem.

Thus, if I pick a \(3 \times 9\) rectangular grid of points, since there are 8 possible ways to colour the points in each column, we must have at least two columns whose colourings are the same. Since there are three points in each column, at least two of them are the same, so we can create our rectangle with vertices the same colour, and we are done.
If we think a bit, we can actually refine our proof. Notice that two of the 8 colourings have all three points coloured the same. One of these three coloured configurations will match with any configuration having two points of that same colour to produce our desired rectangle.

Therefore, we can classify our groups of three as one of two types: type $R$ with at least 2 red points and type $B$ with at least 2 blue points. Now, if we have a $3 \times 7$ rectangular grid of points, at least 4 must be of the same type $R$ or $B$. This extends the original idea of the pigeonhole principle to:

**Generalized pigeonhole principle:**

If you have $n$ pigeonholes and $m > kn$ pigeons, for some integer $k$, then there must be at least one pigeonhole that contains at least $k + 1$ pigeons.

In our case, if we look at the worst case scenario, when I have six things into two groups, I could have three of each. As soon as I add another, there must be some group with four ($7 = 3 \times 2 + 1$, so we must have one with at least $3 + 1 = 4$).

So there must be at least four Rs or at least four Bs. Suppose we have four Bs. We can break the Bs up into three groups: the first and second must be blue; the first and third must be blue; and the second and the third must be blue. These possibilities are pictured below, where the “open” point could be either red or blue. With this classification, the configuration with three blues would fit in any category. Thus, since we have four Bs, but three groups, there must be at least two in the same group, guaranteeing our desired rectangle.

Knowing the nuances of a technique or theorem and how to use it in your solutions is very important. Hopefully, the examples provide some insight into using the pigeonhole principle or problem solving in general. Enjoy the rest of the problems, we may talk about some of them in a future column.