SOLUIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4421. Proposed by Peter Y. Woo. (Correction.)

Show that the area of the largest $30^\circ - 60^\circ - 90^\circ$ triangle that fits inside the unit square is greater than $1/3$.

We originally published a duplicate of problem 4372 under this number. We received 3 more correct submissions to 4372 by Andrea Fanchini, Oliver Geupel and Christóbal Sánchez-Rubio (done independently).

For this problem, we received 14 submissions, 12 of which were correct. We present the solution by Richard Hess.

Consider a $30^\circ - 60^\circ - 90^\circ$ triangle positioned in the unit square as shown below.

Let $B = (1, y)$. Then $AB = \sqrt{1 + y^2}$ and $BC = \sqrt{(1 + y^2)/3}$. From this we find that the $y$-coordinate of $C$ is $1/\sqrt{3} + y$. The triangle will fit inside the square if $1/\sqrt{3} + y \leq 1$. If the $y$-coordinate of $C$ is equal to 1, then $y = 1 - 1/\sqrt{3}$ and the area of the triangle is

$$\frac{1}{2}BC \cdot AB = \frac{1 + y^2}{\sqrt{12}} = \frac{7\sqrt{3}}{18} - \frac{1}{3} = 0.3402 \cdots > \frac{1}{3}.$$


Let $ABC$ be a scalene triangle with incenter $I$ and nine-point center $N$. Find $\angle A$ given that $A, N$ and $I$ are collinear.

We present two of the 15 solutions we received, all of which were correct.

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We shall see that $\angle A = 60^\circ$. First observe that for any point $N$ on the bisector of $\angle BAC$ and for points $C'$ between $A$ and $B$ and $B'$ between $A$ and $C$, if $NC' = NB'$ then either $AC' = AB'$ or $AC'NB'$ is a cyclic quadrangle. Here we are given that $N$ is the center of the nine-point circle of $\triangle ABC$, and that circle contains the midpoints $C'$ of side $AB$ and $B'$ of side $AC$; that is, $NB' = NC'$. We are also given that $N$ is on the bisector of $\angle BAC$ (because it lies on the line $AI$) and, because the triangle is scalene, we have $AB' \neq AC'$. It follows that $AC'NB'$ is a cyclic quadrangle. The midpoint $A'$ of $BC$ is also on the nine-point circle. We also have that $\angle B'A'C' = \angle A$ (corresponding sides are parallel). Because the angle at the center equals twice the angle on the circumference, $\angle B'NC' = 2\angle B'A'C' = 2\angle A$, whence in the isosceles triangle $C'NB'$ we have

$$\angle C'B'N = 90^\circ - \angle A.$$

Finally in the circle $AC'NB'$

$$\angle A/2 = \angle C'AN = \angle C'B'N = 90^\circ - \angle A,$$

so that $\angle A = 60^\circ$, as claimed.

Solution 2, by Ivko Dimitrić.

This solution uses barycentric coordinates with respect to triangle $ABC$ and presupposes knowledge of the homogeneous barycentric coordinates of the nine-point center, namely

$$N = (a \cos(B - C) : b \cos(C - A) : c \cos(A - B)),$$

where $A, B, C$ are the angle measures at the corresponding vertices. [This follows from the homogeneous coordinates of $O = (\sin 2A : \sin 2B : \sin 2C)$, those of the centroid $G = (1 : 1 : 1)$, and the fact that $ON : NG = 3 : (-1)$; see Paul Yiu, Introduction to Triangle Geometry, p. 28. $N$ is the point $X(5)$ in Clark Kimberling’s Encyclopedia of Triangle Centers.]

The line through $A = (1 : 0 : 0)$ and $I = (a : b : c)$ has an equation $cy - bz = 0$. If $A, I$ and $N$ are collinear, the coordinates of $N$ satisfy the above equation of the line $AI$, so that

$$cb \cos(C - A) = bc \cos(A - B),$$

from where we have $(A - B) = \pm(C - A)$. Finally, since the triangle is scalene it follows that

$$A - B = C - A \quad \implies \quad B + C = 2A \quad \implies \quad 180^\circ = A + B + C = 3A,$$

which gives $\angle A = 60^\circ$.

