Captivating audiences for more than 4000 years, juggling has been practiced in countries such as Ancient Egypt, Greece, China and the Polynesian Islands (to name a few) \cite{1}. In Medieval Europe, juggling was at best buffoonery (no help from the pejorative collective noun, “a neverthriving of jugglers”). At worst, it was considered a form of witchcraft \cite{2}! Today, jugglers comprise an active community the world over, a disproportionate number of whom possess an appreciation for mathematics. This curious connection between jugglers and mathematicians is brought to bear by icons like Ronald Graham, who has been the president of both the American Mathematical Society and the International Jugglers Association.

Mathematics aids in the communication and creation of juggling patterns. Its tools help us to distill a system for describing “most” juggling patterns, and to determine what makes one pattern unique from another. Modular arithmetic will be a key tool for us in this investigation (for a review of modular arithmetic, see Donald Rideout’s Problem Solving Vignettes in \textit{Crux} 45(3)). In this investigation, we will study only \textit{simple juggling patterns} — ones for which the following conditions hold:

(A) Throws are made on evenly spaced ‘beats’, a ball is thrown as soon as it is caught;

(B) At most one ball is thrown or caught on each beat;

(C) The throws made by the juggler are a finite repeating pattern, repeating forever (forwards and backwards in time).

Most often, left and right hand will alternate throwing. The number of beats a ball is airborne is called the \textit{height} of the throw; all heights are non-negative integers. If we chronologically list the heights of all the throws made on each beat in a simple juggling pattern, we obtain a bi-infinite sequence of the form

\[ \ldots, t_{n-1}, t_n, t_1, t_2, \ldots, t_{n-1}, t_n, t_1, t_2, \ldots, t_{n-1}, t_n, t_1, t_2, \ldots \]  

\text{(1)}
We call any bi-infinite sequence of the form in (1) a *simple pattern*, and say that it is *generated* by the tuple \((t_1, t_2, \ldots, t_n)\).

One of the first 3-ball juggling patterns a new juggler learns is the 3-ball cascade. Every throw is a throw of height 3, and so this pattern is generated by the singleton tuple \((3)\). The paths that the 3 balls trace through time as they are thrown is visualized in the juggling diagram above. For example, the path of one of the balls, as it is thrown from left hand to right hand to left again, is shown in bold.

![Figure 1: Juggling diagram of (3)](image)

The tuple \((1,5,0)\) also generates a juggling pattern, whose sequence of throw heights is:

\[ \ldots, 1, 5, 0, 1, 5, 0, 1, 5, 0, 1, 5, \ldots \]

We can verify that \((1,5,0)\) generates a simple juggling pattern by drawing its corresponding juggling diagram.

![Figure 2: Juggling diagram of (1,5,0)](image)

Demonstration of 150 pattern.
The dark dots represent beats, and the arcs represent ball throws (left hand, right hand, left hand, right hand, . . . ). A juggling diagram gives a simple way of checking that a tuple generates a simple juggling pattern; i.e. each dot must be incident to exactly 0 or 2 arcs in order for (B) to hold. The simple pattern generated by any tuple will satisfy (A) and (C), but not all simple patterns are simple juggling patterns. For example, (3, 2, 1) generates a pattern with the following juggling diagram.

![Juggling diagram of (3, 2, 1)](image)

Figure 3: Juggling diagram of (3, 2, 1)

The diagram makes it clear that several balls land on a single beat, violating assumption (B). We say a tuple is jugglable if it generates a simple juggling pattern.

In general, a suitably large juggling diagram can always be used to verify if a tuple is jugglable, but this can be unwieldy for longer tuples. Assuming that a throw of height $t_1$ is made on beat 1, the list of beats that a ball is caught in the simple juggling pattern generated by $(t_1, t_2, \ldots, t_n)$ is

$$
\ldots, t_{n-1} - 1, t_n, t_1 + 1, t_2 + 2, \ldots, t_n + n, t_1 + n + 1, \ldots
$$

(2)

The tuple $(t_1, t_2, \ldots, t_n)$ is jugglable if and only if every term in the bi-infinite sequence (2) is distinct.

Problem 1 Prove that a tuple $t = (t_1, t_2, \ldots, t_n)$ is jugglable if and only if $t_1 + 1, t_2 + 2, \ldots, t_n + n$ are all distinct modulo $n$.

Proof. Let $T_n$ be the sequence in (1), where $T_1 = t_1$. Suppose two balls are caught simultaneously, then there exist distinct integers $i, j$ such that $T_i + i = T_j + j$. By the division algorithm, find integers $q_i, q_j$ such that $i = q_i n + r_i$ and $j = q_j n + r_j$ where $1 \leq r_i, r_j \leq n$. Notice that $T_i = t_{r_i}$ and $T_j = t_{r_j}$. We have

$$
t_{r_i} + r_i \equiv t_{r_i} + i = T_i + i = T_j + j \equiv t_{r_j} + j \equiv t_{r_i} + r_j,
$$

where all equivalences above are modulo $n$. On the other hand, if $t_i + i \equiv t_j + j$ (mod $n$) for $1 \leq i \neq j \leq n$, then there is an integer $q$ such that $t_i + i = t_j + j + qn$. This implies $T_i + i = T_j + qn + j + qn$, and so the balls thrown on beats $i$ and $j + qn$ land on the same beat.

