SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(6), p. 256–259; 44(7), p. 302–305; 44(8), p. 340–343.

4351. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle and let ω be its incircle. Let E and F be the tangency points of ω and the sides AC and AB, respectively. Let G be the second intersection point of ω and BE and, similarly, let D be the second intersection point of ω and CF. Prove that

$$\frac{FE \cdot GD}{FG \cdot ED} = 3.$$

We received 6 submissions, all of which were correct, and will feature two of them.

Solution 1, by Richard B. Eden.

Let I be the incenter of $\triangle ABC$ and K the point on FD such that $\angle KGF = \angle DGE$. Since $\angle GFK = \angle GED$, then

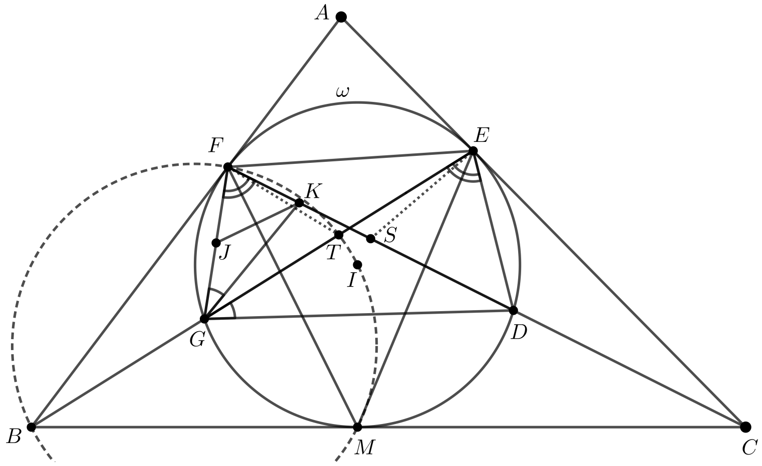
$$\triangle KGF \sim \triangle DGE.$$

Next, since $\angle EGF = \angle EGK + \angle KGE = \angle EGK + \angle DGE = \angle DGK$ and $\angle FEG = \angle KDG$, then

$$\triangle KDG \sim \triangle FEG.$$

From the similarity of these triangles, we get $\frac{FK}{FG} = \frac{ED}{EG}$ and $\frac{DK}{DG} = \frac{EF}{EG}$, so

$$\frac{FE \cdot GD}{FG \cdot ED} = \frac{DK \cdot EG}{FK \cdot EG} = \frac{DK}{FK}.$$



Our goal, therefore, is to prove that $\frac{DK}{FK} = 3$. Let S be the midpoint of DF so that the problem is reduced to showing that K is the midpoint of FS. Let T be the midpoint of EG, and M the point of contact of ω and BC. First, we will show that $\triangle TGF \sim \triangle EMF$. We have $\angle TGF = \angle EGF = \angle EMF$. Since T is the midpoint of chord EG of ω,

$$\angle ITB = \angle ITG = 90^\circ = \angle IFB = \angle IMB,$$

so F, B, M and T are concyclic. Therefore,

$$\angle FTG = \angle FTB = \angle FMB = \angle FEM.$$

Therefore, $\triangle TGF \sim \triangle EMF$. Similarly, $\triangle EDS \sim \triangle EMF$. Therefore, we have $\triangle TGF \sim \triangle EDS$.

It follows that $\dfrac{TG}{GF} = \dfrac{ED}{DS}$, so $\dfrac{ET}{FG} = \dfrac{ED}{FS}$. Since

$$\angle GFS = \angle GFD = \angle GED = \angle TED,$$

it follows that $\triangle GFS \sim \triangle TED$. Now let J be the midpoint of FG. Since $\triangle GFK \sim \triangle GED$ and the medians KJ and DT correspond to each other, then $\triangle JFK \sim \triangle TED$. This implies $\triangle JFK \sim \triangle GFS$, with vertices written in corresponding order. Since J is the midpoint of FG, K is the midpoint of FS.

Solution 2, by Michel Bataille.

We shall use complex numbers, denoting by m (small letter) the complex affix of the point M (capital letter). Without loss of generality, we suppose that ω is the unit circle. We observe that if M and N are points of ω, then

$$MN^2 = |m-n|^2 = (m-n)(\overline{m}-\overline{n}) = (m-n)\left(\frac{1}{m}-\frac{1}{n}\right) = -\frac{(m-n)^2}{mn}.$$

It readily follows that

$$\frac{FE^2 \cdot GD^2}{FG^2 \cdot ED^2} = \frac{(e-f)^2(d-g)^2}{(g-f)^2(d-e)^2}$$

and therefore our problem amounts to showing that the ratio $\rho = \dfrac{(e-f)(d-g)}{(g-f)(d-e)}$ equals 3 or -3.

Let W be the point of tangency of ω and BC. From the respective equations $z + w^2\overline{z} = 2w$ and $z + e^2\overline{z} = 2e$ of the lines BC and CA (which are the tangents to ω at W and E), we obtain $c = \dfrac{2ew}{e+w}$. Since C is on the line DF whose equation is $z + df\overline{z} = d + f$, we have $c + df\overline{c} = d + f$ and a short calculation yields

$$d = \frac{2we - wf - ef}{e + w - 2f} = \frac{w(e-f) + e(w-f)}{(e-f)+(w-f)} = \frac{w(e-f)+e(w-f)}{\alpha}$$

(with $\alpha = (e-f)+(w-f)$). Exchanging e and f gives

$$g = \frac{w(f-e)+f(w-e)}{(f-e)+(w-e)} = \frac{w(f-e)+f(w-e)}{\beta}$$

(with $\beta = (f-e)+(w-e)$). Then we easily calculate

$$\begin{aligned} g-f &= (\beta)^{-1}(f-e)(w-f), \\ d-e &= (\alpha)^{-1}(e-f)(w-e), \quad \text{and} \\ d-g &= (\alpha\beta)^{-1}(e-f)(3w^2-3we-3wf+3ef). \end{aligned}$$

This leads to

$$\rho = \frac{3(e-f)^2(w^2-we-wf+ef)}{(f-e)(w-f)(e-f)(w-e)} = -3,$$

and the result follows.

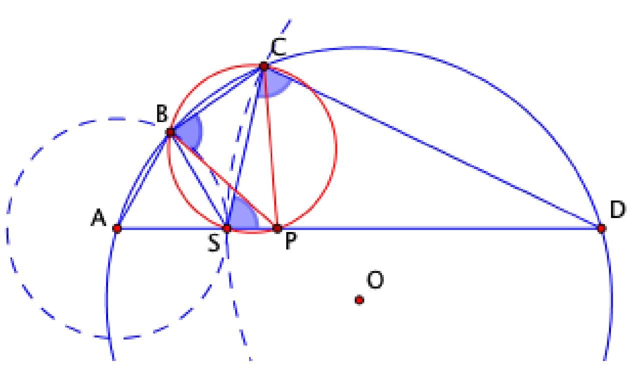
4352. *Proposed by Thanos Kalogerakis.*

Explain how to locate six points A, B, C, D, E, F in that order about the circumference of a circle so that the resulting convex hexagon has an incircle, yet is not regular.

The solutions from C.R. Pranesachar and the proposer were the only submissions we received. We present a composite of their work.

A polygon that has both a circumcircle and an incircle is called *bicentric*. It is clear that for a convex hexagon to have an incircle, it is necessary that the angle bisectors at all six vertices be concurrent. (The point of concurrence will be equidistant from all six sides.) Our construction of a bicentric hexagon will be based on the following theorem, which is interesting in its own right.

Theorem. If $ABCD$ is a cyclic quadrilateral, then the bisectors of the angles at B and at C meet at a point of the chord AD if and only if there exists a point S of AD for which $AS = AB$ and $DS = DC$.



First, assume that for a point S of AD we have $DS = DC$. Assume further (without loss of generality) that the points have been labeled so that $AB \le CD$.

Define P to be the second point where the circle BCS intersects the line AD. We show that BP bisects $\angle ABC$.

From $BSPC$ cyclic, we have $\angle PBC = \angle PSC$.

From ΔDSC isosceles, we have $\angle PSC = \angle DSC = \angle DCS = \dfrac{180^\circ - \angle SDC}{2}$.

From $ABCD$ cyclic, we have $\angle ABC = 180^\circ - \angle SDC$.

It follows that $\angle PBC = \frac{1}{2}\angle ABC$; that is, BP bisects $\angle ABC$, as claimed. Similarly, CP bisects $\angle BCD$.

For the converse, we assume that the bisectors of the angles at B and at C intersect at a point P of the chord AD and, moreover, that S is the point of AD for which $AS = AB$. We will show that $DS = DC$. We have proved above that the quadrilateral $BSPC$ is cyclic. Similarly, for the point S' for which $DC = DS'$ we have $BS'PC$ is also cyclic. But there is just one point of AD other than P that can lie on the circle BCP, whence we conclude that $S' = S$, which finishes the proof.

Finally, for an example of a bicentric hexagon $ABCDEF$ that is not regular, denote its circumcircle by γ, let AD be a diameter, and choose a point S of AD different from its midpoint. Define B and F to be the points where γ intersects the circle (A, AS) (with center A and radius AS); define C and E to be the points where γ intersects the circle (D, DS) labeled so that B and C are on the same side of AD. By our theorem, the bisectors of the angles at B and C meet in a point of AD as do the bisectors of the angles at E and F. By symmetry all four of those angle bisectors must meet in the same point of AD while AD bisects the angles at both endpoints. In other words, all six angle bisectors are concurrent in the center of the incircle of $ABCDEF$. This concludes the required construction.