Editor’s comments. This problem has made several appearances in Crux; see, for example, the article “Recurring Crux Configurations 3: Triangles Whose Angles...
4423. Proposed by Mihaela Berindeanu.

Let \( f : \mathbb{R} \to \mathbb{R} \) be a twice differential function such that
\[
f(x) + f''(x) = -x \cdot c^c \cdot f'(x)
\]
for all real values of \( x \) and an arbitrary constant \( c \). Find \( \lim_{x \to 0} x \cdot f(x) \).

We received 2 submissions, both correct. We present the solution by Michel Bataille.

Being twice differentiable, \( f \) certainly is continuous on \( \mathbb{R} \); in particular, we have
\[
\lim_{x \to 0} f(x) = f(0).
\]
It follows that
\[
\lim_{x \to 0} x \cdot f(x) = 0.
\]

4424. Proposed by Marius Drăgan and Neculai Stanciu.

Let \( k \in \mathbb{N} \) such that \( 9k + 9, 9k + 10 \) and \( 9k + 13 \) are not perfect squares. Prove that
\[
\left\lfloor \sqrt{k + x} + \sqrt{k + x + 1} + \sqrt{k + x + 2} \right\rfloor = \left\lfloor \sqrt{9k + 7} \right\rfloor
\]
for all \( x \in [0, 7/9] \), where \( \lfloor a \rfloor \) denotes the integer part of number \( a \).

We received 7 submissions, all correct. We present the solution by Oliver Geupel.

Let \( f(x) = \sqrt{k + x} + \sqrt{k + x + 1} + \sqrt{k + x + 2} \) for \( x \in [0, 7/9] \). By the AM-GM inequality, we have
\[
\left( \sqrt{k + 1} + \sqrt{k + 2} \right)^6 - (9k + 7)^3 \geq 36k(k+1)(k+2) - (9k+7)^3
\]
\[
= 486k^2 + 135k - 343 > 0,
\]
so
\[
f(0) = \sqrt{k} + \sqrt{k+1} + \sqrt{k+2} > \sqrt{9k+7}
\] (1)

Since the square root function is concave, we have by Jensen’s inequality that
\[
f\left( \frac{7}{9} \right) = \sqrt{k + \frac{7}{9}} + \sqrt{k + \frac{16}{9}} + \sqrt{k + \frac{25}{9}}
\]
\[
\leq \sqrt{\frac{(9k+7) + (9k+16) + (9k+25)}{3}}
\]
\[
= \sqrt{9k+16}.
\]

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Because $f(x)$ is an increasing function, it follows from (1) that
\[
\sqrt{9k+7} < f(0) \leq f\left(\frac{7}{9}\right) < \sqrt{9k+16}. \tag{2}
\]

It is well known that the quadratic residues modulo 9 are 0, 1, 4, and 7. Thus, the numbers $9k+8$, $9k+11$, $9k+12$, $9k+14$, and $9k+15$ are not perfect squares. Moreover, by assumptions, $9k+9$, $9k+10$ and $9k+13$ are not perfect squares either. Hence, if there exists $m \in \mathbb{N}$ such that
\[
\left[\sqrt{9k+7}\right] < m < f(x) < \left[\sqrt{9k+16}\right],
\]
then by (2) we get
\[
9k+7 < m^2 < (f(x))^2 < 9k+16,
\]
a contradiction. Hence $[f(x)] = \left[\sqrt{9k+7}\right]$ for all $x \in [0,7/9]$, completing the proof.

