The problems featured in this section are intended for students at the secondary school level.

To facilitate their consideration, solutions should be received by March 15, 2020.

MA51. Proposed by Nguyen Viet Hung.
Find all non-negative integers $x$, $y$, $z$ satisfying the equation
$$2^x + 3^y = 4^z.$$

MA52. The diagram shows part of a tessellation of the plane by a quadrilateral. Khelen wants to colour each quadrilateral in the pattern.

1. What is the smallest number of colours he needs if no two quadrilaterals that meet (even at a point) can have the same colour?

2. Suppose that quadrilaterals that meet along an edge must be coloured differently, but quadrilaterals that meet just at a point may have the same colour. What is the smallest number of colours that Khelen would need in this case?

3. What is the smallest number of colours needed to colour the edges so that edges that meet at a vertex are coloured differently?

MA53.
Find all positive integers $m$ and $n$ which satisfy the equation
$$\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \ldots \cdot \frac{m^3 - 1}{m^3 + 1} = \frac{n^3 - 1}{n^3 + 2}.$$
MA54. How many six-digit numbers are there, with leading 0s allowed, such that the sum of the first three digits is equal to the sum of the last three digits, and the sum of the digits in even positions is equal to the sum of the digits in odd positions?

MA55. The diagram shows three touching semicircles with radius 1 inside an equilateral triangle, which each semicircle also touches. The diameter of each semicircle lies along a side of the triangle. What is the length of each side of the equilateral triangle?
Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 mars 2020.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA51. Proposé par Nguyen Viet Hung.

Déterminer tous les entiers non négatifs $x, y, z$ satisfaisant à l'équation

$$2^x + 3^y = 4^z.$$

MA52. Le diagramme montre une partie d’un pavage du plan par un quadrilatère. Katherine désire en effectuer un colorage.

1. Déterminer le plus petit nombre de couleurs possible si Katherine exige que deux quadrilatères se touchant, même en un seul point, aient besoin d’être colorés différemment.

2. Supposons maintenant que deux quadrilatères partageant un côté doivent être colorés différemment, mais pas nécessairement ceux se touchant en un point seulement. Déterminer le plus petit nombre de couleurs requises pour colorer les quadrilatères dans ce contexte.

3. Enfin, Katherine désire colorer les côtés seulement, mais de façon à ce que les côtés se rencontrant en un point soient colorés différemment. Déterminer le plus petit nombre de couleurs requises dans ce contexte.
MA53.
Déterminer tous les entiers positifs $m$ et $n$ satis faisant à l’équation

\[
\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{m^3 - 1}{m^3 + 1} = \frac{n^3 - 1}{n^3 + 2}.
\]

MA54. Combien de nombres à six chiffres y a-t-il, tels que la som me des trois premiers chiffres est égale à la som me des trois derniers chiffres et puis que la som me des chiffres en positions paires égale la som me des chiffres en positions impaires? La présence de 0s en première(s) position(s) est permise.


\[\text{Diagramme avec trois demi cercles} \]

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MA26. Nine (not necessarily distinct) 9-digit numbers are formed using each
digit 1 through 9 exactly once. What is the maximum possible number of zeros
that the sum of these nine numbers can end with?

Originally Problem M2430 of Kvant.

We received 3 submissions, all of which were correct and complete. We present
the solution by the Missouri State University Problem Solving Group.

The answer is eight. Since

\[ 8 \times 987654321 + 198765432 = 8100000000, \]

the answer is at least 8. But the maximum value the sum can be is

\[ 9 \times 987654321 = 8888888889, \]

so the only other possibility is to have nine zeros. Now each number whose digits
are a permutation of 1, \ldots, 9 is a multiple of 9, since the sum of their digits is.
Therefore any sum of these numbers must also be a multiple of 9. But the only
10-digit number ending in nine zeros that is a multiple of 9 is 9000000000 and this
is larger than our upper bound.

We note that analogous methods extend this result to base \( b \): if \( b - 1 \) numbers
consisting of permutations of 1, \ldots, \( b - 1 \) are added, the maximum possible number
of zeros that their sum can end in is \( b - 2 \).