Further comments. Once we have constructed one bicentric hexagon, we can construct infinitely many of them according to the Great Poncelet Theorem for Circles: If there is an n-sided polygon inscribed in a circle α and circumscribed about a circle β, then for any point A of α there exists an n-sided polygon, also inscribed in α and circumscribed about β, which has A as one of its vertices. In fact, the theorem applies more generally to families of conics, but the proof, even in the simplest case of a pair of bicentric polygons, is not easy. Note that Poncelet's theorem combined with our result provides a recipe for the construction of all bicentric hexagons — given any bicentric hexagon with circumcircle α and incircle β, there exists a bicentric hexagon $ABCDEF$ inscribed in α and circumscribed about β having AD a diameter of α. Simply start the Poncelet construction with A on the line containing the two centers.

Editor's comments. For the "simplest" proof of Poncelet's theorem known to this editor, see the article "A Simple Proof of Poncelet's Theorem" by Lorenz Halbeisen and Norbert Hungerbühler, *The American Mathematical Monthly* **122**:6 (January 2014) 537-551.

4353. *Proposed by Michel Bataille.*

Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \sum_{j=1}^{n} \frac{1}{k\binom{j+k-1}{j}}.$$

We received four solutions. We present the one by Bao Do, lightly edited.

Let

$$s(j) = \sum_{k=1}^{\infty} \frac{1}{k\binom{j+k-1}{j}} = \sum_{k=1}^{\infty} \frac{j!}{(k+j-1)\cdots(k+1)k^2}.$$

Note that the sum in the definition converges, since

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{j+k-1}{j}} \le \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

We need to evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} s(j)$. Consider

$$\begin{aligned}
\frac{s(j)}{j} &= \sum_{k=1}^{\infty} \frac{(j-1)!}{(k+j-1)\cdots(k+1)k^2} \\
&= (j-2)! \sum_{k=1}^{\infty} \left(\frac{1}{(k+j-2)\cdots(k+1)k^2} - \frac{1}{(k+j-1)\cdots(k+1)k} \right) \\
&= \frac{1}{j-1} \sum_{k=1}^{\infty} \frac{(j-1)!}{(k+j-2)\cdots(k+1)k^2} - (j-2)! \sum_{k=1}^{\infty} \frac{1}{(k+j-1)\cdots k} \\
&= \frac{s(j-1)}{j-1} - \frac{(j-2)!}{(j-1)} \sum_{k=1}^{\infty} \left(\frac{1}{(k+j-2)\cdots k} - \frac{1}{(k+j-1)\cdots(k+1)} \right) \\
&= \frac{s(j-1)}{j-1} - \frac{1}{(j-1)^2}.
\end{aligned}$$

We rewrite this as

$$\frac{s(j)}{j} - \frac{s(j-1)}{j-1} = -\frac{1}{(j-1)^2}.$$

We take the sum from $j = 2$ to n on both sides to obtain

$$\begin{aligned}
\sum_{j=2}^{n} \left(\frac{s(j)}{j} - \frac{s(j-1)}{j-1} \right) &= -\sum_{j=2}^{n} \frac{1}{(j-1)^2} \\
\Longrightarrow \qquad \frac{s(n)}{n} - \frac{s(1)}{1} &= -\sum_{j=1}^{n-1} \frac{1}{j^2} \\
\Longrightarrow \qquad s(n) &= n\Big(s(1) - \sum_{j=1}^{n-1} \frac{1}{j^2}\Big) = n\Big(\frac{\pi^2}{6} - \sum_{j=1}^{n-1} \frac{1}{j^2}\Big).
\end{aligned}$$

Finally, we apply the Stolz-Cesàro theorem twice to calculate the limit [*Ed.: The editor found that most statements of Stolz-Cesàro require the sequence used for the denominators to be strictly monotone and divergent. However, the case when both denominator and numerator sequences converge to* 0 *and the denominator sequence is strictly monotone holds as well.*]:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}s(j) &= \lim_{n\to\infty}\frac{\sum_{j=1}^{n+1}s(j)-\sum_{j=1}^{n}s(j)}{(n+1)-n} = \lim_{n\to\infty}s(n+1)\\ &= \lim_{n\to\infty}(n+1)\left(\frac{\pi^2}{6}-\sum_{j=1}^{n}\frac{1}{j^2}\right) = \lim_{n\to\infty}\frac{\frac{\pi^2}{6}-\sum_{j=1}^{n}\frac{1}{j^2}}{\frac{1}{n+1}}\\ &= \lim_{n\to\infty}\frac{-\sum_{j=1}^{n+1}\frac{1}{j^2}+\sum_{j=1}^{n}\frac{1}{j^2}}{\frac{1}{n+1}-\frac{1}{n}} = \lim_{n\to\infty}\frac{-\frac{1}{(n+1)^2}}{-\frac{1}{n(n+1)}}\\ &= \lim_{n\to\infty}\frac{n(n+1)}{(n+1)^2} = 1.\end{aligned}$$

4354. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square with side length 1. Consider points $M \in AB$, $N \in BC$ and $P \in CA$ such that the triangles BMN and PMN are congruent. Prove that

$$\frac{1}{MB}+\frac{1}{BN}=2+\frac{2}{MB+BN}.$$

Thirteen correct solutions were received, along with one incorrect one. We provide a sample of the variety of approaches used.

Solution 1.

The triangles are congruent through a reflection in the axis MN. Assign coordinates with $A\sim(0,1)$, $B\sim(0,0)$, $C\sim(1,0)$, $M\sim(0,a)$, $N\sim(c,0)$. The line MN has equation $(x/c)+(y/a)=1$ and slope $-a/c$. The line BP is perpendicular to MN and has slope c/a. Hence $P\sim(a(a+c)^{-1},c(a+c)^{-1})$. Since BP and MN intersect at the midpoint of BP, we have

$$\begin{aligned} &\frac{a}{2c(a+c)}+\frac{c}{2a(a+c)}=1\\ \iff\quad & a^2+c^2=2ac(a+c)\\ \iff\quad & (a+c)^2=2ac(a+c+1)\\ \iff\quad & \frac{1}{a}+\frac{1}{c}=\frac{a+c}{ac}=2\left(1+\frac{1}{a+c}\right)\end{aligned}$$

as desired.

Solution 2, by Ivko Dimitrić.

Using the notation and initial results from Solution 1, along with the formula for the distance of a point to the line MN with equation $ax + cy - ac = 0$, we find that

$$\text{dist}(B, MN) = \frac{ac}{\sqrt{a^2 + c^2}}$$

and

$$\text{dist}(P, MN) = \frac{|a^2(a+c)^{-1} + c^2(a+c)^{-1} - ac|}{\sqrt{a^2 + c^2}} = \frac{1}{\sqrt{a^2 + c^2}} \left(\frac{a^2 + c^2}{a + c} - ac \right),$$

since $a^2 + c^2 - ac(a+c) = a^2(1-c) + c^2(1-a) > 0$. (The first distance can also be found by taking the area of triangle MBN in two ways.) Since the two distances are equal,

$$\frac{a^2 + c^2}{a + c} = 2ac \Leftrightarrow (a + c)^2 = 2ac(a + c) + 2ac,$$

from which the result follows.

Solution 3, by I.J.L. Garces.

Let a and c be the respective lengths of BM and BN, and let $\angle BMN = \theta$, so that $c = a \tan \theta$.

$$\frac{1}{a} + \frac{1}{c} - \frac{2}{a+c} = \frac{1}{a} \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{2\cos\theta}{\sin\theta + \cos\theta} \right) = \frac{1}{a \sin\theta (\cos\theta + \sin\theta)}.$$

Observe that $MP = MB$ and $\angle APM = 2\theta - 45^\circ$. Use the Sine Law on triangle AMP to obtain

$$\frac{1}{a\sqrt{2}} = \frac{\sin(2\theta - 45^\circ)}{1 - a},$$

whence

$$\frac{1}{a} = \sin 2\theta - \cos 2\theta + 1 = 2 \sin\theta (\cos\theta + \sin\theta).$$

Hence

$$\frac{1}{a} + \frac{1}{c} - \frac{2}{a+c} = 2,$$

as desired.

Solution 4, by Cristóbal Sánchez-Rubio.

Let Q be the foot of the perpendicular from P to AB, and let a, c, d be the respective lenghts of BM, BN, PQ. The altitudes of triangles PAM and PNC have respective lengths d and $1 - d$. With $[\ldots]$ denoting area, we have that

$$\begin{aligned}
\frac{1}{2} = [ABC] &= [PAM] + [PMBN] + [PNC] \\
&= [PAM] + 2[MBN] + [PNC] \\
&= \frac{1}{2}(1 - a)d + ac + \frac{1}{2}(1 - c)(1 - d)
\end{aligned}$$

Hence $d(a-c) = c(2a-1)$.