4425. Proposed by Nguyen Viet Hung.

Prove the following identities
\[(a) \quad \tan^3 \theta + \tan^3(\theta - 60^\circ) + \tan^3(\theta + 60^\circ) = 27 \tan^3 3\theta + 24 \tan 3\theta, \]
\[(b) \quad \frac{1}{1 + \tan \theta} + \frac{1}{1 + \tan(\theta - 60^\circ)} + \frac{1}{1 + \tan(\theta + 60^\circ)} = \frac{3 \tan 3\theta}{\tan 3\theta - 1}. \]

We received 13 submissions, all of which were correct, and we feature the solution by Aram Tangboonduangjit.

From the triple-angle identity,
\[
\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta},
\]
we have
\[
\tan^3 \theta - (3 \tan 3\theta) \tan^2 \theta - 3 \tan \theta + \tan 3\theta = 0.
\]
This implies that $x = \tan \theta$ is a zero of the polynomial $P(x)$ defined by
\[
P(x) = x^3 - (3 \tan 3\theta)x^2 - 3x + \tan 3\theta,
\]
where $\theta$ is a real number for which $\tan \theta$ and $\tan 3\theta$ are both defined. But since
\[
\tan 3(\theta \pm 60^\circ) = \tan(3\theta \pm 180^\circ) = \tan 3\theta,
\]
it follows that $\tan(\theta - 60^\circ)$ and $\tan(\theta + 60^\circ)$ are zeros of $P(x)$ as well. Because they come from translates of a strictly increasing function, the three numbers $\tan \theta, \tan(\theta \pm 60^\circ)$ are distinct; therefore, they compose all the zeros of the polynomial $P(x)$. We deduce that $P(x)$ factors as
\[
P(x) = (x - p)(x - q)(x - r),
\]
where $p, q, r$ are distinct.
where \( p = \tan \theta \), \( q = \tan(\theta - 60^\circ) \), and \( r = \tan(\theta + 60^\circ) \). Then, comparing the coefficients of the polynomial \( P(x) \) yields

\[
p + q + r = 3 \tan 3\theta, \quad pq + qr + rp = -3, \quad \text{and} \quad pqr = -\tan 3\theta.
\]

For part (a), we have

\[
p^3 + q^3 + r^3 = (p + q + r) ((p + q + r)^2 - 3(pq + qr + rp)) + 3pqr
= 3 \tan 3\theta(9 \tan^2 3\theta - 3(-3)) - 3 \tan 3\theta
= 27 \tan^3 \theta + 24 \tan 3\theta,
\]
as desired.

For part (b), differentiating the polynomial \( P(x) \) with respect to \( x \) on both sides, we obtain \( P'(x) = 3x^2 - (6 \tan 3\theta)x - 3 \) and, from the factored form of \( P(x) \),

\[
\frac{P'(x)}{P(x)} = \frac{1}{x - p} + \frac{1}{x - q} + \frac{1}{x - r}.
\]

Hence,

\[
\frac{1}{1 + p} + \frac{1}{1 + q} + \frac{1}{1 + r} = -\frac{P'(-1)}{P(-1)}
= \frac{3 + 6 \tan 3\theta - 3}{-1 - 3 \tan 3\theta + 3 + \tan 3\theta} = \frac{3 \tan 3\theta}{\tan 3\theta - 1},
\]
as desired.

4426. \textit{Proposed by Michel Bataille.}

Let distinct points \( A, B, C \) on a rectangular hyperbola \( H \) be such that \( \angle BAC = 90^\circ \). A point \( M \) of \( H \), other than \( A, B, C \), is called \textit{good} if the triangles \( MAB \) and \( MAC \) have the same circumradius. Show that either infinitely many \( M \) are good or a unique \( M \) is good. Characterize the triangle \( ABC \) in the former case and find \( M \) and the common circumradius in the latter one.

The two submissions we received were both correct; we will combine them into the featured solution. More precisely, the solution from Walther Janous generalized the problem, but because it relied on computer calculations, we will modify the proposer’s solution to incorporate Janous’ generalization.

