OLYMPIAD CORNER
SOLUTIONS


OC436. In a non-isosceles triangle $ABC$, let $O$ and $I$ be its circumcenter and incenter, respectively. Point $B'$, which is symmetric to point $B$ with respect to line $OI$, lies inside $\angle ABI$. Prove that the tangents to the circumcircle of the triangle $BB'I$ at points $B'$ and $I$ intersect on the line $AC$.

Originally Russia MO, 8th Problem, Grade 10, Final Round 2017 (Geometry).

We received 3 correct submissions. We present all three solutions.

Solution 1, by Lee Jang Yong

Let $\omega$ be the circumcircle of $\triangle ABC$. Let $A'$, $B'$, and $C'$ be the points that are symmetric with respect to line $OI$ to $A$, $B$, and $C$, respectively. Because the symmetry line, $OI$, passes through the centre of $\omega$ we have that the symmetric images $A'$, $B'$, and $C'$ belong to $\omega$, as well.

Let $D$ be the intersection of $AC$ with the tangent at $I$ to the circumcircle of $\triangle BB'I$. We show that $DI = DB'$.

Let $D'$ be the point symmetric to $D$ with respect to line $OI$. The centre of the circumcircle of triangle $BB'I$ lies on $OI$, given that $\triangle BB'I$ is isosceles. Therefore the tangent $ID$ is perpendicular to $OI$, and $I$, $D$, and $D'$ are co-linear with $I$ being the middle point of $DD'$. Moreover, if we extend $DD'$ to intersect $\omega$ at $M$ and $M'$ we find that $I$ is the middle of the new segment, $MM'$.

We are in the setting of the butterfly theorem. Let $X$ be the intersection of $A'I$ with $\omega$, and $Z$ be the intersection of $C'I$ with $\omega$. $D'$ is a point on $A'C'$, $D$, $I$, and $D'$ are co-linear, and $I$ is the middle point of $DD'$. Due to the butterfly theorem we conclude that $D$ belongs to $XZ$.

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However, $XZ$ is the perpendicular bisector of $B'I$, so $DI = DB'$. This is because $I$ is the centre of the incircle and the intersection of the bisectors of $\triangle A'B'C'$. These properties imply the equality of the following $\angle B'IX = \angle IB'X = (\angle C'A'B' + \angle C'B'A')/2$ and that $\triangle B'IX$ is isosceles. Similarly, $\triangle B'IZ$ is isosceles.

Since $DI = DB'$ it follows that $DB'$ is tangent to the circumcircle of $\triangle BB'I$ at $B'$, and that the tangents to the circumcircle of $\triangle BB'I$ at points $B'$ and $I$ intersect on the line $AC$ at the point $D$.

Solution 2, by Ivko Dimitrić.

Consider, without loss of generality, triangle $ABC$ in the plane of complex numbers, whose circumcircle is the unit circle centered at the origin $O$. For any point included in our proof we associate a unique capital letter and a complex number. The capital letter is used to refer to the complex number, as well. Let $A = e^{i\alpha}$, $B = e^{i\beta}$, and $C = e^{i\gamma}$ be the complex numbers that identify the triangle vertices, with $0 < \alpha < \beta < \gamma < 2\pi$. Moreover, denote by $a = e^{i\alpha/2}$, $b = -e^{i\beta/2}$, $c = e^{i\gamma/2}$ so that $\overline{a} = 1/a$, $\overline{b} = 1/b$, $\overline{c} = 1/c$, and $A = a^2$, $B = b^2$, $C = c^2$. Then,

$$I = -(ab + bc + ca) \quad \text{and} \quad \overline{I} = -\frac{a + b + c}{abc}$$  \hspace{1cm} (1)

(see p. 262 of M. Bataille’s article in Crux Mathematicorum, Vol 45:5 (May 2019)).

In general, the orthogonal projection $S$ of a point $X$ to a line $PQ$ is given by

$$S = \frac{1}{2} \left( \frac{PQ - \overline{PQ}}{Q - \overline{P}} + X + \frac{Q - P}{Q - \overline{P}} \overline{X} \right)$$  \hspace{1cm} (2)

and the point $Y$ symmetric to $X$ with respect to $\overrightarrow{PQ}$ is

$$Y = 2S - X = \frac{PQ - \overline{PQ}}{Q - \overline{P}} + \frac{Q - P}{Q - \overline{P}} \overline{X}.$$  \hspace{1cm} (3)

Thus, when $P = O$, $Q = I$ and $X = B$ we get $B' = (I/\overline{I}) \overline{B}$.

