MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(1), p. 4–5.

MA1. How many two-digit numbers are there such that the difference of the number and the number with the digits reversed is a non-zero perfect square? Problem extension: What happens with three-digit numbers? four-digit numbers?

Originally Question 7 from the 1999 W.J. Blundon Contest.

We received 5 submissions of which one was correct and complete. We present the solution by Sophie Bekerman (Los Gatos High School), modified by the editor.

Two-digit numbers.

Let A be such a two digit number. We can express A as $10x + y$ where x is the first digit, y is the second digit, and $x, y \in [0, \ldots, 9]$. Let $\bar{A}$ be A with the digits reversed, note that $\bar{A} = 10y + x$. Given

$$A - \bar{A} = 10x + y - (10y + x) = 9x - 9y,$$

let $9x - 9y = a^2$ where $a \in \mathbb{N}$. It follows that

$$9x - 9y = a^2 \Leftrightarrow x - y = \left(\frac{a}{3}\right)^2$$

and $\left(\frac{a}{3}\right)^2 \in \mathbb{N}$ since it is the difference of two natural numbers. $\left(\frac{a}{3}\right)^2 \leq 9$ since $x - y \leq 9$. The only perfect squares that meet these conditions are 1, 4, and 9. Therefore, the differences of the digits of A are 1, 4, or 9. If $x - y = n$, their difference can be written as $(n + k) - k$ where $n + k = x$ and $k = y$. Since

$$n + k \leq 9 \Leftrightarrow k \leq 9 - n,$$

k can take any value from 0 to $9 - n$. In total, there are $10 - n$ ways to represent each difference. As $n \in [1, 4, 9]$, there are

$$(10 - 1) + (10 - 4) + (10 - 9) = 16$$

possible values of A.

Three-digit numbers.

Let B be such a three digit number. We can express B as $100x + 10y + z$, where x is the first digit, y is the second digit, z is the third digit, and $x, y, z \in [0, \ldots, 9]$. Let $\bar{B}$ be B with the digits reversed, note that $\bar{B} = 100z + 10y + x$. Given

$$B - \bar{B} = 100x + 10y + z - (100z + 10y + x) = 99x - 99z,$$

let $99\,(x-z)=b^2$ where $b\in\mathbb{N}$. It follows that

$$99\,(x-z)=b^2 \Leftrightarrow 11\,(x-z)=\left(\frac{b}{3}\right)^2$$

and $\left(\frac{b}{3}\right)^2\in\mathbb{N}$ since it is the difference of two natural numbers. $\left(\frac{b}{3}\right)^2\le 9$ since $x-z\le 9$. For $11\,(x-z)$ to be a perfect square, $(x-z)$ has to be a factor of 11. This is impossible since $x-z\le 9$. Therefore, there are no possible forms of B.

Four-digit numbers.

Let C be such a four digit number. We can express C as $1000w+100x+10y+z$, where w is the first digit, x is the second digit, y is the third digit, z is the fourth digit, and $w,x,y,z\in[0,\dots,9]$. Let $\bar{C}$ be C with the digits reversed, note that $\bar{C}=1000z+100y+10x+w$. Given

$$\begin{aligned}C-\bar{C}&=1000w+100x+10y+z-(1000z+100y+10x+w)\\&=999w+90x-90y-999z\end{aligned}$$

let $999\,(w-z)+90\,(x-y)=c^2$ where $c\in\mathbb{N}$. It follows that

$$999\,(w-z)+90\,(x-y)=c^2,$$

or, equivalently,

$$111\,(w-z)+10\,(x-y)=\left(\frac{c}{3}\right)^2.$$

If $w=z\Leftrightarrow w-z=0$ then that leaves $10\,(x-y)=\left(\frac{c}{3}\right)^2$. For $10\,(x-y)$ to be a perfect square, $(x-y)$ has to be a factor of 10, which is impossible since $x-y\le 9$. Therefore, $w-z\ne 0$ and $111\le\left(\frac{c}{3}\right)^2\le 1089$.

The perfect squares between 111 and 1089 are 121, 484, 576, 625, 676, and 1089. These are found simply by searching through every perfect square in the range $[111,1089]$ and seeing if the perfect square can be expressed in the form $111m+10n$, where $m,n\in\mathbb{N}$. The only case where $x<y$ is $576=111\cdot 6-10\cdot 9$. For all the other possible squares, $x-y$ happens to be positive.

For 576, $w-z=6$ and $x-y=-9$. There are $10-6=4$ possible pairs of w and z that yield a difference of 6. There is $10-9=1$ possible pair of x and y that yield a difference of 9. Therefore there are 4 possible combinations of w,x,y, and z that will yield 576.

This same methodology applies to the other possible perfect squares for a total of

$$\begin{aligned}&(10-1)\cdot(10-1)+(10-4)\cdot(10-4)+(10-6)\cdot(10-9)+\\&\quad(10-5)\cdot(10-7)+(10-6)\cdot(10-1)+(10-9)\cdot(10-9)=173\end{aligned}$$

combinations of C.

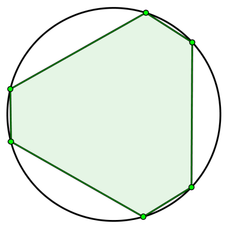
MA2. A sequence $t_1, t_2, \ldots$ beginning with any two positive numbers is defined such that for $n > 2$, $t_n = \dfrac{1 + t_{n-1}}{t_{n-2}}$. Show that such a sequence must repeat itself with a period of 5.

