OLYMPIAD CORNER

No. 376

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by November 30, 2019.

OC446. Given the numbers 2, 3, \ldots, 2017 and the natural number \( n \leq 2014 \), Ivan and Peter play the following game: Ivan selects \( n \) numbers from the given ones, then Peter selects 2 numbers from the remaining numbers, then all the selected \( n + 2 \) numbers are ranked in value:

\[ a_1 < a_2 < \ldots < a_{n+2}. \]

If there exists \( i, \ 1 \leq i \leq n + 1 \) for which \( a_i \) divides \( a_{i+1} \), then Peter wins, otherwise Ivan wins. Find all \( n \) for which Ivan has a winning strategy.

OC447. Let \( m > 1 \) be an integer and let \( N = m^{2017} + 1 \). Positive numbers \( N, N - m, N - 2m, \ldots, m + 1, 1 \) are written in a row. At each step, the leftmost number and all of its divisors (if any) are erased. This process continues until all the numbers are erased. What are the numbers deleted at the last step?

OC448. Let \( x_1 \leq x_2 \leq \ldots \leq x_{2n-1} \) be real numbers whose arithmetic mean is equal to \( A \). Prove that

\[ 2 \sum_{i=1}^{2n-1} (x_i - A)^2 \geq \sum_{i=1}^{2n-1} (x_i - x_n)^2. \]

OC449. A sequence \((a_1, a_2, \ldots, a_k)\) consisting of pairwise distinct squares of an \( n \times n \) chessboard is called a cycle if \( k \geq 4 \) and the squares \( a_i \) and \( a_{i+1} \) have a common side for all \( i = 1, 2, \ldots, k \), where \( a_{k+1} = a_1 \). Subset \( X \) of this chessboard’s squares is mischievous if each cycle on it contains at least one square in \( X \). Determine all real numbers \( C \) with the following property: for each integer \( n \geq 2 \), on an \( n \times n \) chessboard there exists a mischievous subset consisting of at most \( Cn^2 \) squares.

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OC450. Find all pairs \((x, y)\) of real numbers satisfying the system of equations
\[
x \cdot \sqrt{1 - y^2} = \frac{1}{4} \left( \sqrt{3} + 1 \right),
\]
\[
y \cdot \sqrt{1 - x^2} = \frac{1}{4} \left( \sqrt{3} - 1 \right).
\]

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 30 novembre 2019.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d’avoir traduit les problèmes.

OC446. Soit les nombres 2, 3, …, 2017 et le nombre naturel \(n \leq 2014\). Ivan et Peter jouent au jeu suivant : Ivan sélectionne \(n\) nombres parmi ceux donnés, puis Peter sélectionne 2 nombres parmi ceux qui restent, puis tous les \(n + 2\) nombres sélectionnés sont ordonnés en ordre croissant :
\[
a_1 < a_2 < \ldots < a_{n+2}.
\]
S’il existe \(i, 1 \leq i \leq n+1\) pour lequel \(a_i\) divise \(a_{i+1}\), Peter gagne, sinon Ivan gagne. Trouvez toutes les valeurs de \(n\) pour lesquelles Ivan a une stratégie gagnante.

OC447. Soit \(m > 1\) un entier et soit \(N = m^{2017} + 1\). Les nombres positifs \(N, N - m, N - 2m, \ldots, m + 1, 1\) sont écrits sur une rangée. À chaque étape, le nombre le plus à gauche et tous ses diviseurs (s’il en possède) sont effacés. Le processus continue jusqu’à ce que tous les nombres soient effacés. Quels sont les nombres effacés à la dernière étape ?

