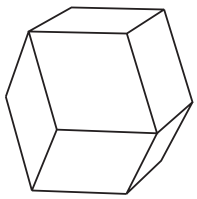
CONTEST CORNER SOLUTIONS

CC339. A *rhombic dodecahedron* has twelve congruent rhombic faces; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\dfrac{p+\sqrt{q}}{r}$, where p, q, and r are natural numbers and r has no factor in common with p or q.

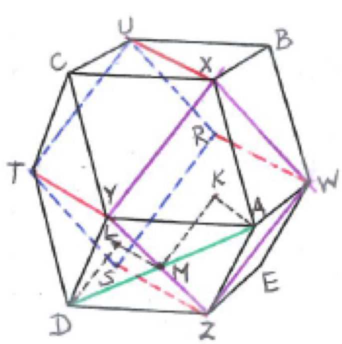


Originally from the 2018 Science Atlantic Math Competition.

The statement of this problem originally appeared in ***Crux*** *44(8). We received no solutions to this problem for the original publication. We have since received a solution by Ivko Dimitrić showing that the statement of the problem is not correct. We present the solution here.*

We have $f = 12$ rhombic faces, so the number of edges is $e = \frac{12\cdot 4}{2} = 24$. By Euler's formula, $v - e + f = 2$, we find the number of vertices to be $v = 14$. Moreover, if x is the number of vertices of degree 4 and y the number of those of degree 3, then from $x + y = 14$ and $e = \frac{4x+3y}{2} = 24$ we get $x = 6$ and $y = 8$.

It is a general fact that a rhombic dodecahedron is a semi-regular polyhedron with symmetries and transitive faces, which means that for each pair α, β of faces there is an isometry of the solid that takes α to β. As a consequence of the symmetries of this polyhedron, the arrangement of edges, the measures of angles in the same position and geometric picture about one vertex is exactly the same as the situation about another vertex of the same degree.



We use the labeling of vertices and points as shown on the diagram. Because of the symmetry, there exists an axis through a selected degree-4 vertex, which is equally inclined to each of the four edges incident to that vertex and there exists an angle so that the rotation through that angle about the axis will take each edge from that vertex to the next one in cyclic order and hence each vertex of the quadrilateral $WXYZ$ to its neighbouring vertex in cyclic order. Thus, these four vertices lie in the same plane perpendicular to the axis of rotation and in that plane each vertex is carried to its neighbour in the same sense by induced rotation in that plane through the same angle, so the quadrilateral is a square. Hence, each degree-4 vertex of the dodecahedron together with four neighboring vertices determines a pyramid with a square base formed by the four neighboring vertices, where the apex (a degree-4 vertex) is projected orthogonally to the center of the square base. Two square pyramids with apexes A and D are joined by the common edge YZ and the faces AYZ and DYZ that share that edge are two congruent triangular halves of rhomboid face $AZDY$ of the dodecahedron. Let a be the edge length of the polyhedron and h be the height of each of the six square pyramids with apex at a degree-4 vertex, such as pyramids $AWXYZ$ and $DTYZS$. The square bases of these six pyramids are congruent and their union forms the surface of a cube, since the dihedral angle at each edge is $90°$, for example $XY \perp XW$ and $XY \perp XU$, so XY is perpendicular to the square face $XURW$. We call this cube the core cube.

Let M be the midpoint of a shorter diagonal YZ of the rhombic face $AZDY$. Because of the congruence of the square pyramids with apexes A and D, the segments AM and DM are equally inclined to the corresponding bases of these pyramids and since the dihedral angle of the core cube at M is $90°$ and $\angle AMD = 180°$, it follows that $\angle KMA = \angle LMD = 45°$, i. e. $\triangle AKM$ and $\triangle DLM$ are two congruent isosceles right triangles and hence $s = h\sqrt{2}$, whereas the edge of the core cube is $2h$.

Then from (a half of) the isosceles $\triangle AYZ$ we get $(h\sqrt{2})^2 + h^2 = a^2$, which gives $h = a/\sqrt{3}$. The rhombic dodecahedron is constructed as the union of the cubical core of edge length $2h$ whose vertices are eight degree-3 vertices of the dodecahedron and six congruent square pyramids of height h, surmounting each of the six faces of the cube on the outside.Thus, its volume is

$$V = \left(\frac{2a}{\sqrt{3}}\right)^3 + 6 \cdot \frac{1}{3}(2h)^2 h = \frac{8a^3}{3\sqrt{3}} + 8\left(\frac{a}{\sqrt{3}}\right)^3 = \frac{16\sqrt{3}}{9}\, a^3.$$

When $a = 1$, the answer can be written as $\frac{0+\sqrt{768}}{9}$ which is the required expression, but it cannot be written in the form $\frac{p+\sqrt{q}}{r}$ where p, q, r are integers with r having no factor in common with p and q since 3 is the common factor of p, q, r.

Remark. Some nice pictures and additional information on rhombic dodecahedron can be found at `https://en.wikipedia.org/wiki/Rhombic.dodecahedron` and at `http://mathworld.wolfram.com/RhombicDodecahedron.html`.