OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(6), p. 243–244; and 44(7), p. 284–285.

OC386. Find all monic polynomials P, Q which are non-constant, have real coefficients and satisfy

$$2P(x) = Q\left(\frac{(x+1)^2}{2}\right) - Q\left(\frac{(x-1)^2}{2}\right)$$

and $P(1) = 1$ for all real x.

Originally Problem 2 of the 2016 Greece National Olympiad.

We received 8 *submissions, of which* 6 *were correct and complete. We present the solution by Oliver Geupel, slightly modified.*

It is straightforward to check that, for every real number c, the following pairs of polynomials P, Q are solutions to the problem:

$$P(x) = x, \qquad Q(x) = x + c, \tag{1}$$

and

$$P(x) = x^3, \qquad Q(x) = x^2 - x + c. \tag{2}$$

We prove that there are no other solutions.

Suppose polynomials P, Q satisfy the conditions of the problem where $n = \deg Q$. Since Q is monic, it has the form $Q(x) = x^n + R(x)$ with a polynomial R of degree less than or equal to $n - 1$. Then,

$$P(x) = \frac{1}{2}\left(\frac{(x+1)^{2n}}{2^n} - \frac{(x-1)^{2n}}{2^n} + R\left(\frac{(x+1)^2}{2}\right) - R\left(\frac{(x-1)^2}{2}\right)\right).$$

Expanding the powers of the binomials and observing that $\deg R \le n - 1$, we obtain that P has the degree $2n - 1$ and leading coefficient $2n/2^n$. Since P is non-constant and monic, we obtain $n > 0$ and $2n = 2^n$, that is, $n \in \{1, 2\}$.

If $n = 1$, we have $Q(x) = x + c$ and

$$P(x) = \frac{1}{2}\left(\frac{(x+1)^2}{2} - \frac{(x-1)^2}{2}\right) = \frac{1}{2} \cdot 2x = x,$$

which gives the solution (1).

If $n = 2$, the polynomial Q has the form $Q(x) = x^2 + ax + c$, with constants a and c. Hence,

$$\begin{aligned} P(x) &= \frac{1}{2}\left(\left(\frac{(x+1)^2}{2}\right)^2 - \left(\frac{(x-1)^2}{2}\right)^2 + a\left(\frac{(x+1)^2}{2} - \frac{(x-1)^2}{2}\right)\right) \\ &= x^3 + (a+1)x. \end{aligned}$$

The hypothesis $P(1) = 1$ leads to $a = -1$ and thus to the solution (2).

OC387. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

Originally Problem 1, Day 1 of the 2016 USA Math Olympiad.

We received 3 *submissions. We present the solution by Oliver Geupel.*

The smallest number of elements in S is 8.

We present a sequence $X_1, X_2, \ldots, X_{100}$ for $S = \{1, 2, \ldots, 8\}$. Let $A_0, \ldots, A_9$ and $B_0, \ldots, B_{10}$ be the following subsets of $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$, respectively:

$$\begin{array}{lllll} A_0 = \emptyset, & A_1 = \{2,4\}, & A_2 = \{1\}, & A_3 = \{3,4\}, & A_4 = \{2\}, \\ A_5 = \{1,4\}, & A_6 = \{3\}, & A_7 = \{1,2\}, & A_8 = \{4\}, & A_9 = \{2,3\}, \end{array}$$

$$\begin{array}{lllll} B_0 = \emptyset, & B_1 = \{5,7\}, & B_2 = \{6,8\}, & B_3 = \{5\}, & B_4 = \{7,8\}, \\ B_5 = \{6\}, & B_6 = \{5,8\}, & B_7 = \{7\}, & B_8 = \{5,6\}, & B_9 = \{8\}, \\ B_{10} = \{6,7\}. & & & & \end{array}$$

For $k \in \{1, 2, \ldots, 100\}$, let $X_k = A_r \cup B_s$ if $k \equiv r \pmod{10}$ and $k \equiv s \pmod{11}$.