OC448. Soit \(x_1 \leq x_2 \leq \ldots \leq x_{2n-1}\) des nombres réels dont la moyenne arithmétique est égale à \(A\). Prouvez que
\[
2 \sum_{i=1}^{2n-1} (x_i - A)^2 \geq \sum_{i=1}^{2n-1} (x_i - x_n)^2.
\]

OC449. Une suite \((a_1, a_2, \ldots, a_k)\) constituée de paires distinctes d’un échiquier \(n \times n\) est appelée un cycle si \(k \geq 4\) et les carrés \(a_i\) et \(a_{i+1}\) ont un côté commun pour
tout \( i = 1, 2, \ldots, k \), où \( a_{k+1} = a_1 \). Le sous-ensemble \( X \) des carrés de cet échiquier est mischievous si chaque cycle contient au moins un carré de \( X \). Trouvez tous les nombres réels \( C \) ayant la propriété suivante : pour tout entier \( n \geq 2 \) sur un échiquier \( n \times n \), il existe un mischievous sous-ensemble consistant en au plus \( Cn^2 \) carrés.

**OC450.** Trouvez toutes les paires \((x, y)\) de nombres réels qui satisfont au système d’équations

\[
x \cdot \sqrt{1 - y^2} = \frac{1}{4} (\sqrt{3} + 1),
\]

\[
y \cdot \sqrt{1 - x^2} = \frac{1}{4} (\sqrt{3} - 1).
\]
OLYMPIAD CORNER

SOLUTIONS


OC421. Mim has a deck of 52 cards, stacked in a pile with their backs facing up. Mim separates the small pile consisting of the seven cards on the top of the deck, turns it upside down, and places it at the bottom of the deck. All cards are again in one pile, but not all of them face down; the seven cards at the bottom do, in fact, face up. Mim repeats this move until all cards have their backs facing up again. In total, how many moves did Mim make?

*Originally Problem 3 from the 2017 Italy Math Olympiad, Final Round.*

*We received no solutions.*

OC422. A $2017 \times 2017$ table is filled with nonzero digits. Among the 4034 numbers whose decimal expansion is formed with the rows and columns of this table, read from left to right and from top to bottom, respectively, all but one are divisible by a prime number $p$, and the remaining number is not divisible by $p$. Find all possible values of $p$.

*Originally problem 5 from the 2017 Moscow Math Olympiad, Grade 11, Second Day, Final Round.*

*We received 1 submission. We present the solution by Oliver Geupel.*

We generalise the problem to an $M \times N$ table, where $M, N > 1$, and show that the possible values of $p$ are exactly 2 and 5.

To see that $p \in \{2, 5\}$ has the desired property, put the top-right table entry equal to 1 and all other entries equal to $p$.

Next, suppose $p \notin \{2, 5\}$ and let an $M \times N$ table be filled with nonzero digits, where $d_{i,j}$ is the digit in row $i$ and column $j$. Then, the numbers whose decimal expansions are formed with the rows and columns of the table are

$$r_i = \sum_{j=1}^{N} 10^{N-j}d_{i,j} \quad \text{and} \quad c_j = \sum_{i=1}^{M} 10^{M-i}d_{i,j},$$

respectively. Note that

$$\sum_{i=1}^{M} \sum_{j=1}^{N} 10^{M+N-i-j}d_{i,j} = \sum_{i=1}^{M} 10^{M-i}r_i = \sum_{j=1}^{N} 10^{N-j}c_j$$

(1)

and that the prime $p$ is not divisible by 10.

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If exactly one of the numbers in \( \{r_1, \ldots, r_M\} \cup \{c_1, \ldots, c_N\} \) were not divisible by \( p \), then exactly one of the numbers \( \sum_{i=1}^{M} 10^{M-i}r_i \) and \( \sum_{j=1}^{N} 10^{N-j}c_j \) were not divisible by \( p \). By (1), this is impossible. This proves that \( p \) fails to have the required property.

**OC423.** There are 100 gnomes with weight 1, 2, \ldots, 100 kg gathered on the left bank of the river. They cannot swim, but they have one boat with capacity 100 kg. Because of the current, it is hard to row back, so each gnome has enough power only for one passage from right side to left as oarsman. Can all gnomes get to the right bank?

