Sometimes a simple idea can be very powerful. In mathematics this happens all the time. In this issue, we revisit an idea, the pigeonhole principle, that I used to solve a problem from the course C&O 380 that I took as an undergraduate from professor Ross Honsberger [2018: 44(4), p. 157-159].

**Pigeonhole principle (a.k.a. Dirichlet box principle):**
If you have \( n \) pigeonholes and \( m > n \) pigeons, then there must be at least one pigeonhole that contains at least two pigeons.

To see this in action, think about the following statement: if 8 people are gathered together in a room, at least two of them were born on the same day of the week. Why does this work? Let’s think of the worst case scenario. Imagine the people arrive at the room one at a time, and we only let people in that were born on a different day of the week than everyone else that is already in the room. We can do this up to a point. Once we reach seven people, every day has been accounted for which means that if we allow an eighth person to enter, this person must have been born on the same day as somebody already present. The pigeonhole principle is an example of an **existence** theorem in mathematics. It tells us something must exist, but it doesn’t really give us any idea how to find it.

The key idea for using the pigeonhole principle is to define our groups in such a way that we are forced to have an overlap. In some problems, like the example above, it is obvious. In others we have to work a bit. Let’s look at problem #7 from C&O 380 [2018: 44(10), p. 419]:

*Of 5 points inside a square of unit side, show that some pair is less than \( \sqrt{2} \) units apart.*

Since we have 5 points and we are trying to force a pair of them to have a condition, it gives us the hint that we want to work with 4 categories. It seems natural to cut the square into four congruent squares as shown below.

```
   3  4
---+---
  1 | 2
```

Thus the pigeonhole principle guarantees that at least two points will be in the same region. How far apart can they be? The furthest apart would be if they were
at opposite vertices of the square, which the Pythagorean theorem tells us are \( \sqrt{2} \) units apart. Since we are picking points from \emph{inside} the square, we are guaranteed that two must be less than \( \frac{\sqrt{2}}{2} \) units apart, so we are done.

Care is needed since we can break the square up into \emph{any} four disjoint regions and the pigeonhole principle will ensure that at least two are in the same region. If we don’t create our regions carefully, we may not get what we are after. For example, we could have created the four regions as shown below.

![Diagram showing four disjoint regions]

We still know for sure that at least two of the points will fall in the same rectangle, but now they can be as far apart as (almost) \( \sqrt{17} \) which is larger than \( \frac{\sqrt{2}}{2} \). We cannot conclude that the problem is impossible just because our configuration gives a larger number. So build your groups carefully.

Next, let’s consider problem #11 from C&O 380 [2019: 45(4), p. 176]: \emph{Prove that no matter how the points of a closed unit square are coloured red or blue, either some two red points or some two blue points are at least \( \sqrt{\frac{5}{2}} \) units apart.}

Consider any three corners of the square. Since we have only two colours, at least two of the points have the same colour. If two of the same coloured corners are diagonally opposite of each other, they are \( \sqrt{2} \) units apart which is larger than \( \frac{\sqrt{5}}{2} \) and we are done.

Suppose the two points that are the same colour are not diagonally opposite. Then we can assume, without loss of generality, that two vertices on the same edge are red and the other vertex is blue, as in the diagram below. Consider the midpoint of the edge nearest one of the red vertices, but not between one of the three coloured vertices, as indicated by a “?” in the diagram. This point, taken with the further red vertex and the blue vertex, forms an isosceles triangle. Once again, we can conclude that at least two vertices will be the same colour. Since the original two vertices are coloured oppositely, the new point is the same colour as one of them. The Pythagorean theorem again yields that these points are \( \frac{\sqrt{5}}{2} \) units apart, and we are done.

![Diagram showing midpoint and isosceles triangle]

Notice that in the first case we were told we had 5 points which gave us the hint that we are looking at four categories. The second problem was a little different. We had to focus in on the two colours, which would be our \emph{categories}. That meant
we had to look at 3 points to force two of them to be the same colour and piece our proof together from there. Let’s look at problems #21–25 from C&O 380 and choose one to focus on.

#21. Suppose each point of the plane is coloured red or blue. Show that some rectangle has its vertices all the same colour.

#22. In chess, is it possible for a knight to go from the lower left corner square of the board to the upper right corner square and in the process to land exactly once on each other square?

#23. Prove that the number of people at the opera next Thursday night who will shake hands an odd number of times is an even number of people.

#24. Prove that no matter what three points of a square lattice are joined, an equilateral triangle will never occur.

#25. Prove that

\[
\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100} < \frac{1}{10}
\]

Let’s looks closely at #21. Again we have two colours, so again we know that for any three points we pick on the plane, at least two will be the same colour. We need four points, all the same colour, forming a rectangle. Let’s investigate all the possibilities of three coloured points. Taking into account the order of the points, there are \(2 \times 2 \times 2 = 8\) possibilities, pictured below.

1. 
2. 
3. 
4. 

Notice that, if I have two identically coloured arrangements of three points, correctly arranged, I will have satisfied the conditions of the problem.

Thus, if I pick a \(3 \times 9\) rectangular grid of points, since there are \(8\) possible ways to colour the points in each column, we must have at least two columns whose colourings are the same. Since there are three points in each column, at least two of them are the same, so we can create our rectangle with vertices the same colour, and we are done.
If we think a bit, we can actually refine our proof. Notice that two of the 8 colourings have all three points coloured the same. One of these three coloured configurations will match with any configuration having two points of that same colour to produce our desired rectangle.

Therefore, we can classify our groups of three as one of two types: type $R$ with at least 2 red points and type $B$ with at least 2 blue points. Now, if we have a $3 \times 7$ rectangular grid of points, at least 4 must be of the same type $R$ or $B$. This extends the original idea of the pigeonhole principle to:

**Generalized pigeonhole principle:**

If you have $n$ pigeonholes and $m > kn$ pigeons, for some integer $k$, then there must be at least one pigeonhole that contains at least $k + 1$ pigeons.

In our case, if we look at the worst case scenario, when I have six things into two groups, I could have three of each. As soon as I add another, there must be some group with four ($7 = 3 \times 2 + 1$, so we must have one with at least $3 + 1 = 4$).

So there must be at least four $R$s or at least four $B$s. Suppose we have four $B$s. We can break the $B$s up into three groups: the first and second must be blue; the first and third must be blue; and the second and the third must be blue. These possibilities are pictured below, where the “open” point could be either red or blue. With this classification, the configuration with three blues would fit in any category. Thus, since we have four $B$s, but three groups, there must be at least two in the same group, guaranteeing our desired rectangle.

Knowing the nuances of a technique or theorem and how to use it in your solutions is very important. Hopefully, the examples provide some insight into using the pigeonhole principle or problem solving in general. Enjoy the rest of the problems, we may talk about some of them in a future column.