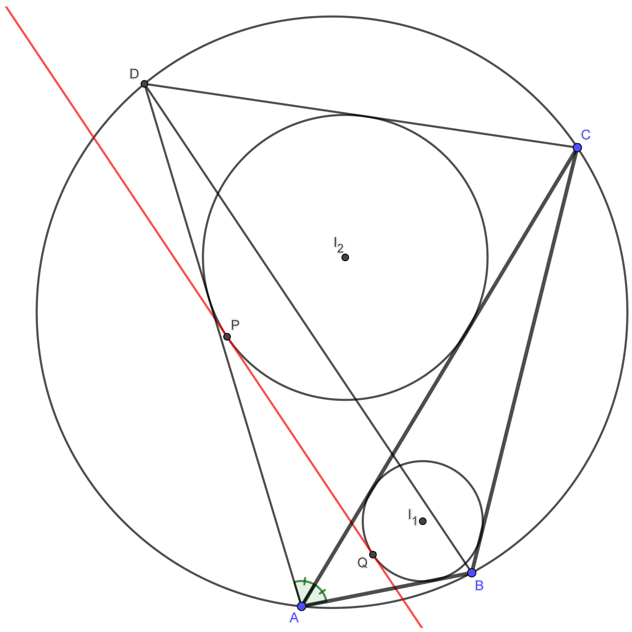
By the Chinese Remainder Theorem, the X_k are mutually distinct. Since the simultaneous congruences $k \equiv 0 \pmod{10}$ and $k \equiv 0 \pmod{11}$ have no solution in the range $k \in \{1, \ldots, 99\}$, the X_k are nonempty. Let denote $A_{10} = A_0$ and $B_{11} = B_0$. For all $i \in \{0, \ldots, 9\}$, the two sets A_i and A_{i+1} are disjoint, and for all $j \in \{0, \ldots, 10\}$, the two sets B_j and B_{j+1} are disjoint. Hence, any two sets X_k and X_{k+1} are disjoint, for all $k \in \{1, \ldots, 99\}$. For all $i \in \{0, \ldots, 9\}$, the set $A_i \cup A_{i+1}$ has at most 3 elements, and for all $j \in \{0, \ldots, 11\}$, the set $B_j \cup B_{j+1}$ has at most 4 elements. Thus, $X_k \cup X_{k+1} \subsetneq S$ for all k. We have proven that $X_1, \ldots, X_{100}$ is a sequence with the required properties.

It remains to show that $|S| \leq 7$ is impossible. The proof is by contradiction. Suppose $X_1, X_2, \ldots, X_{100}$ is a sequence with the desired properties for S with $|S| \leq 7$. We may in fact assume that $|S| = 7$ (in fact, since $2^{|S|} \geq 100$, it must be $|S| \geq 7$). The number of three-element sets X_i is at most $\binom{7}{3} = 35$. Every X_i with at least four elements is followed by X_{i+1} with one or two elements (except when $i = 100$). The number of one- or two-element subsets of S is $\binom{7}{1} + \binom{7}{2} = 28$. It follows that the number of sets X_i with four or more elements is not greater than 29. Hence, the number of distinct sets X_i in the sequence is at most $35+28+29 = 92 < 100$. This is the desired contradiction.

OC388. Let $ABCD$ be a cyclic quadrilateral with $\angle BAC = \angle DAC$. Suppose I_1 and I_2 are the incircles of $\triangle ABD$ and $\triangle ADC$ respectively. Prove that one of the common external tangents of I_1 and I_2 is parallel to BD.

Originally Problem 7, Day 2 of the 2016 China Western Mathematical Olympiad.

We received 2 submissions. We present the solution by Andrea Fanchini.



We use barycentric coordinates with reference to the triangle ABC.

We denote with Γ the circumcircle of triangle ABC, then the point D is

$$D = AAD_\infty \cap \Gamma = \left(2a^2 S_A : b^2(S_B - S_A) : 2S_A(S_A - S_B)\right)$$

so the infinite point of line BD is

$$BD_\infty(a^2 : -b^2 : b^2 - a^2).$$

Centers and radii of the incircles I_1 and I_2 are

$$I_1 : (a : b : c), \qquad \rho_1 = r,$$

$$I_2 : \left(a(c^2 - a^2 + ab) : b^2(a - b) : (b - a)(bc + S_A)\right), \qquad \rho_2 = r\frac{s(b-a)}{c(s-a)}.$$

Then their common external tangent PQ is

$$PQ : b(S_B - S_A)x + a(S_A - S_B)y + ab(a+b)z = 0,$$

that has infinite point

$$PQ_\infty(a^2 : -b^2 : b^2 - a^2) \equiv BD_\infty.$$

OC389. Let n be a positive integer. In a kingdom there are 2^n citizens and a king. In terms of currency, the kingdom uses paper bills with value 2^n and coins with value 2^a with $a = 0, 1 \ldots, n-1$. Every citizen has infinitely many paper bills. Let the total number of coins in the kingdom be S. One fine day, the king decided to implement a policy which is to be carried out every night:

- each citizen must decide on a finite amount of money based on the coins that he/she currently has, and he/she must pass that amount to either another citizen or the king;
- each citizen must pass exactly 1 more than the amount he/she received from other citizens.