Two tuples that differ by a circular permutation generate the same pattern; for example $(1, 5, 0)$ and $(5, 0, 1)$. On the other hand, there is an elementary operation we can apply to a simple juggling pattern that can generate a new simple juggling...
pattern, namely by exchanging the beats that two different balls land on. Precisely, we say \( s = (s_1, s_2, \ldots, s_n) \) is a siteswap of \( t = (t_1, t_2, \ldots, t_n) \) if \( s = (t_2 + 1, t_1 - 1, t_3, \ldots, t_n) \). The ball thrown on the first beat in the pattern generated by \( s \), lands on the same beat as the ball thrown on the second beat in the pattern generated by \( t \). For example, \((4, 2, 3)\) is a siteswap of \((3, 3, 3)\). Applying a siteswap twice returns the original tuple.

**Problem 2** If \( s \) is a siteswap of \( t \), show that \( s \) is jugglable if and only if \( t \) is jugglable.

**Proof.** Notice that \( \{t_1 + 1, t_2 + 2, \ldots, t_n + n\} = \{s_1 + 1, s_2 + 2, \ldots, s_n + n\} \). The result follows by Problem 1.

Constant tuples are always jugglable, and so too are any siteswap and circular permutation of constant tuples. Furthermore, any jugglable tuple can be obtained from a constant tuple via siteswaps and circular permutations.

**Problem 3** Show that for any juggling tuple \( t \), successive siteswaps and circular permutations can be applied to \( t \) to obtain a constant tuple.

**Proof.** If \( t \) is constant, then we are done. Suppose \( t \) is not constant, and by applying circular permutations, assume without loss of generality that \( t_1 \) is a largest entry in \( t \) and \( t_1 > t_2 \). Note \( t_1 \neq t_2 + 1 \), since otherwise by Problem 1, \( t \) would not be jugglable. The siteswapped tuple \((t_2 + 1, t_1 - 1, t_3, \ldots, t_n)\) now has strictly less elements with value \( t_1 \). We can repeat this procedure until we obtain a tuple \( t' \) with largest element strictly less than \( t_1 \). We can continue reducing the value of the largest entry until all entries are the same. This procedure must terminate since the largest entry will never be negative, since the sum of all entries remains constant throughout.

The sum of the entries in a tuple does not change after a siteswap. Thus if a jugglable \( n \)-tuple \( t = (t_1, t_2, \ldots, t_n) \) is obtained from a constant \( n \)-tuple \((a, \ldots, a)\), then \( a = \frac{1}{n} \sum_{i=1}^{n} t_i \). The number of balls used in the pattern generated \((a, \ldots, a)\) is \( a \). Since a siteswap does not change the number of balls used in the pattern it generates, \( a \) is also the number of balls used in the pattern generated by \( t \). This gives a quick way of determining how many balls are required in a juggling pattern; the average of the terms!

We say a tuple is *scramblable* if any permutation of its elements results in a jugglable tuple.

**Problem 4** Show that an \( n \)-tuple \( t = (t_1, t_2, \ldots, t_n) \) is scramblable if and only if there is a nonnegative constants \( c \) and \( q_i \) for \( 1 \leq i \leq n \) such that \( t_i = c + q_i n \) for \( 1 \leq i \leq n \).

**Proof.** By Problem 1, a tuple of the form described will be scramblable since constant tuples are scramblable. On the other hand, suppose \( t \) is scramblable. Subtract multiples of \( n \) from entries in \( t \) to obtain a new tuple \( t' \) such that all entries of \( t' \) are between 1 and \( n \). By Problem 1, since \( t \) is scramblable, so too
is \( t' \). If \( t' \) is constant, we are done. Otherwise if \( 1 \leq a < b \leq n \) are two entries of \( t' \), then the \( n \)-tuple with \( b \) in the first entry, \( a \) in the \( b - a + 1 \) entry, and the remaining entries of \( t' \) in any positions, is a permutation of \( t' \) that is not jugglable, by Problem 1. This is a contradiction since \( t' \) is scramblable, and so \( t' \) must be constant, and \( t \) is of the form claimed.

\[ \square \]

**Problem 5** Show that \((1, 2, \ldots, n)\) is never a juggling sequence if \( n \) is even.

**Problem 6** Let \( n \) be any odd integer. Show that there exists a jugglable \( n \)-tuple with entries from the set \( \{1, 2, \ldots, n\} \) each appearing once. Is there more than one such tuple?

### Juggling Resources

There is no end to excellent online resources for learning to juggle. A good tutorial for learning to juggle three balls is available at [https://www.youtube.com/watch?v=x2_j6kMglco](https://www.youtube.com/watch?v=x2_j6kMglco). A library of juggling tricks, with companion animations is available at [http://www.libraryofjuggling.com/](http://www.libraryofjuggling.com/). Finally, a fun juggling simulator that uses siteswap notation is available at [http://www.gunswap.co/](http://www.gunswap.co/).

For more information on the mathematics of juggling, we refer the avid reader to [1]. This book provides a thorough covering of the mathematics of juggling, and makes reference to other valuable resources on the topic.

### References