MA27. You want to play Battleship on a 10 \times 10 grid with 2 \times 2 squares
removed from each of its corners:
What is the maximum number of submarines (ships that occupy 3 consecutive squares arranged either horizontally or vertically) that you can position on your board if no two submarines are allowed to share any common side or corner?

*Originally Problem 24 of 2018 Savin contest.*

We received 1 submission, which was correct but incomplete. We present the solution by Richard Hess and Taus Brock-Nannestad, and completed by the editor.

Consider the following diagram:

There is no way to place a submarine on the grid without its touching one of the nine marked grid points. No two submarines can touch the same marked grid point so nine submarines is the most that can be placed on the grid without touching.

It is possible to place nine submarines on the grid. There are many ways to do this; here is one:

This is an example of a problem where a construction is a necessary part of the proof. Without actually demonstrating that it is possible to place nine submarines, we know only that we cannot place more than this many.

**MA28.** Prove that for all positive integers $n$, the number

$$\frac{1}{3} \left( 4^{4n+1} + 4^{4n+3} + 1 \right)$$

is not prime.

*Originally Problem 27 of 2017 Savin contest.*

We received 4 submissions which were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

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The statement is false. If $n = 6$, we have
\[
\frac{1}{3}(4^{4n+1} + 4^{4n+3} + 1) = 6\,380\,099\,472\,108\,203,
\]
which is prime. (*Mathematica* claims that $n = 861$ and $n = 5304$ also yield prime values).

However, it is true that if $n \not\equiv 0 \mod 3$, then $(4^{4n+1} + 4^{4n+3} + 1)/3$ is never prime.

If $n \equiv 1 \mod 3$, then $n = 3k + 1, k \in \mathbb{Z}$ and
\[
4^{4n+1} + 4^{4n+3} + 1 = 4^{12k+5} + 4^{12k+7} + 1
= 16 \cdot 64^{4k+1} + 4 \cdot 64^{4k+2} + 1
\equiv 2 \cdot 1 + 4 \cdot 1 + 1 \mod 7
\equiv 0 \mod 7
\]
and
\[
4^{4n+1} + 4^{4n+3} + 1 \geq 4 + 4^3 + 1 = 69 > 7,
\]
so 7 is a non-trivial factor of $(4^{4n+1} + 4^{4n+3} + 1)/3$.

If $n \equiv 2 \mod 3$, then $n = 3k + 2, k \in \mathbb{Z}$ and
\[
4^{4n+1} + 4^{4n+3} + 1 = 4^{12k+9} + 4^{12k+11} + 1
= 64^{4k+3} + 16 \cdot 64^{4k+3} + 1
\equiv 1 + 7 \cdot 1 + 1 \mod 9
\equiv 0 \mod 9
\]
and
\[
4^{4n+1} + 4^{4n+3} + 1 \geq 69 > 9,
\]
so 9 is a non-trivial factor of $(4^{4n+1} + 4^{4n+3} + 1)$ and hence 3 is a non-trivial factor of $(4^{4n+1} + 4^{4n+3} + 1)/3$.

**MA29.** Find all positive integers $n$ satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ can be split into pairs so that if numbers in each pair are added and all the sums are multiplied together, the result is a perfect square.  

*Originally Problem 2 of Fall Junior A-level of XL Tournament of Towns 2017.*

We received 3 submissions, all of which were correct and complete. We present the solution by the Missouri State University Problem Solving Group, modified by the editor.

We claim that $n$ satisfies the condition if $n > 1$.

*Cruix Mathematicorum*, Vol. 46(1), January 2020
We first observe that \( n = 1 \) fails the condition. For \( n = 1 \) the only pairing is \( \{1, 2\} \), the sum of which is the non-perfect square 3.

There are two cases:

1. \( n = 2k \) where \( k \geq 1 \). By pairing \( i \) with \( 2n + 1 - i \) for \( i = 1, 2, \ldots, n \) gives a product of \((2n + 1)^k\)^2.