Since $QP \| BN$ and $BP \perp MN$,

$$\angle QPB = \angle PBN = 90^\circ - \angle MBP = \angle BMN.$$

It follows that the right triangles PQB and MBN are similar, so $d/(1-d) = a/c$ and $d = a/(a+c)$. Plugging this into the previous equation yields that $a(a-c) = c(a+c)(2a-1)$ which can be unravelled to the desired result

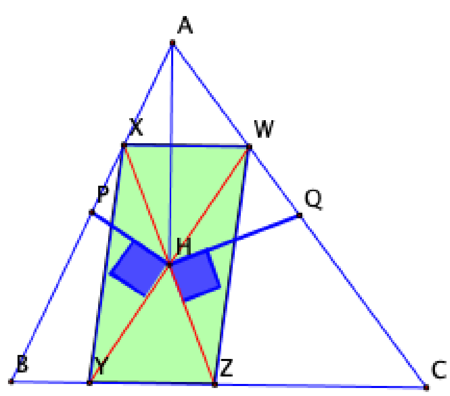
$$\frac{1}{a} + \frac{1}{c} = 2 + \frac{2}{a+c}.$$

Editor's comments. For the situation to be viable, P must be distant no more than 1 from each of the vertices A and C. If the triangles BMN and PMN are congruent with each of M and N corresponding to the other, then $BNPM$ is a rectangle and $MB + BN = 1$. In this case, the equation holds only when $MB = BN$ and P is the midpoint of AC.

4355. *Proposed by Mihaela Berindeanu.*

Let H be the orthocenter of triangle ABC with P the midpoint of AB and Q the midpoint of AC. If WY is the line perpendicular to HP at H with $W \in AC$ and $Y \in BC$, while XZ is the perpendicular to HQ at H with $X \in AB$ and $Z \in BC$, prove that the quadrilateral $WXYZ$ is a parallelogram.

In rewording the proposer's problem, the editors mistakenly omitted the necessary condition that WY and XZ both had to pass through H. All four submissions corrected that error. We present the solution by Leonard Giugiuc.



To prove that $WXYZ$ is a parallelogram it is sufficient to prove that H is the midpoint of both diagonals. To that end, we introduce Cartesian coordinates, choosing (without loss of generality)

$$A(0,1), \quad B(-u,0), \quad \text{and} \quad C(v,0),$$

where $u = \cot B$ and $v = \cot C$. We therefore have

$$H(0, uv), \quad P\left(-\frac{u}{2}, \frac{1}{2}\right), \quad \text{and} \quad Q\left(\frac{v}{2}, \frac{1}{2}\right).$$

The slope of the line PH is $\frac{2uv-1}{u}$, so the equation of WY is $(1-2uv)(y-uv) = ux$. Also, BC is $y = 0$ and AC is $x + vy = v$. Thus, from

$$(1-2uv)(y-uv) = ux \quad \text{and} \quad x + vy = v$$

we get $W\,(v(1-2uv), 2uv)$, while from

$$(1-2uv)(y-uv) = ux \quad \text{and} \quad y = 0$$

we get $Y\,(v(2uv-1), 0)$. We deduce that H is the midpoint of the segment WY. Similarly (by interchanging the roles of $-u$ and v), we find $X\,(u(2uv-1), 2uv)$ and $Z\,(u(1-2uv), 0)$, so that H is the midpoint of XZ, which concludes the proof.

4356. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Solve the following system over reals:

$$\begin{cases} a+b+c+d = 6, \\ a^2+b^2+c^2+d^2 = 12, \\ abc+abd+acd+bcd = 8+abcd. \end{cases}$$

We received 14 submissions including that from the proposers. All are correct and we present the solution by Ramanujan Srihari, modified slightly by the editor.

From the first two equations we readily obtain

$$ab+ac+ad+bc+bd+cd = 12.$$

Let $abcd = k$ and let

$$f(x) = x^4 - 6x^3 + 12x^2 - (k+8)x + k \tag{1}$$

be the polynomial with a, b, c, d as its roots.

Then

$$f'(x) = 4x^3 - 18x^2 + 24x - (k+8). \tag{2}$$

Suppose $f'(r) = 0$ where $r \in \mathbb{R}$. Then the slope of $y = f(x)$ at $x = r$ is 0.

Now, by straightforward computations we have

$$\begin{aligned} f(r) &= f(r) + (1-r)f'(r) \\ &= r^4 - 6r^3 + 12r^2 - (6+8)r + k + (1-r)(4r^3 - 18r^2 + 24r - (k+8)) \\ &= -3r^4 + 16r^3 - 30r^2 + 24r - 8 \\ &= -(r^2 - 4r + 4)(3r^2 - 4r + 2) \\ &= -(r-2)^2(3r^2 - 4r + 2). \end{aligned}$$

Since $3r^2 - 4r + 2 = 3((r - \frac{2}{3})^2 + \frac{2}{9}) > 0$ we see that $f(r) \le 0$ with $f(r) = 0$ if and only if $r = 2$.

Assume that $f'(2) \neq 0$. Then for all t with $f'(t) = 0$ we have $f(t) < 0$. This implies that f has only two simple roots which is a contradiction since $f(x)$ has four roots, a, b, c, and d.

Thus, $f'(2) = 0$ and $t = 2$ is the only multiple root of f. From (2), $f'(2) = 0 \Longrightarrow k = 0$.

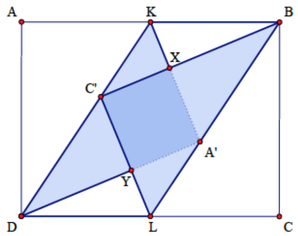
Hence we get from (1) that $f(x) = x^4 - 6x^3 + 12x^2 - 8x = x(x-2)^3$ and so

$$(0, 2, 2, 2), \quad (2, 0, 2, 2), \quad (2, 2, 0, 2), \quad \text{and} \quad (2, 2, 2, 0)$$

are the only solutions.

4357. *Proposed by Arsalan Wares.*

Suppose $ABCD$ represents a 9 by 12 rectangular sheet. Points K and L are midpoints of sides AB and DC, respectively. First, edge AD, of the rectangular sheet $ABCD$, is folded over by making a crease along DK. Then edge BC is folded over by making a crease along BL. Folded corners of the sheet overlap over a polygonal region $C'XA'Y$ as shown.



Find the area of the overlapping polygon $C'XA'Y$.

We received 14 correct solutions and 4 incorrect submissions. We present the solution by Skidmore College Problem Group.

We claim that the area of $C'XA'Y$ is $\dfrac{2160}{169}$ square units.

Note first that
$$\angle CBL = \angle C'BL = \angle ADK = \angle KDA'.$$

Thus
$$\angle ABL = 90^\circ - \angle CBL = \angle AKD = 90^\circ - \angle ADK.$$

But $\angle ABL = \angle AKD$, implying that DK and BL are parallel. Thus, by alternating interior angles, $\angle KC'B = \angle C'BL$. Now

$$\angle C'XK = 180^\circ - \angle KCB' - \angle C'KX = 90^\circ$$

and $\angle XC'L = \angle KA'D = 90^\circ$, so $C'B$ and DA' are parallel, implying that $C'XA'Y$ is a rectangle.

We know that $AB = DC = 12$, $BC = AD = 9$, and $CL = 6$. Set $a = A'Y = C'X$ and $b = XA' = C'Y$. Then $DY^2 + YL^2 = DL^2$, so

$$(9-a)^2 + (6-b)^2 = 36,$$

i.e.,

$$a^2 - 18a + b^2 - 12b + 81 = 0.$$

But $\Delta C'DY$ is similar to $\Delta KDA'$, so

$$\frac{9-a}{9} = \frac{b}{6},$$

giving $a = 9 - \frac{3}{2}b$. Thus

$$\left(9 - \frac{3}{2}b\right)^2 - 18\left(9 - \frac{3}{2}b\right) + b^2 - 12b + 81 = 0,$$

which, after simplifying, is $\dfrac{13}{4}b^2 - 12b = 0$, implying that $b = \dfrac{48}{13}$ and thus $a = \dfrac{45}{13}$. This gives the area of $\dfrac{2160}{169}$.

4358. *Proposed by George Stoica.*

Let $f : [0, \infty) \to [0, \infty)$ be a non-increasing function such that

$$\int_0^\infty f(x)\sin(2\pi x)dx = 0.$$

Prove that f is constant on each of the intervals $(n, n+1)$, $n \in \mathbb{N}$.

We received 13 submissions, including the one from the proposer. Among them 12 are correct. The other solver apparently misread the question and assumed that the function is non-decreasing instead, but arrived at the same conclusion! We present the solution by Kee-Wai Lau.