Choosing the asymptotes of \( H \) as the coordinate axes, we take the equation of \( H \) to be \( xy = 1 \). We begin with \( A, B, C, M \) four arbitrarily chosen points of \( H \), and let \( a, b, c, m \) denote, respectively, their abscissas. At the end we will investigate what happens when \( \angle BAC \) is a right angle (as the proposer intended). From the law of Sines, we see that the triangles \( MAB \) and \( MAC \) have the same circumradius if and only if \( \sin(\angle MBA) = \sin(\angle MCA) \); that is, if and only if \( \cos^2(\angle MBA) = \cos^2(\angle MCA) \), which itself is equivalent to

\[
\left( \frac{MB^2 + AB^2 - MA^2}{2MB \cdot AB} \right)^2 = \left( \frac{MC^2 + AC^2 - MA^2}{2MC \cdot AC} \right)^2 \quad (1)
\]

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By easy calculations, we find

\[ MB^2 + AB^2 - MA^2 = (m - b)^2 + \left( \frac{1}{m} - \frac{1}{b} \right)^2 + (b - a)^2 + \left( \frac{1}{b} - \frac{1}{a} \right)^2 - (m - a)^2 - \left( \frac{1}{m} - \frac{1}{a} \right)^2 \]

\[ = 2(b - a)(b - m) \left( 1 + \frac{1}{ab^2m} \right) \]

\[ = 2(b - a)(b - m) \left( \frac{ab^2m + 1}{ab^2m} \right), \]

and

\[ MB^2 \cdot AB^2 = \left( (b - m)^2 + \left( \frac{1}{b} - \frac{1}{m} \right)^2 \right) \left( (b - a)^2 + \left( \frac{1}{b} - \frac{1}{a} \right)^2 \right) \]

\[ = (b - a)^2(b - m)^2 \left( 1 + \frac{1}{b^2m^2} \right)^2 \left( 1 + \frac{1}{a^2b^2} \right) \]

\[ = (b - a)^2(b - m)^2 \left( \frac{b^2m^2 + 1}{b^2m^2} \right) \left( \frac{a^2b^2 + 1}{a^2b^2} \right). \]

Replacing \( b \) by \( c \), we obtain \( MC^2 + AC^2 - MA^2 \) and \( MC^2 \cdot AC^2 \), and then (1), written as

\[ \left( \frac{MB^2 + AB^2 - MA^2}{MC^2 + AC^2 - MA^2} \right)^2 = \frac{MB^2 \cdot AB^2}{MC^2 \cdot AC^2}, \]

becomes

\[ \frac{(ab^2m + 1)^2}{(ac^2m + 1)^2} = \frac{(b^2m^2 + 1)(a^2b^2 + 1)}{(c^2m^2 + 1)(a^2c^2 + 1)}, \]

which, after expanding and arranging, rewrites as

\[ 0 = a^2(b^2 - c^2) - 2am(b^2 - c^2) + m^2(b^2 - c^2) - a^4b^2c^2m^2(b^2 - c^2) - a^2b^2c^2m^4(b^2 - c^2) \]

\[ - (b^2 - c^2) \left( (a^2 - 2am + m^2) - a^2b^2c^2m^2(a^2 - 2am + m^2) \right), \]

and finally,

\[ (b - c)(b + c)(a - m)^2(abcm + 1)(abcm - 1) = 0. \]  \hspace{1cm} (2)

Because we assume that \( A, B, C, M \) are distinct points, neither \( b - c \) nor \( a - m \) can be zero, which leaves us with two cases.

- **Case 1.** When \( b + c = 0 \), equation (2) has infinitely many solutions \( m \) with corresponding good points \( M \). The triangles \( ABC \) are given by a arbitrary points \( A \) and \( B \) on \( \mathcal{H} \) and point \( C \) the reflection of \( B \) in the origin.