*Originally problem 3 from the 2017 Russia Math Olympiad, Grade 9, Final Round.*

*We received 2 submissions. We present the solution by Oliver Geupel.*

The answer is No. Let us generalize the problem to \( N \) gnomes \( G_1, \ldots, G_N \) with mass 1, \ldots, \( N \) kilograms, respectively, and a boat with capacity \( N \) kilograms. We show that the passage is possible if and only if \( N \) is an odd number.

First assume \( N \) is odd, say, \( N = 2n + 1 \). A passage is:

- step \( k \) (\( 1 \leq k \leq n \)): transfer \( G_k \) and \( G_{2n+1-k} \) to right (R), then \( G_k \) to L
- step \( n + 1 \): transfer \( G_{2n+1} \) to R, then \( G_{2n} \) to L
- step \( n + 1 + k \) (\( 1 \leq k < n \)): transfer \( G_k \), \( G_{2n+1-k} \) to R, then \( G_{2n-k} \) to L
- step \( 2n + 1 \): transfer \( G_n \) and \( G_{n+1} \) to R.

Next, assume \( N \) is even. We prove by contradiction that there is no passage.

Suppose there exists a passage with \( t \) transfers to R. There are \( t - 1 \) transfers to L with total mass at least \( 1 + 2 + \cdots + (t - 1) \) kilograms. Hence, the total mass transferred to R is at least

\[
(1 + 2 + \cdots + N) + (1 + 2 + \cdots + (t - 1))
\]

kilograms and at most \( Nt \) kilograms by the given capacity. Thus,

\[
N(N + 1) + (t - 1)t \leq 2Nt,
\]

which we rewrite in the form

\[
(2N + 1 - 2t)^2 \leq 1.
\]

We obtain \( t \in \{N, N+1\} \), and in all transfers to R the boat is full. If \( t = N+1 \) then every gnome paddles twice to R, while in the case \( t = N \) the gnome \( G_N \) crosses the river only once. Since the boat is full in all transfers to R, \( G_{N-1} \) shares the boat with \( G_1 \) twice. Also \( G_{N-2} \) shares the boat with \( G_2 \) twice. Continuing this way, there remains no partner for \( G_{N/2} \). This is the desired contradiction.
OC424. Let \( n \) be a nonzero natural number, let \( a_1 < a_2 < \ldots < a_n \) be real numbers and let \( b_1, b_2, \ldots, b_n \) be real numbers. Prove that:

(a) if all the numbers \( b_i \) are positive, then there exists a polynomial \( f \) with real coefficients and having no real roots such that \( f(a_i) = b_i \) for \( i = 1, 2, \ldots, n \);

(b) there exists a polynomial \( f \) of degree at least 1 having all real roots and such that \( f(a_i) = b_i \) for \( i = 1, 2, \ldots, n \).

Originally problem 2 from the 2017 Russia Math Olympiad, Grade 12, Final Round.

We received 2 submissions. We present the solution by the Missouri State University Problem Solving Group.

(a) Let

\[
h_i(x) = \prod_{j=1}^{i-1} (x - a_j)^2 \prod_{j=i+1}^{n} (x - a_j)^2,
\]

\[
g_i(x) = \frac{h_i(x)}{h_i(a_i)}.
\]

Then

\[
g_i(a_i) = 1,
\]

\[
g_i(a_j) = 0 \text{ for } i \neq j
\]

\[
g_i(x) > 0 \text{ for } x \neq a_j.
\]

Take

\[
f(x) = \sum_{i=1}^{n} b_i g_i(x).
\]

Note that \( f(a_i) = b_i, i = 1, \ldots, n \), as desired.

Since \( b_i > 0, f(x) > 0 \) for \( x \neq a_i, i = 1, \ldots, n \) and \( f(a_i) = b_i > 0 \). Therefore \( f(x) > 0 \) for all \( x \) and hence \( f \) has no real roots.