For any $k \in \mathbb{N}$, let $a_k = \int_0^k f(x)\sin(2\pi x)dx$. Then

$$\begin{aligned} a_k &= \sum_{n=0}^{k-1}\left(\int_n^{n+1/2} f(x)\sin(2\pi x)dx + \int_{n+1/2}^{n+1} f(x)\sin(2\pi x)dx\right) \\ &= \sum_{n=0}^{k-1}(J_n + K_n) = \sum_{n=0}^{k-1} I_n, \end{aligned}$$

where $I_n = J_n + K_n$, $J_n = \int_n^{n+1/2} f(x)\sin(2\pi x)dx$, and $K_n = \int_{n+1/2}^{n+1} f(x)\sin(2\pi x)dx$. Substituting $x = n + y$ and $x = n + y + 1/2$ into J_n and K_n, respectively, we then obtain

$$I_n = \int_0^{1/2} (f(n+y) - f(n+y+1/2))\sin(2\pi y)dy. \qquad (1)$$

Since f is non-increasing, we have $I_n \geq 0$ and since $\lim_{k\to\infty} a_k = 0$, we obtain $I_n = 0$ for any $n \in \mathbb{N} \cup \{0\}$. It follows from (1) that except for a set of measure zero,

$$f(n+y) - f(n+y+1/2) = 0 \text{ for } y \in (0, 1/2).$$

Let $\epsilon \in (0, 1/2)$. Then there exist

$$t_0 \in (n+1/2, n+1/2+\epsilon) \quad \text{and} \quad t_1 \in (n+1-\epsilon, n+1)$$

such that $f(t_0) = f(t_0 - 1/2)$ and $f(t_1) = f(t_1 - 1/2)$. Since f is non-increasing,

$$\begin{aligned} f(y) = f(n+1/2) \text{ for } y &\in [n+\epsilon, n+1/2] \cup [n+1/2, n+1-\epsilon] \\ &= [n+\epsilon, n+1-\epsilon]. \end{aligned}$$

Letting $\epsilon \to 0^+$ we then obtain $f(y) = f(n+1/2)$ for all $y \in (n, n+1)$. Thus, f is constant on each of the intervals $(n, n+1)$, $n \in \mathbb{N}$.

4359. *Proposed by Daniel Sitaru.*

Let a, b and c be positive real numbers. Prove that

$$3\ln(a^b + b^c + c^a) + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c + \ln 27.$$

We received 10 submissions, including the one from the proposer, all of which are correct. We present a composite based on the solutions given by Richard B. Eden and Ramanujan Srihari, which are similar to all the other solutions submitted.

Let $f(x) = \ln x$, $x > 0$. Then $f''(x) = -\dfrac{1}{x^2} < 0$ so f is concave. By Jensen's Inequality we then have

$$3\ln\left(\frac{a^b + b^c + c^a}{3}\right) \geq b\ln a + c\ln b + a\ln c \tag{1}$$

with equality if and only if $a = b = c$.

Next, consider $g(x) = \ln x + \frac{1}{x} - 1, x > 0$. Then $g'(x) = \dfrac{x-1}{x^2}$ which implies $g(x) \geq g(1) = 0$ so $\ln x \geq 1 - \frac{1}{x}$ for all $x > 0$. Hence,

$$b\ln a \geq b(1 - \tfrac{1}{a}), \quad c\ln b \geq c(1 - \tfrac{1}{b}), \quad a\ln c \geq a(1 - \tfrac{1}{c}). \tag{2}$$

From (1) and (2), we then obtain

$$3\ln(a^b + b^c + c^a) \geq b(1 - \tfrac{1}{a}) + c(1 - \tfrac{1}{b}) + a(1 - \tfrac{1}{c}) + \ln 27$$

so

$$3\ln(a^b + b^c + c^a) + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c + \ln 27$$

follows, completing the proof.

4360. *Proposed by H. A. ShahAli.*

Let a, b, c be non-negative real numbers such that $a+b+c=1$. Find the minimum and maximum values of the expression

$$\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac}.$$

When do those extreme values occur?

We received 9 correct solutions and 3 incorrect submissions. We present the solution by Paolo Perfetti, modified by the editor.

We note first that

$$\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac} \leq a+b+b+c+c+a=2,$$

with equality for $a=1$, $b=c=0$ and its permutations, so the maximum is 2.

The minimum is $9/5$ and it attained for $a=b=c=1/3$ or $a=b=1/2$, $c=0$ and its permutations. To prove this we show that

$$\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac} \geq \left[\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac}\right]\Big|_{\frac{a+b}{2},\frac{a+b}{2},c=0}.$$

Equivalently, we need to prove that

$$\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac} \geq \frac{a+b}{1+\frac{(a+b)^2}{4}}+\frac{a+b}{2}+\frac{a+b}{2}=\frac{a+b}{1+\frac{(a+b)^2}{4}}+1.$$

To do so, it suffices to show that

$$\frac{a+b}{1+ab} \geq \frac{a+b}{1+\frac{(a+b)^2}{4}} \quad \text{and} \quad \frac{b+c}{1+bc}+\frac{a+c}{1+ac} \geq 1.$$

The first of these follows from the AM-GM inequality since $ab \leq \frac{(a+b)^2}{4}$. The second can be successively rewritten as

$$\begin{aligned}
&(b+c)(1+ac)+(a+c)(1+bc) \geq (1+bc)(1+ac)\\
&(a+b)+2c+2abc+(a+b)c^2-1-c(a+b)-abc^2 \geq 0\\
&1+c+2abc+(a+b)c^2-1-c(a+b)-abc^2 \geq 0\\
&c[1+ab(2-c)+c(1-c)-(1-c)] \geq 0\\
&c[ab(2-c)+c(2-c)] \geq 0\\
&c(2-c)(ab+c) \geq 0,
\end{aligned}$$

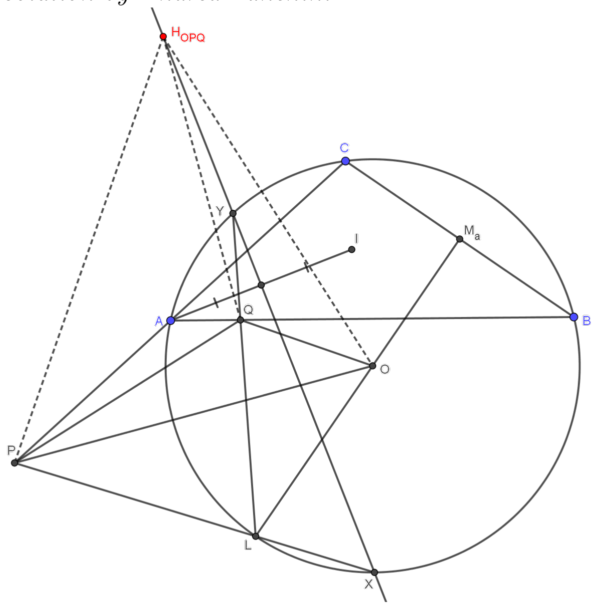
which is true. It follows that

$$\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac} \geq \left[\frac{a+b}{1+ab}+\frac{b+c}{1+bc}+\frac{a+c}{1+ac}\right]\Big|_{\frac{1}{2},\frac{1}{2},0}=\frac{9}{5}.$$

4361. *Proposed by Andrew Wu.*

Let ABC be a scalene triangle with circumcircle Γ, circumcenter O and incenter I. Suppose that L is the midpoint of the arc BAC of Γ. The perpendicular bisector of AI meets at X the arc AC that contains B, and at Y the arc AB that contains C. Let XL and AC meet at P; let YL and AB meet at Q. Show that the orthocenter of triangle OPQ lies on XY.

Three of the four submissions that we received were correct, and one was flawed. We feature the solution by Andrea Fanchini.



We use barycentric coordinates with respect to triangle ABC, where its circumcircle Γ is described by the equation $a^2yz + b^2zx + c^2xy = 0$. It will turn out that the orthocenter of ΔOPQ, call it H', is the intersection of XY and BC.

The midpoint L of the arc BAC of Γ is

$$L = OM_a \cap \Gamma = \left(a^2 : -b(b-c) : c(b-c)\right).$$

The perpendicular bisector of AI is

$$XY : bcx - c(a+c)y - b(a+b)z = 0,$$

whence the points X and Y are

$$X = XY \cap \Gamma = \left(a(a+b) : b(a+b) : -c^2\right),$$
$$Y = XY \cap \Gamma = \left(a(a+c) : -b^2 : c(a+c)\right).$$

Therefore, the lines XL and YL are

$$XL : c(c-b)x + acy + a(a+b)z = 0,$$
$$YL : b(b-c)x + a(a+c)y + abz = 0,$$

so that the points P and Q are

$$P = XL \cap AC = (a(a+b) : 0 : c(b-c)),$$
$$Q = YL \cap AB = (a(a+c) : b(c-b) : 0).$$

Finally, two altitudes of ΔOPQ are given by the polars with respect to Γ of Q and of P, namely

$$POQ_{\infty\perp} \quad : \quad bc(b-c)x - ac(a+c)y - ab(a+b)z = 0, \quad \text{and}$$
$$QOP_{\infty\perp} \quad : \quad bc(b-c)x + ac(a+c)y + ab(a+b)z = 0.$$

Their intersection is the orthocenter of triangle OPQ, namely

$$H'\,(0 : -b(a+b) : c(a+c)).$$

One easily checks that it lies on XY (obtained above), as desired. As a bonus, we see that H' also lies on the line BC.

