# PROBLEM SOLVING VIGNETTES

No.6

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Repdigit Recreations

In this issue we will look at a couple of problems from the course C&O 380 that I took from Ross Honsberger that has been featured in previous columns. The next set of problems are:

#16. $A$ is the integer $666\cdots 66$, containing 666 sixes. $B$ is the integer $333\cdots 33$, containing 666 threes. State the value of $AB$.

#17. Show that a positive integer, with more than one digit, all of whose digits are the same, cannot be a perfect square.

#18. Show that the sum of the squares of 83 consecutive natural numbers is never a perfect square.

#19. Devise a method of trisecting a given line segment, using only straightedge and compasses, which does not involve parallel lines.

#20. Construct an equilateral triangle so that it has one vertex on each of three given parallel lines.

Problems #16 and #17 both deal with *repdigit* numbers. That is, numbers that are comprised of a single digit repeated a number of times. Don Rideout, in problem #3 of his vignette [2019: 45(3), p. 120], looked at *repunit* numbers, that is, repdigit numbers made up of only the digit 1. We will run into a couple of the properties of these numbers as we solve the two problems. We will look at #17 first.

How do we know if a number is a perfect square? Looking at the first few squares we start to see a pattern in the units digit.

| | | | | |
|---|---|---|---|---|
| $1^2 = 1$ | $2^2 = 4$ | $3^2 = 9$ | $4^2 = 16$ | $5^2 = 25$ |
| $6^2 = 36$ | $7^2 = 49$ | $8^2 = 64$ | $9^2 = 81$ | $10^2 = 100$ |
| $11^2 = 121$ | $12^2 = 144$ | $13^2 = 169$ | $14^2 = 196$ | $15^2 = 225$ |
| $16^2 = 256$ | $17^2 = 289$ | $18^2 = 324$ | $9^2 = 361$ | $20^2 = 400$ |

The unit digits follow the pattern

$$1, 4, 9, 6, 5, 6, 9, 4, 1, 0, 1, 4, 9, \ldots$$

If we use modular arithmetic, as in some recent columns, we would get the following:

| $n \pmod{10}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n^2 \pmod{10}$ | 0 | 1 | 4 | 9 | 6 | 5 | 6 | 9 | 4 | 1 |

which tells us the same thing: the units digit of a perfect square is $0, 1, 4, 5, 6$, or 9. Hence the repdigit numbers $222\cdots22$, $333\cdots33$, $777\cdots77$, and $888\cdots88$ cannot be perfect squares.

We write the other repdigit numbers in the form

$$\begin{aligned}111\cdots11 &= 1 \times 111\cdots11\\ 444\cdots44 &= 4 \times 111\cdots11\\ 555\cdots55 &= 5 \times 111\cdots11\\ 666\cdots66 &= 6 \times 111\cdots11\\ 999\cdots99 &= 9 \times 111\cdots11.\end{aligned}$$

Clearly $555\cdots55$ is not a perfect square since $5 \mid 555\cdots55$, but $5 \nmid 111\cdots11$. Similarly, $666\cdots66$ is not a perfect square.

The remaining three candidates are written as a perfect square times $111\cdots11$. Thus if $111\cdots11$ is a perfect square, then so is $444\cdots44$ and $999\cdots99$. If $111\cdots11$ is not a perfect square then neither are $444\cdots44$ and $999\cdots99$.

To determine if $111\cdots11$ is a perfect square we will go back to modular arithmetic and look at numbers modulo 4.

| $n \pmod 4$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $n^2 \pmod 4$ | 0 | 1 | 0 | 1 |

So if a number is a perfect square it must be congruent to 0 or 1 modulo 4. Taking into account that $4 \mid 100$ and hence $4 \mid 10^n$ when $n \geq 2$ (so $10^n \equiv 0 \pmod 4$) we get

$$111\cdots11 \equiv 11 \equiv 3 \pmod 4$$

and so $111\cdots11$ is not a perfect square and therefore no repdigit number, of more than one digit, is a perfect square.

Next, we will look at problem #16. A few computations suggest a pattern:

$$\begin{aligned}6 \times 3 &= 18 & 66 \times 33 &= 2\,178\\ 666 \times 333 &= 221\,778 & 6\,666 \times 3\,333 &= 22\,217\,778\\ 66\,666 \times 33\,333 &= 2\,222\,177\,778 & 666\,666 \times 333\,333 &= 222\,221\,777\,778\end{aligned}$$

that is,

$$\overbrace{666\cdots66}^{n \text{ 6s}} \times \overbrace{333\cdots33}^{n \text{ 3s}} = \overbrace{222\cdots22}^{n-1 \text{ 2s}}1\overbrace{777\cdots77}^{n-1 \text{ 7s}}8. \qquad (1)$$

It is one thing to see a pattern and be certain it is true. It is another thing to *prove* that the pattern does indeed hold. The pattern seems to call out for mathematical induction like we saw in the last issue [2019: 45(5), p. 236-240].

To make our lives easier, we will introduce the sequence of repunit numbers

$$\{U_n\}_{n=1}^{\infty} = \{1, 11, 111, 1111, \ldots\}.$$

Our proposition that we would like to prove is

$$P_n : \quad (6U_n)(3U_n) = 10^n[2U_n - 1] + 7U_n + 1. \tag{2}$$

You may want to convince yourself that (2) is equivalent to (1).