(b) If \( b_i = 0 \) for all \( i \), let

\[
f(x) = \prod_{i=1}^{m} (x - a_i).
\]

Otherwise, if necessary, insert additional data points so that the non-zero \( y \)-coordinates alternate in sign making sure that all the data points with \( y \)-coordinate 0 lie between two data points whose \( y \)-coordinates have opposite signs. Next, if necessary, insert additional data points with \( y \)-coordinate 0, so that the number of such points between two points with non-zero \( y \)-coordinates is even. Denote the

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x-coordinates of the points with non-zero y-coordinate by \( a'_i, i = 1, \ldots, m \), with \( a'_1 < \ldots < a'_m \) and denote the number of data points with y-coordinate 0 whose x-coordinates lie between \( a'_i \) and \( a'_{i+1} \) by \( 2k_i \).

Let \( f \) be the interpolating polynomial through this augmented set of data points having minimal degree. There are \( m + \sum_{i=1}^{m-1} 2k_i \) data points, so

\[
\deg f \leq m - 1 + \sum_{i=1}^{m-1} 2k_i.
\]

Now for any polynomial, the sum of the multiplicities of its zeros between a point where the polynomial takes a positive value and one where it takes a negative value must be odd. In our case, for each interval \([a'_i, a'_{i+1})\), the sum of the multiplicities of the zeros in that interval must be at least \( 2k_i + 1 \). Therefore the sum of the multiplicities of the zeros of \( f \) must be at least

\[
\sum_{i=1}^{m-1} (2k_i + 1) = m - 1 + \sum_{i=1}^{m-1} 2k_i.
\]

Therefore

\[
\deg f = m - 1 + \sum_{i=1}^{m-1} 2k_i
\]

and all of its roots are real, as desired.

For example, suppose \( a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 6 \) and \( b_1 = 5, b_2 = 0, b_3 = 0, b_4 = 3 \). We could insert the data point \((1, -2)\) and obtain \( a'_1 = 0, a'_2 = 1, a'_3 = 6 \) and \( 2k_1 = 0, 2k_2 = 2 \). On the other hand, if we inserted the data point \((3, -4)\), we would need to add two more zeros, say at \( x = 1 \) and \( x = 5 \) to obtain \( a'_1 = 0, a'_2 = 3, a'_3 = 6 \) and \( 2k_1 = 2, 2k_2 = 2 \).

\textbf{OC425.} Consider a triangle \( ABC \) with \( \angle A < \angle C \). Point \( E \) is on the internal angle bisector of \( \angle B \) such that \( \angle EAB = \angle ACB \). Let \( D \) be a point on line \( BC \) such that \( B \in CD \) and \( BD = AB \). Prove that the midpoint \( M \) of the segment \( AC \) is on the line \( DE \).

\textit{Originally problem 4 from the 2017 Romania Math Olympiad, Grade 7, District Round.}

\textit{We received 7 submissions. We present the solution by Sushanth Sathish Kumar.}

Define \( T = BC \cap AE \). The key is to apply Menelaus’s theorem to triangle \( TCA \). We want to show that

\[
\frac{AM}{MC} \cdot \frac{TE}{EA} \cdot \frac{CD}{DT} = -1,
\]

where lengths are taken to be directed.
Clearly $AM/MC = 1$. By the angle bisector theorem, we have $TE/EA = BT/BA$. However, triangles $TBA$ and $ABC$ are similar since $\angle BAT = \angle BCA$ and $\angle B$ is shared. It follows that 

$$TE/EA = BA/BC = c/a.$$ 

To compute the last ratio, note that 

$$CD = CB + BD = a + c$$ 

and 

$$DT = DB + BT = c + BT.$$ 

But by the similar triangles found above, 

$$BT = BA^2/BC = c^2/a.$$ 

Hence, 

$$\frac{CD}{DT} = -\frac{a + c}{c + c^2/a} = -\frac{a}{c}.$$ 

Taking the product of the three ratios in question, we get $-1$, as desired.