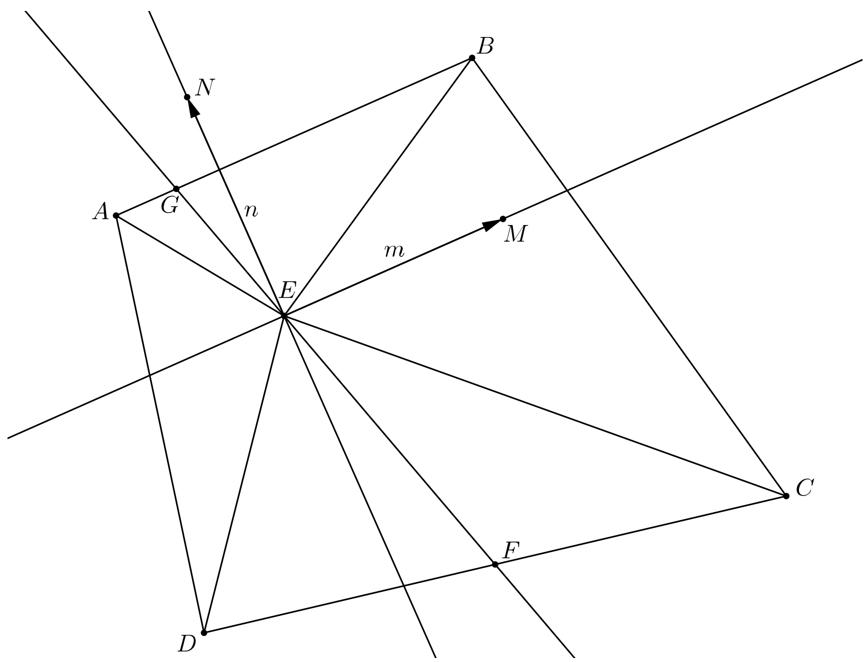
Editor's comments. Note the relationship between our Problem 4361 and Brocard's theorem that the center of a circle is the orthocenter of triangles that are self-polar with respect to the circle (meaning each side of the triangle is the pole of the opposite vertex): $\Delta PQH'$ is an appropriate self-polar triangle with respect to Γ, therefore its orthocenter is O (which implies, of course, that H' is the orthocenter of ΔOPQ). Bataille observed that the polar of H', namely the line PQ, contains the incenter I. That can easily be confirmed using the coordinates of our featured solution: subtract the coordinates of Q from those of P and you get the coordinates of $I(a : b : c)$.

4362. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral and let F be the midpoint of CD. Consider a point E inside $ABCD$ such that $AE \cdot CE = BE \cdot DE$. The lines EF and AB intersect at G. If $\angle AED + \angle CEB = 180°$, prove that $\angle AED = \angle AGE$.

We received 4 solutions and will feature just one of them here, by Michel Bataille.

Let lines m and n pass through E and be parallel and perpendicular to AB, respectively. Let M on m and N on n be such that $EM = EN = 1$. Consider m as the x-axis and n as the y-axis of a system of axes with origin at E. Then to each point is assigned a complex affix. If y_0 denotes the distance from E to the line AB, the affixes a and b of A and B are $a = x_1 + iy_0$ and $b = x_2 + iy_0$ for some real numbers x_1, x_2. Let $\alpha = \angle AED$. Then $\alpha \in (0, \pi)$ and we may suppose that M and N are chosen such that the oriented angle $\angle(\overrightarrow{EA}, \overrightarrow{ED})$ equals α, and then, by assumption $\angle(\overrightarrow{EC}, \overrightarrow{EB}) = \pi - \alpha$ (see the figure on the next page).



Let c and d denote the affixes of C and D. Since $AE \cdot CE = BE \cdot DE$, we have $\left|\frac{d}{a}\right| = \left|\frac{c}{b}\right| = \rho$, say, so that

$$\frac{d}{a} = \rho e^{i\alpha} \quad \text{and} \quad \frac{b}{c} = \frac{1}{\rho} e^{i(\pi-\alpha)}.$$

We deduce

$$\frac{c}{b} = -\rho e^{i\alpha}$$

and the affix f of F is given by

$$f = \frac{c+d}{2} = \rho e^{i\alpha}(a-b) = \rho e^{i\alpha}(x_1 - x_2).$$

From the equation

$$e^{-i\alpha} z - e^{i\alpha}\overline{z} = 0$$

of EF and $z - \overline{z} = 2iy_0$ of AB, we readily obtain the affix g of G:

$$g = \frac{y_0 e^{i\alpha}}{\sin\alpha}.$$

Observing that $y_0 > 0$ and $\sin\alpha > 0$, we see that α is an argument of g, which means that $\angle(\overrightarrow{EM}, \overrightarrow{EG}) = \alpha$. Since $AG \parallel EM$, it follows that $\angle(\overrightarrow{GA}, \overrightarrow{GE}) = \alpha$ and so $\angle AGE = \angle ADE$, as required.

4363. *Proposed by Michel Bataille.*

Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 > 0$ and the recursion

$$a_{n+1} = \frac{a_n}{1+(n+1)a_n^2}.$$

Prove that the series $\sum\limits_{n=0}^{\infty} a_n^2$ is convergent and find $\lim\limits_{n\to\infty}\left(n\cdot\sum\limits_{k=n}^{\infty} a_k^2\right)$.

We received 7 solutions. We present the solution by Oliver Geupel, lightly edited.

The sequence $b_n = (n+1)a_n$, where $n \geq 0$, satisfies the recursion

$$b_{n+1} = (n+2)b_n/(n+1+b_n^2).$$

Note that $b_n > 0$ for all n. We show by induction that $b_n \leq 1$ for $n \geq 1$. First, we have $b_1 = 2b_0/(1+b_0^2) \leq 1$, which is the base case. Assuming the assertion holds for some index n, we obtain

$$b_{n+1} = \frac{nb_n + 2b_n}{n+1+b_n^2} \leq \frac{n+(1+b_n^2)}{n+1+b_n^2} = 1,$$

and the induction is complete. Hence $a_n \leq \frac{1}{n+1}$ for $n \geq 1$, and we obtain

$$\sum_{n=0}^{\infty} a_n^2 \leq a_0^2 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = a_0^2 + \frac{\pi^2}{6} - 1;$$

that is, $\sum\limits_{n=0}^{\infty} a_n^2$ converges.

For $n \geq 1$, note that

$$b_{n+1} = \frac{(n+2)b_n}{n+1+b_n^2} \geq \frac{(n+2)b_n}{n+2} = b_n.$$

The sequence $(b_n)_{n\geq 1}$ is thus increasing and bounded above by 1, which implies that it has a limit $0 < L \leq 1$. From the recursion formula for b_{n+1} (taking the limit as $n \to \infty$ of both sides) we get $L = \dfrac{(n+2)L}{n+1+L^2}$. The only positive root of this equation is $L = 1$, which must be the desired limit.

Note that

$$n \cdot \sum_{k=n}^{\infty} a_k^2 = n \cdot \sum_{k=n}^{\infty} \frac{b_k^2}{(k+1)^2}.$$

For every $\varepsilon > 0$ there is an index n_0 such that, whenever $n > n_0$, we have $1-\varepsilon < b_n \leq 1$. Moreover, for $n > 0$,

$$\frac{1}{n+1} = \int_{n+1}^{\infty} x^{-2}\,dx < \sum_{k=n}^{\infty} \frac{1}{(k+1)^2} < \int_{n}^{\infty} x^{-2}\,dx = \frac{1}{n}.$$

Hence for $n > n_0$,

$$\frac{n}{n+1} \cdot (1-\varepsilon)^2 < n \cdot \sum_{k=n}^{\infty} a_k^2 \leq 1,$$

and so $\lim\limits_{n\to\infty} \left(n \cdot \sum\limits_{k=n}^{\infty} a_k^2 \right) = 1$.

4364. *Proposed by George Stoica.*

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a binary operation, and define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(a,b) = \begin{cases} a \text{ if } b = 1 \\ f(g(a,b-1),a) \text{ if } b \geq 2. \end{cases}$$

If f is associative and g is commutative, prove that $f(a,b) = a+b$ and $g(a,b) = ab$.

We received 6 solutions. We present the solution by the Missouri State University Problem Solving Group.

Note that, since g is commutative, from the definition of g it follows that $g(1,a) = a$ for all $a \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Then

$$n + 1 = g(1, n+1) = f(g(1,n),1) = f(n,1).$$

That is,

$$f(n,1) = n + 1 \tag{1}$$

for every $n \in \mathbb{N}$.

We now show that $f(n,m) = n + m$. Fix n and use induction on $m \geq 1$.

$$\begin{aligned} f(n,m+1) &= f(n,f(m,1)) && \text{(by (1), the base case)} \\ &= f(f(n,m),1) && \text{(by associativity of } f\text{)} \\ &= f(n,m) + 1 && \text{(by (1))} \\ &= n + m + 1 && \text{(by the induction hypothesis).} \end{aligned}$$

Next we show $g(n,m) = nm$. Once again fix n and use induction on $m \geq 1$. Note that $g(n,1) = n \cdot 1$ by the definition of g.

$$\begin{aligned} g(n,m+1) &= f(g(n,m),n) && \text{(by the definition of } g\text{)} \\ &= f(nm,n) && \text{(by the induction hypothesis)} \\ &= nm + n && \text{(by the formula for } f \text{ proved above)} \\ &= n(m+1). \end{aligned}$$

This completes the proof.

4365. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let a and b be real numbers such that $a+b, a^4$ and b^4 are rational numbers and $a+b \neq 0$. Prove that a and b are rational numbers.

We received 7 solutions and will feature the solution by Madhav Modak.