- **Case 2.** For \( b + c \neq 0 \), (2) reduces to \( (abcm + 1)(abcm - 1) = 0 \). It follows that there will generally be exactly two good points, namely

\[ \left( \frac{1}{abc}, abc \right) \quad \text{and} \quad \left( -\frac{1}{abc}, -abc \right). \]
Of course, should any of \(a, b,\) or \(c\) equal \(\pm \frac{1}{abc}\), there will be only one candidate for a good point. The computer tells us that when \(m = \pm \frac{1}{abc}\) the common circumradius of triangles \(ABM\) and \(ACM\) is

\[
\frac{\sqrt{(a^2b^2 + 1)(b^2c^2 + 1)(c^2a^2 + 1)}}{2abc}.
\]

3

Solution to the original problem. We now assume that \(\angle BAC = 90^\circ\). Expressing that the dot product \(\overrightarrow{AB} \cdot \overrightarrow{AC}\) vanishes, we deduce that \(a, b, c\) satisfy

\[
a^2bc = -1.\]

After replacing \(abc\) by \(-\frac{1}{a}\) and multiplying the equation by \(-a^2\), equation (2) becomes

\[
(b - c)(b + c)(m + a)(m - a)^3 = 0.
\]

Thus, any \(M \neq A, B, C\) on \(\mathcal{H}\) is good if \(b + c = 0\); otherwise, only one point is good: the one with abscissa \(-a\). As with an arbitrary triangle, the former case occurs if and only if \(B\) and \(C\) are symmetrical about the centre \(O\) of \(\mathcal{H}\), that is, if and only if the hypotenuse is a diameter of \(\mathcal{H}\) (that is, if and only if the legs of \(\Delta ABC\) are parallel to the axes of \(\mathcal{H}\), as in figure 1). In the latter case, the only good point is the reflection of \(A\) in \(O\). It is known that if a triangle is inscribed in a rectangular hyperbola, then the centre \(O\) is on the nine-point circle of the triangle (see for example C.V. Durell, \textit{A Concise Geometrical Conics}, MacMillan, 1952, p. 72), hence the reflection in \(O\) of the orthocentre is on the circumcircle of the triangle. Here, the orthocentre of \(ABC\) being \(A\), it follows that the only good point is on the circumcircle of \(ABC\) and therefore the common circumradius of \(\Delta MAB\) and \(\Delta MAC\) is the circumradius of \(\Delta ABC\) (see figure 2). Because the circumradius of a right triangle equals half the hypotenuse \(BC\), equation (3) reduces to

\[
\frac{|b - c|}{2bc} \sqrt{b^2c^2 + 1}
\]

when \(a^2bc = -1\) (and, therefore, \(a^2b^2 = \frac{b}{c}, a^2c^2 = \frac{c}{b},\) and \(-a^2 = \frac{1}{bc}\)).

![Figure 1](image-url)
Proposed by Max A. Alekseyev.

Prove that the equation
\[ u^8 + v^9 + w^{14} + x^{15} + y^{16} = z^8 \]
has infinitely many solutions in positive integers with \( \gcd(u, v, w, x, y, z) = 1 \).

There was only one solution, by the proposer. However, the problem remains open in case a different family of solutions can be found.

Let \( z = s + t \) and \( u = |s - t| \) for integers \( s \) and \( t \). Then
\[ z^8 - u^8 = 2^4st^7 + 2^4 \cdot 7s^3t^5 + 2^4 \cdot 7s^5t^3 + 2^4s^7t. \]

We make the choice \( s = 2^a \) and \( t = 7^b \) for suitable choices of \( a \) and \( b \) to arrange that
\[
\begin{align*}
v^9 &= 2^4st^7 = 2^{4+a} \cdot 7^b; \\
w^{14} &= 2^4 \cdot 7s^3t^5 = 2^{4+3a} \cdot 7^{1+3b}; \\
x^{15} &= 2^4s^7t = 2^{4+7a} \cdot 7^b; \\
y^{16} &= 2^4 \cdot 7s^5t^3 = 2^{4+5a} \cdot 7^{1+3b}. \\
\end{align*}
\]