Let $D = \frac{1}{2}(I + B') = \frac{1}{2} \left( 1 + \frac{\overline{B}}{I} \right)$ be the midpoint of segment $B'I$ and let $K$ be the center of the circumcircle of $BB'I$. The perpendicular bisector of $B'I$ consists of points $Z$ for which

$$\left( \frac{Z - D}{I - B'} \right) = -\frac{Z - D}{I - B'}$$

yielding

$$\left[ Z - \frac{I}{2} \left( 1 + \frac{\overline{B}}{I} \right) \right] I \left( 1 - \frac{B}{I} \right) + \left[ Z - \frac{I}{2} \left( 1 + \frac{B}{I} \right) \right] I \left( 1 - \frac{\overline{B}}{I} \right) = 0,$$

After multiplying out, simplifying, and dividing by $I\overline{I}$ the equation of the bisector, $KD$, reduces to

$$\left( 1 - \frac{B}{I} \right) \frac{Z}{I} + \left( 1 - \frac{\overline{B}}{I} \right) \frac{\overline{Z}}{I} = 1 - \frac{1}{I\overline{I}}.$$  \hspace{1cm} (4)

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Since $KB' = KI$ and $KD$ is the angle bisector of $\angle B'KI$, the two tangents to the circumcircle of $BB'I$ at points $B'$ and $I$ intersect on the bisector $KD$. Therefore, to prove that the two tangents intersect on the line $AC$ it suffices to show that the lines $DK, AC$ and the perpendicular to $IO$ at $I$ intersect at one point.

The line perpendicular to $IO$ at $I$ is the locus of points $Z$ such that $Z - I$ is a real multiple of $i(I - O)$, i.e.,

$$\frac{Z - I}{I} = -\frac{Z - I}{I} \iff \overline{IZ} = 2I\overline{I}.$$

The line through arbitrary two points $P$ and $Q$ has an equation

$$(Q - P)Z - (Q - P)\overline{Z} = P\overline{Q} - \overline{P}Q,$$

so that the line $AC$ through $A = a^2$ and $C = c^2$ is

$$Z + c^2a^2Z = a^2 + c^2.$$

Using (??) and combining (??) and (??), we find the intersection of the perpendicular to $IO$ at $I$ and the line $AC$ to be the point $Z$ whose affix satisfies

$$Z = \frac{2I\overline{I} - (a^2 + c^2)I}{I - c^2a^2\overline{I}} = \frac{(a + b + c)(a + 2b + c)}{ca(ca - b^2)},$$

by factoring out $a + c$ on the top and the bottom. Consequently,

$$Z = \frac{(ab + bc + ca)(ab + bc + 2ca)}{b^2 - ca} = \frac{I(ca - I)}{ca - b^2}.$$ 

It can be now shown that this point satisfies the equation (??) so the lines $DK, AC$ and $IZ$ are concurrent at $Z$. Namely, using (??) we compute

$$1 - \frac{1}{I\overline{I}} = \frac{(a + c)(ab + bc + ca + b^2)}{(a + b + c)(ab + bc + ca)} = \frac{-a + c}{abcI} \cdot \frac{I - B}{I},$$

so that

$$\left(1 - \frac{B}{I}\right) = -\frac{abcI}{a + c} \left(1 - \frac{1}{I\overline{I}}\right) \quad \text{and} \quad \left(1 - \frac{B}{I}\right) = \frac{-I}{b(a + c)} \left(1 - \frac{1}{I\overline{I}}\right).$$

Also, from (??) we get

$$\frac{Z}{I} = \frac{ca - I}{ca - b^2} \quad \text{and} \quad \frac{Z}{\overline{I}} = \frac{b^2(caI - 1)}{ca - b^2}.$$
Substituting these into the left-hand side of (??) we get
\[
\left[ \frac{abc \sqrt{I}}{a + c} \frac{ca - I}{ca - b^2} + \frac{I}{b(a + c)} \frac{b^2(1 - ca \sqrt{I})}{ca - b^2} \right] \left( 1 - \frac{1}{\sqrt{I}} \right) = 1 - \frac{1}{\sqrt{I}}. 
\]
This verifies the equation (??) and proves the claim.

Solution 3, by Andrea Fanchini.

We use barycentric coordinates with reference to \( \triangle ABC \). The line
\[
IO : bc(cS_C - bS_B)x + ac(aS_A - cS_C)y + ab(bS_B - aS_A)z = 0 
\]
has infinite perpendicular point
\[
IO_{\infty \perp} (a(b - c) : b(c - a) : c(a - b)). 
\]
Therefore, the tangent to the circumcircle of \( \triangle BB'I \) at point \( I \) is given by
\[
IIO_{\infty \perp} : bc(b + c - 2a)x + ac(a + c - 2b)y + ab(a + b - 2c)z = 0. 
\]
Point \( B' \), which is the symmetric image of point \( B \) with respect to line \( OI \) is identified by
\[
B' \left( 2a(s - b)(c - a)(c - b) : b^2(b - a)(b - c) : 2c(b - a)(c - a) \right). 
\]
Therefore, the tangent to the circumcircle of \( \triangle BB'I \) at \( B' \) is given by

\[
B'X : b^2 c (b - a)(b - c)(b + c - 2a)x - 2ac(s - b)(c - a)^2(a + c - 2b)y \\
+ ab^2 (b - a)(b - c)(a + b - 2c)z = 0.
\]

In conclusion, the two tangent lines intersect at the point identified by

\[
X = IIO_{\infty \perp} \cap B'X = (a(2c - a - b) : 0 : c(b + c - 2a)).
\]

Clearly, this point lies on the line \( AC \), since its second coordinate is 0.