2. \( n = 2k + 1 \) where \( k \geq 1 \). When \( k \geq 1 \), we pair 1 and 5, 2 and 4, 3 and 6, and \( 6 + i \) with \( 2n + 1 - i \) for \( i = 1, 2, \ldots, n - 3 = 2k - 2 \). The product is then

\[
(1 + 5)(2 + 4)(3 + 6)(2n + 7)^{2k-2} = (18(2n + 1)^{k-1})^2.
\]

**MA30.** Consider the two marked angles on a grid of equilateral triangles.

Prove that these angles are equal.

*Originally Problem 18 of 2017 Savin contest.*

We received 6 solutions, all of which were correct. We present the solution of Missouri State University Problem Solving Group, modified by the editor.

Let the side lengths of the equilateral triangles be 1.

**Method I.** Consider the figure below.

Let \( \alpha = m(\angle ACB) \) and \( \beta = m(\angle AED) \). Since \( AB = 2 \) and \( BC = 5 \), the Law of Cosines gives

\[
AC = \sqrt{2^2 + 5^2 + 2 \cdot 5} = \sqrt{39}.
\]

Applying the Law of Cosines again

\[
\cos \alpha = \frac{\sqrt{39^2 + 5^2 - 2^2}}{2 \cdot 5\sqrt{39}} = \sqrt{\frac{12}{13}}
\]
Similarly, $DE = \sqrt{3}$ and applying the Law of Cosines to $\triangle AFE$ we have

\[ AE = \sqrt{1^2 + 3^2 + 1 \cdot 3} = \sqrt{13}. \]

One more use of the Law of Cosines gives

\[ \cos \beta = \frac{\sqrt{13^2 + 3^2 - 2^2}}{2\sqrt{3\sqrt{39}}} = \frac{\sqrt{12}}{13}, \]

so the angles in question are congruent.

**Method II.** Consider the figure below.

Triangle $ABD$ in this figure is congruent to triangle $DAE$ in the figure in Method I. Thus, we wish to show that $\angle ACB \cong \angle ADB$. The point marked $O$ is equidistant from each of $A, B, C, D$ (it lies on the intersection of the perpendicular bisectors of $AD, AB, and BC$). Therefore, these points lie on a circle centered at $O$. Since $\angle ACB$ and $\angle ADB$ are subtended by the same arc, they must be congruent.
In the Calendar Problem, your goal is to figure out the day of the week on which you were born.

There are various YouTube videos of mathematicians (or “mathemagicians”) performing this trick in their heads. For example, an audience member will call out her birthday (e.g. May 25, 2004), and the mathematician will instantly reply, “Tuesday”.

In this article, we will unpack this problem and determine an algorithm to solve this problem.

First, let’s investigate the day of the week that our birthday falls on in the year 2020. To do this, all we need is the knowledge that January 1, 2020 is Wednesday. Whenever I have presented this problem in a class, either with high school students or undergraduates, one student always knows the number of days in each month:

<table>
<thead>
<tr>
<th>Month</th>
<th>Days</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>31</td>
</tr>
<tr>
<td>February</td>
<td>29 (since 2020 is a leap year)</td>
</tr>
<tr>
<td>March</td>
<td>31</td>
</tr>
<tr>
<td>April</td>
<td>30</td>
</tr>
<tr>
<td>May</td>
<td>31</td>
</tr>
<tr>
<td>June</td>
<td>30</td>
</tr>
<tr>
<td>July</td>
<td>31</td>
</tr>
<tr>
<td>August</td>
<td>31</td>
</tr>
<tr>
<td>September</td>
<td>30</td>
</tr>
<tr>
<td>October</td>
<td>31</td>
</tr>
<tr>
<td>November</td>
<td>30</td>
</tr>
<tr>
<td>December</td>
<td>31</td>
</tr>
</tbody>
</table>

Notice that January 1 must be the same day of the week as January 8, January 15, January 22, and January 29. This is because each of these numbers in \{1, 8, 15, 22, 29\} gives the same remainder when divided by 7.

Thus, for some birthdays, the Calendar Problem can be easily solved. Let’s consider someone born on January 17. For the year 2020, since January 1 is a Wednesday we know that January 15 is a Wednesday, which implies that January 16 is a Thursday, from which it follows that January 17 is a Friday.