To obtain integer values of the variables, we require that \( a \) be congruent to 5 (mod 9), 8 (mod 14), 8 (mod 15), 12 (mod 16), and that \( b \) be congruent to 0 (mod 9), 11 (mod 14), 0 (mod 15) and 5 (mod 16). Therefore, modulo 5040,
\[ a \equiv 428 \quad \text{and} \quad b \equiv 1845. \]

For arbitrary nonnegative integers \( m \) and \( n \), we let
\[ a = 5040m + 428 \quad \text{and} \quad b = 5040n + 1845. \]
This yields the family of solutions given by

\[
\begin{align*}
\begin{array}{l}
u = 2^{5040m+48} \cdot 7^{3920n+1435} \\
w = 2^{1080m+92} \cdot 7^{1800n+659} \\
x = 2^{2352m+200} \cdot 7^{336n+123} \\
y = 2^{1575m+134} \cdot 7^{945n+346} \\
z = 2^{5040m+428} + 7^{5040n+1845}.
\end{array}
\end{align*}
\]

Any common divisor of \(u\) and \(z\) must divide \(u \pm z\) as well, and so is a common divisor of a power of 2 and a power of 7. It follows that the greatest common divisor of \(u\) and \(z\), and so of all the variables, is 1.


Let \(ABC\) be a triangle and let \(O\) be an arbitrary point in the same plane. Let \(A', B'\) and \(C'\) be the reflections of \(A, B\) and \(C\) in \(O\). Prove that

\[
\frac{AB' \cdot BC'}{AB \cdot BC'} + \frac{BC' \cdot C'A}{BC \cdot CA} + \frac{CA' \cdot A'B}{CA \cdot AB} \geq 1.
\]

There were 4 correct solutions, all of which used complex numbers and the remaining one used vectors. One additional solution was submitted, but relied on an unproven inequality that seems as difficult as the proposed problem. We present two solutions.

Solution 1, by Michel Bataille and Sushanth Sathish Kumar (independently).

Locate the points \(O, A, B, C, A', B', C'\), respectively, in the complex plane at 0, \(a, b, c, -a, -b, -c\). It is required to show that

\[
\left| \frac{(a+b)(b+c)}{(a-b)(b-c)} \right| + \left| \frac{(b+c)(c+a)}{(b-c)(c-a)} \right| + \left| \frac{(c+a)(a+b)}{(c-a)(a-b)} \right| \geq 1.
\]

Observe that

\[
\begin{align*}
&|(a+b)(b+c)(c-a)| + |(b+c)(c+a)(a-b)| + |(c+a)(a+b)(b-c)| \\
\geq & |(a+b)(b+c)(c-a) + (b+c)(c+a)(a-b) + (c+a)(a+b)(b-c)| \\
= & |(ab+bc+ca)(c-a) + (a-b)(b-c) + b^2(c-a) + c^2(a-b) + a^2(b-c)| \\
= & |0 - (a-b)(b-c)(c-a)| \\
= & |(a-b)(b-c)(c-a)|.
\end{align*}
\]

Dividing by \(|(a-b)(b-c)(c-a)|\) yields the desired inequality.

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Solution 2, by the proposer.

Locate the points in the complex plane as in the previous solution. Let

\[ u = \frac{b+c}{b-c}; \quad v = \frac{c+a}{c-a}; \quad w = \frac{a+b}{a-b}. \]