**OC437.** The magician and his helper have a deck of cards. The cards all have the same back, but their faces are coloured in one of 2017 colours (there are 1000000 cards of each colour). The magician and the helper are going to show the following trick. The magician leaves the room; volunteers from the audience place \( n > 1 \) cards in a row on a table, all face up. The helper looks at these cards, then he turns all but one card face down (without changing their order). The magician returns, looks at the cards, points to one of the face-down cards and states its colour. What is the minimum number \( n \) such that the magician and his helper can have a strategy to do the magic trick successfully?

*Originally Russia MO, 4th Problem, Grade 11, Final Round 2017 (Game Theory).*

*No solutions were received.*

**OC438.** A teacher gives the students a task of the following kind. He informs them that he thought of a monic polynomial \( P(x) \) of degree 2017 with integer coefficients. Then he tells them \( k \) integers \( n_1, n_2, \ldots, n_k \) and the value of the expression \( P(n_1)P(n_2) \cdots P(n_k) \). According to these data, the students should then find teacher’s polynomial. Find the smallest \( k \) for which the teacher can compose such a problem so that the polynomial found by the students must necessarily coincide with the one he thought of.

*Originally Russia MO, 3rd Problem, Grade 11, Regional Round 2017 (Algebra).*

*No solutions were received.*

**OC439.** Let \((G, \cdot)\) be a group and let \( m \) and \( n \) be two nonzero natural numbers that are relatively prime. Prove that if the functions \( f : G \to G, f(x) = x^{m+1} \) and \( g : G \to G, g(x) = x^{n+1} \) are surjective endomorphisms, then the group \( G \) is abelian.

*Originally Romania MO, 2nd Problem, Grade 12, District Round 2017 (Abstract Algebra).*

We received 2 correct submissions. We present the solution by Oliver Geupel. Independently, Corneliu Manescu-Avram submitted a similar solution.

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Let $a$ be an arbitrary element of $G$. Since $f$ is a surjective endomorphism, we deduce that, for every $b \in G$, there is a $c \in G$ such that $b = f(c)$, and it holds
\[
a^m b = a^{-1} f(a) f(c) = a^{-1} f(ac) = a^{-1} (ac)^{m+1} = (ca)(ac)^m \cdots (ac) = (ca)\cdots (ca)^{a^{-1}} = (ca)^{m+1} a^{-1} = f(ca)a^{-1} = f(c)f(a)a^{-1} = ba^m.
\]
Hence, $a^m$ commutes with every element of the group. Similarly, $a^n$ commutes with every element of the group.

It is well-known and easy to verify that the set of elements of a group, that commute with every element of the group, is a subgroup of $G$, called the centre $Z(G)$ of the group. Thus, for integers $q$ and $r$, we have $a^{mq+nr} \in Z(G)$. Since $m$ and $n$ are co-prime, integers $q$ and $r$ can be chosen such that $mq + nr = 1$. Consequently, $Z(G) = G$, that is, $G$ is abelian.

**OC440.** Let $f : [a, b] \to [a, b]$ be a differentiable function with continuous and positive first derivative. Prove that there exists $c \in (a, b)$ such that
\[
f(f(b)) - f(f(a)) = (f'(c))^2(b - a).
\]

*Originally Romania MO, 4th Problem, Grade 11, Final Round 2017.*

We received 4 correct submissions. We present the solution by Ioko Dimitrić. Similar solutions were submitted independently by Brian Bradie and Corneliu Manescu-Avram.

Since $f([a, b]) \subset [a, b]$ and $f$ is increasing and differentiable, the Mean Value Theorem for $f$ applied to the interval $[f(a), f(b)]$ guarantees the existence of a number $q, \ a \leq f(a) < q < f(b) \leq b$, such that
\[
f(f(b)) - f(f(a)) = f'(q)(f(b) - f(a)).
\]

Another application of the same theorem on the interval $[a, b]$ tells us that
\[
f(b) - f(a) = f'(p)(b - a)
\]
for some number $p, \ a < p < b$. Combining the two formulas we get
\[
f(f(b)) - f(f(a)) = f'(p)f'(q)(b - a), \quad (1)
\]
where $p, q \in (a, b)$.

Next, we can assume that $f'(p) \leq f'(q)$. Since $f'$ is positive we have
\[
f'(p) \leq \sqrt{f'(p)f'(q)} \leq f'(q).
\]
Then, the value $\sqrt{f'(p)f'(q)}$ is between $f'(p)$ and $f'(q)$. Since $f'$ is continuous, by the Intermediate Value Theorem for $f'$ on the interval $[p, q]$, there exists $c \in [p, q] \subset (a, b)$ such that $f'(c) = \sqrt{f'(p)f'(q)}$. Combining with (1)
\[
f(f(b)) - f(f(a)) = (f'(c))^2(b - a),
\]
and the statement follows.