Let $a+b=r$, $a^4=c$ and $b^4=d$. Then by assumption, r, c, d are rational numbers and $r \neq 0$. Eliminating b from $a+b=r$ and $b^4=d$, we get $(r-a)^4=d$ or

$$a^4-4ra^3+6r^2a^2-4r^3a+r^4=d.$$

Hence

$$-4ar(a^2+r^2)=t-6a^2r^2, \quad \text{where} \quad t=d-r^4-c. \tag{1}$$

Squaring (1), we get:

$$\begin{aligned}
&16a^2r^2(c+r^4+2a^2r^2)=t^2+36cr^4-12a^2r^2t,\\
&a^2(16cr^2+16r^6)+32cr^4=t^2+36cr^4-12a^2r^2t,\\
&a^2(16cr^2+16r^6+12r^2t)=t^2+4cr^4,\\
&a^2(16cr^2+16r^6+12r^2d-12r^6-12cr^2)=t^2+4cr^4,\\
&a^2(4cr^2+4r^6+12r^2d)=t^2+4cr^4. \qquad (2)
\end{aligned}$$

Now $r \neq 0$ so that $r^6>0$ and so $4cr^2+4r^6+12r^2d>0$ as $c, d \geq 0$. Hence by (2),

$$a^2=\frac{t^2+4cr^4}{4cr^2+4r^6+12r^2d},$$

so that a^2 is rational. Then by (1),

$$a=(6a^2r^2-t)/4r(a^2+r^2),$$

as $4r(a^2+r^2) \neq 0$, we see that a is rational, hence $b=r-a$ is rational, as was to be shown.

4366. *Proposed by Daniel Sitaru.*

Let x_n be the base angle of a right triangle with base n and altitude 1. Find

$$\sum_{k=1}^{\infty} x_{k^2+k+1}.$$

There were 15 correct solutions, all variants of the following two solutions.

Solution 1.

The arms of the right triangle have lengths 1 and n, and x_n is the angle adjacent to the latter arm. Thus, $x_n=\arctan\frac{1}{n}$. Observe that

$$\tan(x_k-x_{k+1})=\frac{\frac{1}{k}-\frac{1}{k+1}}{1+\frac{1}{k(k+1)}}=\frac{1}{k^2+k+1}=\tan(x_{k^2+k+1}).$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} x_{k^2+k+1} &= \sum_{k=1}^{n}(x_k - x_{k+1})\\ &= x_1 - x_{n+1}\\ &= \frac{\pi}{4} - \arctan\frac{1}{n+1},\end{aligned}$$

so that

$$\sum_{k=1}^{\infty} x_{k^2+k+1} = \frac{\pi}{4}.$$

Solution 2.

Let

$$u_n = \tan\left(\sum_{k=1}^{n} x_{k^2+k+1}\right).$$

Checking the values of u_n for small values of n, we are led to the conjecture that $u_n = n(n+2)$. Suppose that this holds for $n = m-1$. Then

$$\begin{aligned}u_n &= \tan\left(x_{m^2+m+1} + \sum_{k=1}^{m-1} x_{k^2+k+1}\right)\\ &= \frac{\dfrac{1}{m^2+m+1} + \dfrac{m-1}{m+1}}{1 - \dfrac{m-1}{(m+1)(m^2+m+1)}} = \frac{m(m^2+1)}{(m+2)(m^2+1)} = \frac{m}{m+2}.\end{aligned}$$

Thus, an induction argument, along with $u_1 = 1/3$, establishes that $u_n = n/(n+2)$ for each positive integer n. Since the limit as n tends to infinity of u_n is 1, the sum of the given series is $\frac{\pi}{4}$.

4367. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let a, b and c be distinct complex numbers such that $|a| = |b| = |c| = 1$ and $|a+b+c| \le 1$. Prove that

$$\left|\frac{(a+b)(b+c)}{(a-b)(b-c)}\right| + \left|\frac{(b+c)(c+a)}{(b-c)(c-a)}\right| + \left|\frac{(c+a)(a+b)}{(c-a)(a-b)}\right| = 1.$$

We received 3 correct solutions. We present the solution by Michel Bataille.

Let $u = \dfrac{(c+a)(a+b)}{(c-a)(a-b)}$, $v = \dfrac{(a+b)(b+c)}{(a-b)(b-c)}$, $w = \dfrac{(b+c)(c+a)}{(b-c)(c-a)}$.

If for example $b+c = 0$, then $v = w = 0$ and $u = -1$, hence $|u| + |v| + |w| = 1$ is satisfied.

In what follows, we suppose that $b+c, c+a, a+b$ are nonzero complex numbers.

Since the conjugate of a, b, c are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, respectively, an easy calculation gives $\overline{u} = u, \overline{v} = v, \overline{w} = w$, meaning that u, v, w are real numbers.

Similarly, $z = \dfrac{(a+b)(b+c)(c+a)}{(a-b)(b-c)(c-a)}$ is a nonzero purely imaginary number and so $uvw = z^2$ is a negative real number.

The three following identities are readily checked:

$$(a+b)(b+c)(c-a)+(b+c)(c+a)(a-b)+(c+a)(a+b)(b-c) = -(a-b)(b-c)(c-a) \quad (1)$$

$$(a+b)(b-c)(c-a)+(b+c)(c-a)(a-b)+(c+a)(a-b)(b-c) = 8abc-(a+b)(b+c)(c+a) \quad (2)$$

$$(a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc. \quad (3)$$

Identity (1) shows that $u+v+w = -1$ while (2) easily gives

$$uv + vw + wu = z^2 \left(\frac{8abc}{(a+b)(b+c)(c+a)} - 1 \right). \quad (4)$$

But the condition $|a+b+c| \leq 1$ writes as

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \leq 1,$$

that is,

$$\frac{(a+b)(b+c)(c+a)+abc}{abc} \leq 1$$

(using (3)) or finally

$$\frac{(a+b)(b+c)(c+a)}{abc} \leq 0.$$

From this and (4), we deduce that $uv + vw + wu > 0$.

Now, let

$$p(x) = (x-u)(x-v)(x-w) = x^3 + x^2 + (uv+vw+wu)x - uvw.$$

The derivative

$$p'(x) = 3x^2 + 2x + (uv + vw + wu)$$

is positive on $[0, \infty)$ (since $uv + vw + wu > 0$), hence p is an increasing function on $[0, \infty)$ and therefore

$$p(x) > p(0) = -uvw > 0$$

for $x \in [0, \infty)$. Thus, the roots u, v, w of $p(x)$ are negative real numbers and so

$$|u| + |v| + |w| = -u - v - w = -(u+v+w) = -(-1) = 1,$$

as desired.

4368. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate
$$\sum_{n=2}^{\infty}\left[2^n\left(\zeta(n)-1\right)-1\right],$$
where ζ denotes the Riemann zeta function defined as $\zeta(z)=\sum\limits_{n=1}^{\infty}\frac{1}{n^z}$.

We received 9 submissions, all correct, although one used WolframAlpha software. We present the solution by Paolo Perfetti, which is short, simple, and similar to the other seven non-WolframAlpha solutions.

$$\begin{aligned}\sum_{n=2}^{\infty}[2^n(\zeta(n)-1)-1] &= \sum_{n=2}^{\infty}\left[2^n\sum_{k=2}^{\infty}\frac{1}{k^n}-1\right]\\ &=\sum_{n=2}^{\infty}\sum_{k=3}^{\infty}\frac{2^n}{k^n}\\ &=\sum_{k=3}^{\infty}\sum_{n=2}^{\infty}\frac{2^n}{k^n}\\ &=\sum_{k=3}^{\infty}\frac{4}{k^2}\frac{1}{1-\frac{2}{k}}\\ &=2\sum_{k=3}^{\infty}\left[\frac{1}{k-2}-\frac{1}{k-1}+\frac{1}{k-1}-\frac{1}{k}\right]\\ &\underbrace{=}_{telescoping}\quad 2(1+\frac{1}{2})=3.\end{aligned}$$

4369. *Proposed by Mihaela Berindeanu.*

On the sides of triangle ABC, take points
$$A_1,\ A_2\in(BC)\,,\ B_1,\ B_2\in(AC)\,,\ C_1,\ C_2\in(AB)\,,$$
so that
$$BA_1=A_2C,\ CB_1=B_2A,\ AC_1=C_2B.$$
On $B_2C_1,\ A_1C_2,\ A_2B_1$ take $A_3,\ B_3,\ C_3$ so that
$$\frac{C_1A_3}{A_3B_2}=\frac{A_1B_3}{B_3C_2}=\frac{B_1C_3}{C_3A_2}=k.$$
Find all values of k for which $AA_3,\ BB_3,\ CC_3$ are concurrent lines.

We received 5 correct submissions. We present the solution by Michel Bataille.

Let $\alpha > 0$ be such that $\overrightarrow{BA_1} = \alpha\overrightarrow{BC} = \overrightarrow{A_2C}$. Then,

$$A_1 = (1-\alpha)B + \alpha C \quad \text{and} \quad A_2 = \alpha B + (1-\alpha)C.$$

In a similar way, there exist $\beta, \gamma > 0$ such that

$$\begin{aligned} B_1 &= \beta A + (1-\beta)C, \\ B_2 &= (1-\beta)A + \beta C, \\ C_1 &= (1-\gamma)A + \gamma B, \\ C_2 &= \gamma A + (1-\gamma)B. \end{aligned}$$

Expressing that $\overrightarrow{C_1A_3} = k\overrightarrow{A_3B_2}$, we obtain $(1+k)A_3 = kB_2 + C_1$, that is,

$$(1+k)A_3 = \rho A + \gamma B + k\beta C,$$

where $\rho = 1 + k - \gamma - k\beta$. Similarly,

$$\begin{aligned} (1+k)B_3 &= k\gamma A + \sigma B + \alpha C, \\ (1+k)C_3 &= \beta A + k\alpha B + \tau C, \end{aligned}$$

where $\sigma, \tau \in \mathbb{R}$. It readily follows that the equations of the lines AA_3, BB_3, CC_3 respectively are

$$k\beta y - \gamma z = 0, \quad \alpha x - k\gamma z = 0, \quad k\alpha x - \beta y = 0.$$

Now, the common point of BB_3 and CC_3 is $J = (k\beta\gamma : k^2\alpha\gamma : \alpha\beta)$ (not at infinity since $k\beta\gamma + k^2\alpha\gamma + \alpha\beta$ is positive, hence not zero), and this point J is on AA_3 if and only if $k\beta(k^2\alpha\gamma) - \gamma(\alpha\beta) = 0$, that is, if and only if $k^3 = 1$.