The homogeneous system

\[
\begin{align*}
(1-u)y + (1+u)z &= 0 \\
(1+v)x + (1-v)z &= 0 \\
(1-w)x + (1+w)y &= 0
\end{align*}
\]

has a nontrivial solution \((x,y,z) = (a,b,c)\). Therefore, the determinant of its coefficients, \(2(1 + (wu + uv + vw))\), is zero. Hence

\[
\begin{vmatrix}
(a+b)(b+c) \\
(a-b)(b-c)
\end{vmatrix} + \begin{vmatrix}
(b+c)(c+a) \\
(b-c)(c-a)
\end{vmatrix} + \begin{vmatrix}
(c+a)(a+b) \\
(c-a)(a-b)
\end{vmatrix} = |wu| + |uv| + |vw|
\]

\[
\geq |wu + uv + vw| = |1| = 1.
\]

4429. Proposed by Lorian Saceanu.

Let \(a, b, c\) be positive real numbers. Prove that

\[
\sqrt{\frac{a^2 + b^2 + c^2}{2(ab + bc + ca)}} \geq \frac{a + b + c}{\sqrt{a(b+c) + b(a+c) + c(a+b)}}
\]

We received 4 solutions, of which one was incorrect and another is incomplete. We present the proof by Vasile Cirtoaje, modified and enhanced by the editor.

The proposed inequality is equivalent to

\[
\sqrt{a(b+c) + b(c+a) + c(a+b)} \geq (a+b+c)\sqrt{\frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}}
\]

or

\[
\left(\sqrt{a(b+c) + b(c+a) + c(a+b)}\right)^2 \geq \frac{2(a+b+c)^2(ab + bc + ca)}{a^2 + b^2 + c^2}
\]

By AM-GM inequality, we have

\[
\begin{align*}
b + c &\geq 2\sqrt{bc} \implies \\
ab + ac &\geq 2a\sqrt{bc} \implies \\
(a+b)(a+c) &\geq a^2 + 2a\sqrt{bc} + bc \implies \\
\sqrt{(a+b)(a+c)} &\geq a + \sqrt{bc}.
\end{align*}
\]
Similarly, $\sqrt{(b+c)(b+a)} \geq b + \sqrt{ca}$ and $\sqrt{(c+a)(c+b)} \geq c + \sqrt{ab}$. Hence,

$$\left( \sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \right)^2$$

$$= 2(ab + bc + ca) + 2 \sum_{\text{cyc}} \sqrt{bc(a+b)(a+c)}$$

$$\geq 2(ab + bc + ca) + 2 \sum_{\text{cyc}} \sqrt{bc(a + \sqrt{bc})}$$

$$= 4(ab + bc + ca) + 2\sqrt{abc} \sqrt{a + \sqrt{b} + \sqrt{c}}.$$ \hfill (2)

From (1) and (2), we see that it suffices to show that

$$2(ab + bc + ca) + \sqrt{abc} \sqrt{a + \sqrt{b} + \sqrt{c}} \geq \frac{(a + b + c)^2(ab + bc + ca)}{a^2 + b^2 + c^2},$$

which is equivalent to

$$\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c})(a^2 + b^2 + c^2) \geq (ab + bc + ca)(2(ab + bc + ca) - a^2 - b^2 - c^2).$$ \hfill (3)

Since

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$$

$$= ((\sqrt{a} + \sqrt{b})^2 - c)(-\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$$

$$= ((\sqrt{a} + \sqrt{b})^2 - c)(\sqrt{a} - \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$$

$$= 2(a + b)c - c^2 - (a - b)^2$$

$$= 2(ab + bc + ca) - a^2 - b^2 - c^2,$$

we see from (3) that it now suffices to prove that

$$\sqrt{abc}(a^2 + b^2 + c^2) \geq (ab + bc + ca)(\sqrt{a} + \sqrt{b} - \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c}),$$

which follows from the trivial fact that

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

and the known inequality that

$$\sqrt{abc} \geq (\sqrt{a} + \sqrt{b} - \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c}).$$ \hfill (4)

Finally, equality holds if $a = b = c$.