Originally Question 9 from the 2002 W.J. Blundon Contest.

We received 5 solutions. We present the solution by Richard Hess.

Start with a and b. Then the next terms are $(1+b)/a$, $(a+b+1)/(ab)$, $(a+1)/b$, a, b, $\ldots$. This sequence has a period of five since terms six and seven duplicate terms one and two.

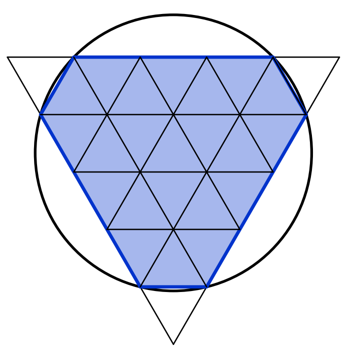
MA3. A hexagon H is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Find the area of H.



Originally Question 10 from the 2000 W.J. Blundon Contest.

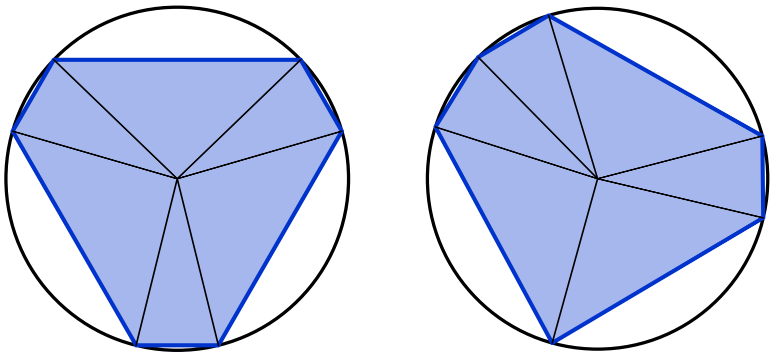
We received seven solutions, out of which we present the one by Valcho Milchev, lightly edited.

By symmetry, all the internal angles of the hexagon H are equal and thus $120°$. This means that H may be tiled by equilateral triangles as shown in the figure:



H is composed of 22 equilateral triangles of side length 1, each of which has an area of $\dfrac{\sqrt{3}}{4}$. Therefore the area of H is $\dfrac{11\sqrt{3}}{4}$.

Editor's Comments. The statement of Problem MA3 did not specify in which order the segments appear in the hexagon, even though the picture suggested a specific arrangement. However, it turns out that all cyclic hexagons with three edges of length 1 and three edges of length 3 have the same area. This can be seen by drawing the radii from the centre of the circle to the six vertices of the hexagon (see figure below). This splits the hexagon into six isosceles triangles with leg lengths equal to the radius of the circles. Three of the isosceles triangles have base length 3 and three have base length 1, irrespective of the arrangement of the edges in the hexagon.



MA4. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

Originally Question 9 from the 2002 W.J. Blundon Contest.

We received seven submissions, all of which were correct and complete. We present the joint solution by Amit Kumar Basistha (Anundoram Borooah Academy High School) and Sophie Bekerman (Los Gatos High School), done independently, slightly modified by the editor.

$x + y = a$ is tangent to $x^2 + y^2 = b$ when the system

$$\begin{cases} x + y = a \\ x^2 + y^2 = b \end{cases}$$

has exactly one solution. Given that $x + y = a \Leftrightarrow y = a - x$, by substitution we see that

$$x^2 + y^2 = b \Leftrightarrow x^2 + (a - x)^2 = b \Leftrightarrow 2x^2 - 2ax + a^2 - b = 0$$

A quadratic equation has one solution if and only if the discriminant is equal to 0. By construction, our expression has only one solution, thus by setting the discriminant Δ of the above expression to 0 we see that

$$\Delta = 4a^2 - 4\left(2\left(a^2 - b\right)\right) = 0 \Leftrightarrow a^2 = 2b$$

When there is one solution for x, there is also only one solution for y since $y = a - x$. Hence, when $a^2 = 2b$, the line $x + y = a$ is tangent to the circle $x^2 + y^2 = b$.

MA5. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

Originally Question 8 from the 2002 W.J. Blundon Contest.

We received 5 submissions of which 3 were correct and complete. We present the solution by Vitthal Ingle and Konstantine Zelator, done independently.

Let θ_1, θ_2 be the angles made between the x-axis and the lines $y = 3x$ and $y = 2x$ respectively. Clearly, $\tan\theta_1 = 3$ and $\tan\theta_2 = 2$. Let $\alpha = \theta_1 - \theta_2$, the angle between the lines $y = 2x$ and $y = 3x$. By the angle subtraction identity for tangent:

$$\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} = \frac{3-2}{1+3\cdot 2} = \frac{1}{7}.$$

Let O be the origin. We have

$$\tan\alpha = \frac{1}{7} = \frac{\overline{PQ}}{\overline{OP}} = \frac{5}{\overline{OP}},$$

and so $\overline{OP} = 35$. Let P have coordinates $(a, 2a)$, and let M be the projection of P onto the x-axis. Now $\overline{OP}^2 = \overline{OM}^2 + \overline{MP}^2$, and so $35^2 = a^2 + 4a^2 = 5a^2$. It follows that $a = 7\sqrt{5}$ and so P has coordinates $(7\sqrt{5}, 14\sqrt{5})$. Note that the solution $a = -7\sqrt{5}$ gives a point in the third quadrant, and so can not be the answer.