We conclude that AA_3, BB_3, CC_3 are concurrent if and only if $k = 1$ (which means that A_3, B_3, C_3 are the midpoints of B_2C_1, C_2A_1, A_2B_1, respectively).

4370. *Proposed by Leonard Giugiuc and Sladjan Stankovik.*

Solve the following system of equations:

$$\begin{cases} a+b+c+d = 4, \\ a^2+b^2+c^2+d^2 = 7, \\ abc+abd+acd+bcd-abcd = \frac{15}{16}. \end{cases}$$

We received 10 submissions, including the one from the proposers, 9 of which are correct, and the other one, incomplete. We present the same solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith (jointly); and Ramanujan Srihari.

Note first that

$$\begin{aligned} 16 &= (a+b+c+d)^2 \\ &= a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd) \\ &= 7+2(ab+ac+ad+bc+bd+cd) \end{aligned}$$

so

$$ab + ac + ad + bc + bd + cd = \frac{9}{2}.$$

Letting $w = a - 1$, $x = b - 1$, $y = c - 1$, and $z = d - 1$, we then have

$$w + x + y + z = 0$$

and

$$\begin{aligned} wxyz &= (a-1)(b-1)(c-1)(d-1) \\ &= (abcd) - (abc + abd + acd + bcd) \\ &\quad + (ab + ac + ad + bc + bd + cd) - (a + b + c + d) + 1 \\ &= -\frac{15}{16} + \frac{9}{2} - 4 + 1 \\ &= \frac{9}{16}. \end{aligned}$$

Also,

$$\begin{aligned} w^2 + x^2 + y^2 + z^2 &= (a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 \\ &= a^2 + b^2 + c^2 + d^2 - 2(a + b + c + d) + 4 \\ &= 7 - 8 + 4 \\ &= 3. \end{aligned}$$

By the AM-GM Inequality, we have

$$3 = w^2 + x^2 + y^2 + z^2 \geq 4\sqrt[4]{w^2x^2y^2z^2} = 4\sqrt{\frac{9}{16}} = 3$$

with equality if and only if

$$w^2 = x^2 = y^2 = z^2 = \frac{3}{4}.$$

Hence,

$$(w, x, y, z) = \left(\pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}\right).$$

Since $w + x + y + z = 0$, we then easily see that exactly two of w, x, y, and z must be $\frac{\sqrt{3}}{2}$, and the other two equal -$\frac{\sqrt{3}}{2}$. Therefore, there are six solutions given by

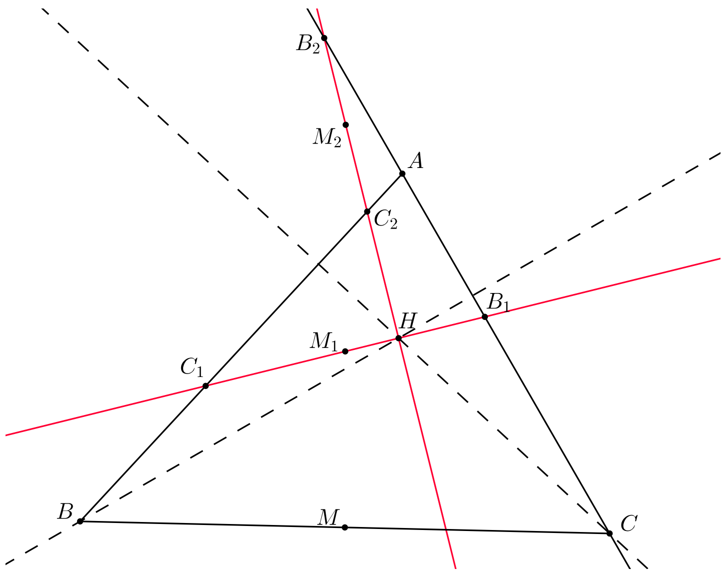
$$(a, b, c, d) = (1 + \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2})$$

together with its five permutations.

4371. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let two perpendicular lines pass through the orthocenter H of a triangle ABC. Suppose they meet the sides AB and AC in C_1, B_1 and C_2, B_2, respectively. Define M_i as the midpoint of B_iC_i $(i = 1, 2)$ and M as the midpoint of BC. Prove that M, M_1 and M_2 are collinear.

We received 6 submissions, of which 5 were complete and correct. We present the solution by Michel Bataille.



The perpendicular lines through H define a system of orthonormal axes with origin at H. We assign the coordinates

$$B_1(2b_1, 0),\ C_1(2c_1, 0),\ B_2(0, 2b_2),\ C_2(0, 2c_2)$$

for some real numbers b_1, b_2, c_1, c_2. Then, $M_1(b_1 + c_1, 0)$, $M_2(0, b_2 + c_2)$ and the equation of the line M_1M_2 is

$$(b_2 + c_2)x + (b_1 + c_1)y = (b_1 + c_1)(b_2 + c_2). \tag{1}$$

The equation of the line $AB = C_1C_2$ is $c_2x + c_1y = 2c_1c_2$. The altitude from B is perpendicular to $AC = B_1B_2$ and passes through the origin H, so its equation is $b_1x - b_2y = 0$. The lines AB and BH intersect in B, so we calculate

$$B\left(\frac{2c_1c_2b_2}{b_1c_1 + b_2c_2}, \frac{2c_1c_2b_1}{b_1c_1 + b_2c_2}\right).$$

Similarly (*mutatis mutandis*), we have

$$C\left(\frac{2b_1b_2c_2}{b_1c_1 + b_2c_2}, \frac{2b_1b_2c_1}{b_1c_1 + b_2c_2}\right).$$

Thus the midpoint M of BC has coordinates

$$x_M = \frac{b_2c_2(b_1+c_1)}{b_1c_1+b_2c_2} \quad \text{and} \quad y_M = \frac{b_1c_1(b_2+c_2)}{b_1c_1+b_2c_2};$$

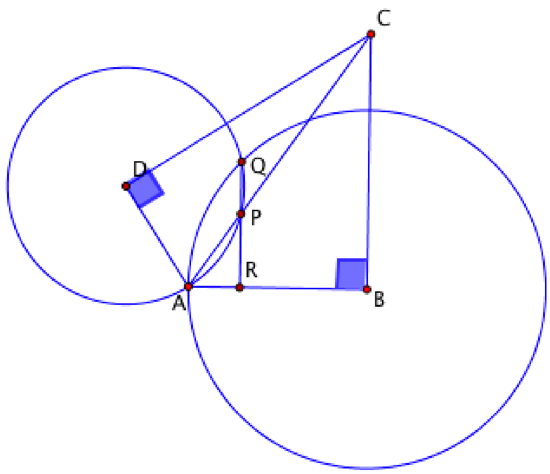
and

$$\begin{aligned}(b_2+c_2)x_M + (b_1+c_1)y_M &= \frac{b_2c_2(b_1+c_1)(b_2+c_2)}{b_1c_1+b_2c_2} + \frac{b_1c_1(b_1+c_1)(b_2+c_2)}{b_1c_1+b_2c_2} \\ &= (b_1+c_1)(b_2+c_2)\end{aligned}$$

and from (1) we obtain that M is on the line M_1M_2, as desired.

4372. *Proposed by J. Chris Fisher.*

Given a quadrangle $ABCD$ with right angles at B and D, let the circle (D, DA) (with center D and radius DA) intersect the line AC at P and the circle (B, BA) at Q. Prove that PQ is perpendicular to AB.



We received 7 submissions, of which 5 were correct and complete. We present a solution based on the submission by Sushanth Sathish Kumar, completed and corrected by the editor.

Since AQ is the radical axis of circles (B, BA) and (D, DA), the line DB bisects $\angle QDA$. By the inscribed angle theorem,

$$\angle QPA = \frac{360^\circ - \angle QDA}{2} = 180^\circ - \angle BDA.$$

Since $\angle QPC$ and $\angle QPA$ are supplementary, it follows that $\angle QPC = \angle BDA$. Quadrilateral $ABCD$ is cyclic, so also $\angle BDA = \angle ACB$. Therefore, we have $\angle QPC = \angle PCB$ (since $\angle PCB$ and $\angle ACB$ are the same angle), which implies $\overline{PQ} \parallel \overline{CB}$. From this we can conclude $PQ \perp AB$, as desired.

4373. *Proposed by Michel Bataille.*

Let p be an odd prime. Let q and r be the quotient and the remainder in the division of the positive integer n by p and let $S_n = \sum_{k=r}^{n}(-1)^{k-r}\binom{k}{r}$. Show that $S_n \equiv 0 \pmod{p}$ if and only if q is odd and $2^r \equiv 1 \pmod{p}$.