**Editor’s Comment.** The proposer in private communication pointed out that (4) is known as Schur’s inequality of first degree and could be proved as follows. Assume without loss of generality that $a \leq b \leq c$ and consider $a, b$ and $c$ as the sides of a triangle. Then using the transformation

$$x = \frac{1}{2}(b + c - a), \quad y = \frac{1}{2}(c + a - b), \quad z = \frac{1}{2}(a + b - c),$$

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and applying the inequality

$$(x + y)(y + z)(z + x) \geq 8xyz,$$

(which is an immediate consequence of the AM-GM inequality), we obtain

$$abc \geq (a + b - c)(b + c - a)(c + a - b),$$

so (4) follows.


Let $s \geq \frac{28}{3}$ be a fixed real number. Consider the real numbers $a, b, c$ and $d$ such that

$$a + b + c + d = 4 \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 = s. $$

Find the maximum value of the product $abcd$.

We received 7 solutions, 3 of which were correct. We present the solution by Walther Janous, modified by the editor.

We shall use Lagrange multipliers to determine the desired maximum.

At the boundary of the set

$$B = \{(a, b, c, d) : a + b + c + d = 4 \text{ and } a^2 + b^2 + c^2 + d^2 = s\},$$

at least one of the variables $a, b, c, d$ is zero, whence $abcd = 0$.

We now set

$$F = abcd - \lambda(a + b + c + d - 4) - \mu(a^2 + b^2 + c^2 + d^2 - s),$$

From $\frac{d}{da}F = 0$, it follows that

$$-2a \mu + bcd - \lambda = 0.$$

Setting

$$f(t) = 2\mu t^2 + \lambda t,$$

we have

$$f(a) = abcd.$$

Similarly, we get $f(b) = f(c) = f(d) = abcd$. Since $f(t)$ is quadratic in $t$, we infer that among the four numbers $a, b, c, d$, there are at most four values. We consider three cases.

Case 1: $a = b = c = d$. Here, $a = b = c = d = 1$, implying that

$$a^2 + b^2 + c^2 + d^2 = 4 < \frac{28}{3},$$
a contradiction.

Case 2: \( a = b = c = A \) and \( d = B \), with \( A \neq B \). Solving the system

\[
\begin{align*}
3A + B &= 4 \\
3A^2 + B^2 &= s
\end{align*}
\]

gives

\[
\begin{align*}
A &= \frac{\sqrt{3} \left( 2\sqrt{3} - \sqrt{s - 4} \right)}{6} \\
B &= \frac{2 + \sqrt{3} \cdot \sqrt{s - 4}}{2}
\end{align*}
\]

or

\[
\begin{align*}
A &= \frac{\sqrt{3} \left( 2\sqrt{3} + \sqrt{s - 4} \right)}{6} \\
B &= \frac{2 - \sqrt{3} \cdot \sqrt{s - 4}}{2}
\end{align*}
\]

We obtain

\[
abcd = A^3B = \frac{-16\sqrt{3}(s - 4)^{3/2} - 3s^2 - 48s + 384}{144}.
\]

Case 3: \( a = b = A \) and \( c = d = B \), with \( A \neq B \). Solving the system

\[
\begin{align*}
2A + 2B &= 4 \\
2A^2 + 2B^2 &= s
\end{align*}
\]

gives

\[
\begin{align*}
A &= \frac{2 - \sqrt{s - 4}}{2} \\
B &= \frac{2 + \sqrt{s - 4}}{2}
\end{align*}
\]

or

\[
\begin{align*}
A &= \frac{2 + \sqrt{s - 4}}{2} \\
B &= \frac{2 - \sqrt{s - 4}}{2}
\end{align*}
\]

This gives

\[
abcd = A^2B^2 = \left( \frac{s - 8}{4} \right)^2.
\]

It remains to determine which of the two obtained maxima is the greater one. Straightforward calculation shows that the maximum in Case 3 is greater than or equal to the maximum in Case 2 if and only if \( s \geq \frac{28}{3} \). Hence, the maximum for \( s \geq \frac{28}{3} \) is \( \left( \frac{s - 8}{4} \right)^2 \).