For birthdays in the month of January, notice that the answer can be found by simply taking the date, dividing by 7, and calculating the remainder. Then we can use this table to read off the answer:
Here are two common approaches for solving the Calendar Problem.

**Approach One:** Count the number of days that have elapsed from the start of the year (January 0) until the target date. For example, March 23 consists of $31 + 29 + 23$ days, since we need to add up all the days in January and February and then the twenty-three days in March. This adds up to 83. We divide by 7. Since $83 = 7 \times 11 + 6$, the remainder is 6. From the above table, we see that a remainder of 6 corresponds to Monday.

**Approach Two:** Determine the day of the week for the first date of each month, showing that if January 1 falls on Wednesday, then February 1 must be a Saturday, March 1 must be a Sunday, and so on. From this, students can solve their problem for any given date by adding or subtracting increments of seven. For example, March 23 has to be the same date as March 16, March 9, and March 2. Thus, March 23 has to be a Monday, since March 1 is a Sunday.

A clever approach combines these two paradigms, using the first date of each month to determine the appropriate “shift”. For example, March 1 is $31 + 29 = 60 = 8 \times 7 + 4$ days after January 1, and so March 1 is “shifted” by 4 days compared to January 1. Thus, if we know that the shift number of March is 4, then we can determine the day of week of March 23 by adding the date to the shift number ($4 + 23 = 27 = 3 \times 7 + 6$), dividing the number by 7 and taking the remainder (which is 6), and then reading the above table to conclude that the answer is Monday.

For leap years such as the year 2020, the shift dates of each month are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Notice this table forms four sets of three digits that can be remembered this way:

$$034 = 5^2 + 3^2, \quad 025 = 5^2, \quad 036 = 6^2, \quad 146 = 12^2 + 2^2.$$  

For example, the shift number for June is 5, since the number of days until the start of June is $31 + 29 + 31 + 30 + 31 = 152 = 21 \times 7 + 5$, which has a remainder of 5 upon division by 7. In other words, June 1 is exactly 21 weeks and 5 days after January 1 which implies that the shift for June is 5.

Let $D$ be the date and $S$ be the shift number. For example, June 15 would have $D = 15$ and $S = 5$. To perform this algorithm in our head, we just need to
add $D + S$, divide by 7, and the remainder gives us our answer to the Calendar Problem. Since this remainder is 6, we can conclude that June 15, 2020 will be a Monday.

Now let’s extend this by replacing the year 2020 with our birth year. In solving this harder problem, we realize that each 365-day year contributes one extra day (52 weeks plus 1 day) and each 366-day leap year contributes two extra days (52 weeks plus 2 days). Thus, if January 1, 2020 is a Wednesday, then January 1, 2019 was a Tuesday, since we have shifted back one day. And similarly, January 1, 2021 will be a Friday since we will need to shift forward two days.

In one of my school visits (in 2019), one student made the powerful insight that her birthday in 2002 must be the same day of week as her birthday in 2019, since there are 17 “extra days” in addition to the four Feb 29 “leap days” that occurred in 2004, 2008, 2012, and 2016. Since $17 + 4 = 21$, the calendar shifted 21 days between her birthday in 2002 and her birthday in 2019. And since 21 is a multiple of 7, if her birthday fell on a Tuesday in 2019, then it must have fallen on a Tuesday in 2002. This student provided a clear method for how to handle the tricky concept of leap years.

A different student from the same class observed that the calendar repeats itself every 28 years, since each year contributes one extra day (52 weeks plus 1 day), and there are 7 occurrences of February 29 during any 28-year period. Thus, the calendar shifts by $28 + 7 = 35$ days, which is a multiple of 7. This observation enabled the student to determine the day of the week on which her parents were born.

Through this process of solving the Calendar Problem and determining an algorithm that works for any birthday, students demonstrate the four principles of the Computational Thinking process.

(i) Decomposition: break down the problem into smaller tasks
(ii) Pattern recognition: identify similarities, differences, and patterns within the problem
(iii) Abstraction: identify general principles and filter out unnecessary information
(iv) Algorithmic design: identify and organize the steps needed to solve the problem

As mathematicians we use these four principles in our research endeavours, and the Calendar Problem offers a challenge for enabling our students to have similar experiences.