We received three submissions, including the one from the proposer. We present the solution by Modhav R. Modak, modified slightly by the editor.

By assumption, we have $n = qp + r$. For $i = 0, 1, 2, \ldots, q-1$, let

$$A_i = \sum_{k=0}^{p-1}(-1)^k\binom{r+ip+k}{r}.$$

Since p is odd, $(-1)^{ip} = 1$ or -1 depending on whether i is even or odd. Hence we can write

$$\begin{aligned} S_n &= \sum_{k=0}^{p-1}(-1)^k\binom{r+k}{r} + \sum_{k=p}^{2p-1}(-1)^k\binom{r+k}{r} + \cdots + \sum_{k=q}^{qp-1}(-1)^k\binom{r+k}{r} \\ &= A_0 - A_1 + \cdots + (-1)^{qp-1}A_{q-1} + (-1)^{qp}\binom{r+qp}{r}. \qquad (1) \end{aligned}$$

Since $p > 2$ and $r < p$, we have for $r \le k$,

$$(k+p)(k+p-1)\cdots(k+p-r+1) \equiv (k)(k-1)\cdots(k-r+1) \pmod{p}$$

$$\text{so that } \binom{k+p}{r} \equiv \binom{k}{r} \pmod{p} \quad \text{since } p \not| r!. \qquad (2)$$

By (2) we then have, for $i = 0, 1, 2, \ldots, q-1$, that

$$A_i \equiv \binom{r}{r} - \binom{r+1}{r} + \cdots + \binom{r+p-1}{r} \pmod{p}$$

$$\text{i.e.} \quad A_i \equiv A_0 \pmod{p}.$$

Hence from (1) we get, modulo p,

$$S_n \equiv \begin{cases} 1, & \text{if } q \text{ is even,} \\ A_0 - 1 & \text{if } q \text{ is odd.} \end{cases} \qquad (3)$$

We next establish the following identity:

$$\sum_{k=0}^{m}(-1)^k\binom{r+k}{r} = \frac{1}{2^{r+1}}\left[1 + (-1)^m\left\{\sum_{k=0}^{r}2^k\binom{m+k}{k}\right\}\right], \qquad (4)$$

To prove (4), note that

$$\sum_{k=0}^{\infty}(-1)^k\binom{m+k}{k}x^k = \frac{1}{(1+x)^{m+1}}.$$

Hence

$$\sum_{k=0}^{m}(-1)^k\binom{r+k}{r} = \text{coefficient of } x^m \text{ in } f(x) = \frac{1}{(1-x)(1+x)^{m+1}}.$$

Now $f(x)$ can be decomposed into partial fractions as

$$f(x) = \frac{1}{2^{r+1}}\left[\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{(1+x)^2} + \frac{2^2}{(1+x)^3} + \cdots + \frac{2^r}{(1+x)^{r+1}}\right] \qquad (5)$$

since the right hand side of (5) can be written as

$$= \frac{1}{2^{r+1}}\left[\frac{1}{1-x} + \frac{1}{2}\sum_{k=1}^{r+1}\frac{2^k}{(1+x)^k}\right]$$

$$= \frac{1}{2^{r+1}}\left[\frac{1}{1-x} + \frac{1}{2}\cdot\frac{2}{1+x}\cdot\frac{(2/(1+x))^{r+1}-1}{(2/(1+x))-1}\right]$$

$$= \frac{1}{2^{r+1}}\left[\frac{1}{1-x} + \frac{2^{r+1}-(1+x)^{r+1}}{(1-x)(1+x)^{r+1}}\right] = f(x).$$

Hence (4) follows by taking the coefficient of x^m in (5).

Letting $m = p-1$ in (4), we then have

$$A_0 = \sum_{k=0}^{p-1}(-1)^k\binom{r+k}{r} = \frac{1}{2^{r+1}}\left(1 + (-1)^{p-1}\left(\sum_{k=0}^{p-1}2^k\binom{p-1+k}{k}\right)\right)$$

$$= \frac{1}{2^{r+1}}\left(1 + 1 + 2\binom{p}{1} + 2^2\binom{p+1}{2} + \cdots + 2^{p-1}\binom{2p-2}{p-1}\right), \qquad (6)$$

so

$$A_0 \equiv \frac{1}{2^r} \pmod{p},$$

since

$$\binom{p}{1}, \binom{p+1}{2}, \ldots, \binom{2p-2}{p-1}$$

are all congruent to zero modulo p.

Hence, (3) becomes

$$S_n \equiv \begin{cases} 1, \text{ if } q \text{ is even,} \\ \frac{1}{2^r}\cdot(1-2^r), \text{ if } q \text{ is odd.} \end{cases} \qquad (7)$$

Thus if q is odd and $2^r \equiv 1 \pmod{p}$, then (6) shows that $S_n \equiv 0 \pmod{p}$. Further, if q is even or if $2^r \not\equiv 1 \pmod{p}$, then $S_n \not\equiv 0 \pmod{p}$, completing the proof.

4374. *Proposed by Šefket Arslanagić.*

For a fixed positive integer n, solve the equation

$$|1 - |2 - |3 - \cdots - |\, n - x \underbrace{|\cdots\cdots|}_{n \text{ vertical bars}} = 1.$$

There were 6 correct solutions. We present the standard approach.

Let E_n denote the given equation, S_n the set of its solutions, and $T_n = \frac{1}{2}n(n+1)$, the sum of the first n positive integers. It is readily checked that $S_1 = \{0, 2\}$, $S_2 = \{0, 2, 4\}$, $S_3 = \{-1, 1, 3, 5, 7\}$. We prove by induction that

$$S_n = \{-T_{n-2}, -T_{n-2} + 2, -T_{n-2} + 4, \ldots, -T_{n-2} + 2(T_{n-1} + 1) = T_n + 1\}$$

for $n \geq 3$.

Suppose that this holds for $n = m$. Then x is a solution of E_{m+1} if and only if $|m + 1 - x|$ is a nonnegative solution of E_m, if and only if

$$|m + 1 - x| = -T_{m-2} + 2k$$

for $\frac{1}{2}T_{m-2} \leq k \leq T_{m-1} + 1$.

The smallest value assumed by $|m + 1 - x|$ for a solution x is equal to 0 when $m \equiv 1, 2 \pmod 4$ and 1 when $m \equiv 0, 3 \pmod 4$. It follows that the numbers in S_{m+1} are all consecutively even when $m \equiv 0, 1 \pmod 4$ and all consecutively odd when $m \equiv 2, 3 \pmod 4$.

The solution x assumes its smallest value when

$$x = (m + 1) - (T_m + 1) = -(T_m - m) = -T_{m-1}$$

and its largest value when

$$x = (m + 1) + (T_m + 1) = T_{m+1} + 1.$$

Thus, S_{m+1} consists of all integers between $-T_{m-1}$ and $T_{m+1} + 1$ inclusive that share the same parity.

4375. *Proposed by George Stoica.*

Consider two sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n\} \cup \{y_n\} = \mathbb{N}$. Let $l > 1$ be given. Prove that $\liminf x_n/n \geq l$ if and only if $\limsup y_n/n \leq l/(l - 1)$.

The problem is incorrect as it stands. A counterexample was offered by Oliver Geupel: let $x_n = n^2$, $y_{2n-1} = n^2$ and $y_{2n} = n$. If you want one where the two sequences are exhaustive but not overlapping take $x_n = 3n$ and $y_{2^m} = 2^{m+1}$ with the other y_n picking up the missing numbers. In this case, take $l = 3$ and $l/(l - 1) = 3/2$.

The following solution is essentially that of the proposer, with additional hypotheses to make it work.

We assume that both $\{x_n\}$ and $\{y_n\}$ are increasing, nonoverlapping and jointly exhaustive sequences of positive integers. Let t be a positive integer and let $N(t)$ be the number of indices n for which $y_n \leq t$ and $M(t)$ the number of indices n for which $x_n \leq t$. Then $M(t) + N(t) = t$.

Suppose $x_m \leq t < x_{m+1}$. Then $M(t) = m$ and

$$\frac{x_m}{m} \leq \frac{t}{M(t)} < \left(\frac{x_{m+1}}{m+1}\right)\left(\frac{m+1}{m}\right)$$

so that

$$\liminf_{t\to\infty} \frac{t}{M(t)} = \liminf_{m\to\infty} \frac{x_m}{m} = a \geq 1.$$

Similarly

$$\limsup_{t\to\infty} \frac{t}{N(t)} = \limsup_{m\to\infty} \frac{y_m}{m} = b \geq 1.$$

Since

$$\frac{M(t)}{t} + \frac{N(t)}{t} = 1$$

and $\limsup M(t)/t = 1 - \liminf N(t)/t$, we find that $\frac{1}{a} + \frac{1}{b} = 1$ and $b = \frac{a}{a-1}$.

Let l be as in the hypothesis. Then,

$$\begin{aligned}\liminf x_n/n = \liminf t/M(t) \geq l &\Leftrightarrow \limsup M(t)/t \leq 1/l \\ &\Leftrightarrow \liminf N(t)/t \geq 1 - (1/l) = (l-1)/l \\ &\Leftrightarrow \limsup t/N(t) = \limsup y_n/n \leq l/(l-1).\end{aligned}$$