MATHEMATTIC SOLUTIONS


MA26. Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with?

Originally Problem M2430 of Kvant.

We received 3 submissions, all of which were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

The answer is eight. Since

\[ 8 \times 987654321 + 198765432 = 8100000000, \]

the answer is at least 8. But the maximum value the sum can be is

\[ 9 \times 987654321 = 8888888889, \]

so the only other possibility is to have nine zeros. Now each number whose digits are a permutation of 1, \ldots, 9 is a multiple of 9, since the sum of their digits is. Therefore any sum of these numbers must also be a multiple of 9. But the only 10-digit number ending in nine zeros that is a multiple of 9 is 9000000000 and this is larger than our upper bound.

We note that analogous methods extend this result to base \( b \): if \( b - 1 \) numbers consisting of permutations of 1, \ldots, \( b - 1 \) are added, the maximum possible number of zeros that their sum can end in is \( b - 2 \).

MA27. You want to play Battleship on a 10 \times 10 grid with 2 \times 2 squares removed from each of its corners:
What is the maximum number of submarines (ships that occupy 3 consecutive squares arranged either horizontally or vertically) that you can position on your board if no two submarines are allowed to share any common side or corner?

*Originally Problem 24 of 2018 Savin contest.*

*We received 1 submission, which was correct but incomplete. We present the solution by Richard Hess and Taus Brock-Nannestad, and completed by the editor.*

Consider the following diagram:

There is no way to place a submarine on the grid without its touching one of the nine marked grid points. No two submarines can touch the same marked grid point so nine submarines is the most that can be placed on the grid without touching.

It is possible to place nine submarines on the grid. There are many ways to do this; here is one:

This is an example of a problem where a construction is a necessary part of the proof. Without actually demonstrating that it is possible to place nine submarines, we know only that we cannot place more than this many.

**MA28.** Prove that for all positive integers \( n \), the number

\[
\frac{1}{3} \left( 4^{4n+1} + 4^{4n+3} + 1 \right)
\]

is not prime.

*Originally Problem 27 of 2017 Savin contest.*

*We received 4 submissions which were correct and complete. We present the solution by the Missouri State University Problem Solving Group.*
The statement is false. If \( n = 6 \), we have

\[
\frac{1}{3}(4^{4n+1} + 4^{4n+3} + 1) = 6380099472108203,
\]

which is prime. (Mathematica claims that \( n = 861 \) and \( n = 5304 \) also yield prime values).

However, it is true that if \( n \not\equiv 0 \mod 3 \), then \( (4^{4n+1} + 4^{4n+3} + 1)/3 \) is never prime.

If \( n \equiv 1 \mod 3 \), then \( n = 3k + 1, k \in \mathbb{Z} \) and

\[
4^{4n+1} + 4^{4n+3} + 1 = 4^{12k+5} + 4^{12k+7} + 1
\]

\[
= 16 \cdot 64^{4k+1} + 4 \cdot 64^{4k+2} + 1
\]

\[
\equiv 2 \cdot 1 + 4 \cdot 1 + 1 \mod 7
\]

\[
\equiv 0 \mod 7
\]

and

\[
4^{4n+1} + 4^{4n+3} + 1 \geq 4 + 4^3 + 1 = 69 > 7,
\]

so 7 is a non-trivial factor of \( (4^{4n+1} + 4^{4n+3} + 1)/3 \).

If \( n \equiv 2 \mod 3 \), then \( n = 3k + 2, k \in \mathbb{Z} \) and

\[
4^{4n+1} + 4^{4n+3} + 1 = 4^{12k+9} + 4^{12k+11} + 1
\]

\[
= 64^{4k+3} + 16 \cdot 64^{4k+3} + 1
\]

\[
\equiv 1 + 7 \cdot 1 + 1 \mod 9
\]

\[
\equiv 0 \mod 9
\]

and

\[
4^{4n+1} + 4^{4n+3} + 1 \geq 69 > 9,
\]

so 9 is a non-trivial factor of \( (4^{4n+1} + 4^{4n+3} + 1) \) and hence 3 is a non-trivial factor of \( (4^{4n+1} + 4^{4n+3} + 1)/3 \).

**MA29.** Find all positive integers \( n \) satisfying the following condition: numbers 1, 2, 3, \ldots, 2n can be split into pairs so that if numbers in each pair are added and all the sums are multiplied together, the result is a perfect square.

*Originally Problem 2 of Fall Junior A-level of XL Tournament of Towns 2017.*

We received 3 submissions, all of which were correct and complete. We present the solution by the Missouri State University Problem Solving Group, modified by the editor.

We claim that \( n \) satisfies the condition if \( n > 1 \).

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We first observe that \( n = 1 \) fails the condition. For \( n = 1 \) the only pairing is \( \{1, 2\} \), the sum of which is the non-perfect square 3.

There are two cases:

1. \( n = 2k \) where \( k \geq 1 \). By pairing \( i \) with \( 2n + 1 - i \) for \( i = 1, 2, \ldots, n \) gives a product of \( ((2n + 1)^k)^2 \).

2. \( n = 2k + 1 \) where \( k \geq 1 \). When \( k \geq 1 \), we pair 1 and 5, 2 and 4, 3 and 6, and \( 6 + i \) with \( 2n + 1 - i \) for \( i = 1, 2, \ldots, n - 3 = 2k - 2 \). The product is then

\[
(1 + 5)(2 + 4)(3 + 6)(2n + 7)^{2k-2} = (18(2n + 7)^{k-1})^2.
\]

**MA30.** Consider the two marked angles on a grid of equilateral triangles.

![Diagram](image)

Prove that these angles are equal.

*Originally Problem 18 of 2017 Savin contest.*

We received 6 solutions, all of which were correct. We present the solution of Missouri State University Problem Solving Group, modified by the editor.

Let the side lengths of the equilateral triangles be 1.

**Method I.** Consider the figure below.

![Diagram](image)

Let \( \alpha = m(\angle ACB) \) and \( \beta = m(\angle AED) \). Since \( AB = 2 \) and \( BC = 5 \), the Law of Cosines gives

\[
AC = \sqrt{2^2 + 5^2 + 2 \cdot 5} = \sqrt{39}.
\]

Applying the Law of Cosines again

\[
\cos \alpha = \frac{\sqrt{39^2 + 5^2 - 2^2}}{2 \cdot 5 \sqrt{39}} = \frac{\sqrt{12}}{13}.
\]
Similarly, $DE = \sqrt{3}$ and applying the Law of Cosines to $\triangle AFE$ we have

$$AE = \sqrt{1^2 + 3^2 + 1 \cdot 3} = \sqrt{13}.$$ 

One more use of the Law of Cosines gives

$$\cos \beta = \frac{\sqrt{13^2} + \sqrt{3^2} - 2^2}{2 \sqrt{3} \sqrt{39}} = \frac{12}{13},$$

so the angles in question are congruent.

**Method II.** Consider the figure below.

Triangle $ABD$ in this figure is congruent to triangle $DAE$ in the figure in Method I. Thus, we wish to show that $\angle ACB \cong \angle ADB$. The point marked $O$ is equidistant from each of $A, B, C, D$ (it lies on the intersection of the perpendicular bisectors of $AD, AB, and BC$). Therefore, these points lie on a circle centered at $O$. Since $\angle ACB$ and $\angle ADB$ are subtended by the same arc, they must be congruent.