During the 2018-2019 sabbatical year, I worked with the Callysto Project, a federally-funded initiative to bring computational thinking and mathematical problem solving into Grade 5-12 Canadian classrooms (www.callysto.ca). Through my work with Callysto, I visited over a dozen schools and worked with 700+ students, sharing rich math problems that incorporated the Callysto technology (a
A web-based platform known as a Jupyter Notebook, freely accessible to anyone with an Internet connection. I created a Notebook for the Calendar Problem, to be used by teachers and students. This free resource, which also includes a lesson plan for teachers, can be found at [www.bit.ly/CallystoCalendar](http://www.bit.ly/CallystoCalendar).

We end with three questions for consideration.

Communications, including solutions, concerning these questions are welcomed via email at richard.hoshino@questu.ca.

**Question #1**

Here is an algorithm that determines the correct day of week for any date in the 20th century (Jan 1, 1901 to Dec 31, 2000).

Let $Y$ be the last two digits of the year, $D$ be the day, and $S$ be the “shift” value according to the following table that is correct for non-leap years:

<table>
<thead>
<tr>
<th>Month</th>
<th>Shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>0</td>
</tr>
<tr>
<td>Feb</td>
<td>3</td>
</tr>
<tr>
<td>Mar</td>
<td>3</td>
</tr>
<tr>
<td>Apr</td>
<td>6</td>
</tr>
<tr>
<td>May</td>
<td>1</td>
</tr>
<tr>
<td>Jun</td>
<td>4</td>
</tr>
<tr>
<td>Jul</td>
<td>6</td>
</tr>
<tr>
<td>Aug</td>
<td>2</td>
</tr>
<tr>
<td>Sep</td>
<td>5</td>
</tr>
<tr>
<td>Oct</td>
<td>0</td>
</tr>
<tr>
<td>Nov</td>
<td>3</td>
</tr>
<tr>
<td>Dec</td>
<td>5</td>
</tr>
</tbody>
</table>

For example, the author’s birthday (June 15, 1978) has $Y = 78$, $D = 15$, and $S = 4$.

Now calculate the sum $T = Y + \lfloor Y/4 \rfloor + D + S$.

If the year corresponds to a leap year (i.e., $Y$ is a multiple of 4) and the month is January or February, subtract 1 from $T$. (Why do we need to do this?)

Divide $T$ by 7 and determine its remainder. The remainder tells us our answer:

<table>
<thead>
<tr>
<th>Remainder</th>
<th>Day</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Sunday</td>
</tr>
<tr>
<td>1</td>
<td>Monday</td>
</tr>
<tr>
<td>2</td>
<td>Tuesday</td>
</tr>
<tr>
<td>3</td>
<td>Wednesday</td>
</tr>
<tr>
<td>4</td>
<td>Thursday</td>
</tr>
<tr>
<td>5</td>
<td>Friday</td>
</tr>
<tr>
<td>6</td>
<td>Saturday</td>
</tr>
</tbody>
</table>

For example, October 29, 1929 has $T = 29 + 7 + 29 + 0 = 65$, which gives a remainder of 2 when divided by 7. Therefore, this date in history (known as Black Tuesday) was indeed a Tuesday.

Here is the question: why does this algorithm work?

**Question #2**

What day of the week would it be on your 100th birthday?

_Crux Mathematicorum_, Vol. 46(1), January 2020
Question #3

Create your own algorithm for other famous dates before the 20th century, and apply it to the famous dates such as the following:

(i) July 1, 1867 (Confederation Day in Canada)
(ii) July 4, 1776 (Independence Day in the USA)
(iii) April 23, 1616 (Death of William Shakespeare)
(iv) September 30, 1207 (Birthday of Rumi)

Note that you will need to be careful about ensuring the correct calculation of leap years, due to the quirky rules that occur when the year is a multiple of 100 but not a multiple of 400. Specifically, the years 1600 and 2000 are leap years, while the years 1700, 1800, 1900 are not leap years.