To aid us in our proof we will need the following properties of the repunit numbers:

$$10 \times U_n + 1 = U_{n+1} \tag{3}$$

$$U_a + 10^a \times U_b = U_{a+b} \tag{4}$$

We leave the proofs of these as exercises. Now on to our proof by induction.

If we look at $P_1$, we get

$$(6U_1)(3U_1) = 6 \times 3 = 18$$

and

$$10^1[2U_1 - 1] + 7U_1 + 1 = 10 \times (2 - 1) + 7 + 1 = 18$$

so the proposition is true for $n = 1$.

Suppose $P_n$ is true for some $n = k \in \mathbb{N}$, then

$$(6U_k)(3U_k) = 10^k[2U_k - 1] + 7U_k + 1. \tag{5}$$

So, using (3) we get

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= (6(10U_k + 1))(3(10U_k + 1)) \\ &= 100((6U_k)(3U_k)) + 360U_k + 18 \end{aligned} \tag{6}$$

Combining (5) with (6) yields

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= 100(10^k[2U_k - 1] + 7U_k + 1) + 360U_k + 18 \\ &= 10^{k+2}[2U_k - 1] + 1060U_k + 118 \end{aligned} \tag{7}$$

Breaking the right side of (7) into two parts and using the properties (3) and (4), yields

$$\begin{aligned}10^{k+2}[2U_k - 1] &= 10^{k+1}[2(10U_k + 1 - 1) - 10] \\ &= 10^{k+1}[2(U_{k+1} - 1) - 10] \\ &= 10^{k+1}[2U_{k+1} - 12] \\ &= 10^{k+1}[2U_{k+1} - 1 - 11] \\ &= 10^{k+1}[2U_{k+1} - 1] - 11 \times 10^{k+1} \end{aligned} \tag{8}$$

and

$$\begin{aligned}1060U_k + 118 &= 1000U_k + 60U_k + 118\\ &= 1000(U_{k-2} + 10^{k-2}U_2) + 60(U_2 + 10^2U_{k-2}) + 118\\ &= 7000U_{k-2} + 11 \times 10^{k+1} + 778\\ &= 7000U_{k-2} + 777 + 1 + 11 \times 10^{k+1}\\ &= 7U_{k+1} + 1 + 11 \times 10^{k+1} \qquad (9)\end{aligned}$$

Putting (8) and (9) back into (7) yields

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= 10^{k+1}[2U_{k+1} - 1] - 11 \times 10^{k+1} + 7U_{k+1} + 1 + 11 \times 10^{k+1}\\ &= 10^{k+1}[2U_{k+1} - 1] + 7U_{k+1} + 1\end{aligned}$$

which shows that $P_{k+1}$ is true and completes the induction. So for the problem at hand we have

$$\overbrace{666\cdots66}^{666\ \text{6s}} \times \overbrace{333\cdots33}^{666\ \text{3s}} = \overbrace{222\cdots22}^{665\ \text{2s}}1\overbrace{777\cdots77}^{665\ \text{7s}}8.$$

It is nice to have an opportunity to use a new tool, but it is always nice to find a slick solution such as

$$\begin{aligned}\overbrace{666\cdots66}^{666\ \text{6s}} \times \overbrace{333\cdots33}^{666\ \text{3s}} &= 6 \times 3 \times \overbrace{111\cdots11}^{666\ \text{1s}} \times \overbrace{111\cdots11}^{666\ \text{1s}}\\ &= 9 \times 2 \times \overbrace{111\cdots11}^{666\ \text{1s}} \times \overbrace{111\cdots11}^{666\ \text{1s}}\\ &= \overbrace{999\cdots99}^{666\ \text{9s}} \times \overbrace{222\cdots22}^{666\ \text{2s}}\\ &= (10^{666} - 1) \times \overbrace{222\cdots22}^{666\ \text{2s}}\\ &= \overbrace{222\cdots22}^{666\ \text{2s}}\overbrace{000\cdots00}^{666\ \text{0s}} - \overbrace{222\cdots22}^{666\ \text{2s}}\\ &= \overbrace{222\cdots22}^{665\ \text{2s}}1\overbrace{777\cdots77}^{665\ \text{7s}}8.\end{aligned}$$

The biggest hammer isn't always the best tool for the job.

A little manipulation tells us that the repdigit number $\overbrace{ddd\cdots dd}^{n\ \text{ds}}$, where $d \in \{1, 2, \ldots, 9\}$ can be written as

$$\overbrace{ddd\cdots dd}^{n\ \text{ds}} = \frac{d}{9}\left(\overbrace{999\cdots99}^{n\ \text{9s}}\right) = \frac{d}{9}(10^n - 1)$$

which make sense as

$$\overbrace{ddd\cdots dd}^{n\text{ ds}} = d + 10d + 100d + \cdots + d \times 10^{n-1}$$

is a geometric series.

Enjoy the rest of the problems from the problem set, and you may enjoy the following problem from the vault:

> Find a quadratic polynomial $f(x)$ such that, if $n$ is a positive integer consisting of the digit 5 repeated $k$ times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

This was Mayhem problem M256 that appeared in [2006: 32(5), p. 265-266] and the solution is in [2007: 33(5), p. 271] which can be generalized to

> Find a degree $d$ polynomial $f(x)$ such that, if $n$ is a positive integer consisting of the digit 5 repeated $k$ times, then $f(n)$ consists of the digit 5 repeated $dk$ times.