Find the minimum value of S which will guarantee that the king will be able to collect money every night eternally.

Originally Problem 3 of the 2016 Japan Mathematical Olympiad Finals.

We received 1 *incomplete submission.*

OC390. Let $n \geq 2$ be an integer. Find the least value of γ which satisfies the inequality

$$x_1 x_2 \cdots x_n \leq \gamma\,(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

for any positive real numbers $x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \ldots + x_n = 1$ and any real numbers $y_1, y_2, \ldots, y_n$ with $y_1 + y_2 + \ldots + y_n = 1$ and $0 \leq y_1, y_2, \ldots, y_n \leq \frac{1}{2}$.

Originally Problem 6, Day 2 of the 2016 Spain Mathematical Olympiad.

We received 2 *submissions. We present the solution by the IISER Mohali Problem Solving Group.*

For $n = 2$, one can easily verify that $\gamma = \frac{1}{2}$. Henceforth, we shall assume $n > 2$. Without any loss of generality, we may assume that $x_n \leq x_{n-1} \leq \cdots \leq x_1$. Let S denote the set

$$\{(x_1, x_2, \ldots, x_n,\, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{2n} :$$
$$0 < x_n \leq x_{n-1} \leq \cdots \leq x_1 < 1, x_1 + x_2 + \cdots + x_n = 1 \text{ and}$$
$$0 \leq y_1, y_2, \ldots, y_n \leq \frac{1}{2}, y_1 + y_2 + \cdots + y_n = 1\}.$$

Now consider the following lemma:

Lemma. For all $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in S$, we have

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq \frac{1}{2}(x_n + x_{n-1}).$$

Proof. We will show that if y_n and y_{n-1} are less than $\frac{1}{2}$, then the value of the expression $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ cannot be smaller than $\frac{1}{2}(x_n + x_{n-1})$. For real numbers $s_1, s_2, \ldots, s_{n-2}$ and $t_1, t_2, \ldots, t_{n-2}$ with $0 \leq s_i, t_i \leq \frac{1}{2}$ for all i, we see that

$$\begin{aligned} & \left(\frac{1}{2} - \sum_{i=1}^{n-2} s_i\right) x_n + \left(\frac{1}{2} - \sum_{i=1}^{n-2} t_i\right) x_{n-1} + \sum_{i=1}^{n-2} (s_i + t_i) x_i \qquad (1) \\ = \; & \frac{1}{2}(x_n + x_{n-1}) + \sum_{i=1}^{n-2} s_i(x_i - x_n) + \sum_{i=1}^{n-2} t_i(x_i - x_{n-1}) \\ \geq \; & \frac{1}{2}(x_n + x_{n-1}) \end{aligned}$$

where the last inequality follows from assumption $x_n \leq x_{n-1} \leq \cdots \leq x_1$. □

Define a function $F : S \to \mathbb{R}$ as

$$F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \frac{x_1x_2\cdots x_n}{x_1y_1 + x_2y_2 + \cdots + x_ny_n}. \qquad (2)$$

Then, in view of the above lemma, we have

$$F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \leq \frac{x_1x_2\cdots x_n}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}}. \qquad (3)$$

Notice that F is bounded. Indeed, from the AM-GM inequality we get

$$\frac{x_nx_{n-1}}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}} \leq \sqrt{x_nx_{n-1}}$$

so that combining this with inequality (2) gives

$$F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \leq x_1x_2\cdots x_{n-2}\sqrt{x_nx_{n-1}} < 1.$$

This shows that the supremum of F exists. Further, finding such a least number γ (as in the problem) amounts to finding the supremum of the function F.

Writing $\alpha = x_n + x_{n-1}$, we see that $x_1 + x_2 + \cdots + x_{n-2} = 1 - \alpha$, and so from the AM-GM inequality, we have

$$x_nx_{n-1} \leq \frac{\alpha^2}{4} \quad \text{and} \quad x_1x_2\cdots x_{n-2} \leq \left(\frac{1-\alpha}{n-2}\right)^{n-2}.$$

Therefore,

$$\frac{x_1x_2\cdots x_n}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}} \leq \frac{\alpha}{2}\left(\frac{1-\alpha}{n-2}\right)^{n-2} \qquad (4)$$

with equality if and only if

$$x_n = x_{n-1} = \frac{\alpha}{2} \quad \text{and} \quad x_1 = x_2 = \cdots = x_{n-2} = \frac{1-\alpha}{n-2}.$$

Since $x_n \le x_1$, we must have $\frac{\alpha}{2} \le \frac{1-\alpha}{n-2}$ or $\alpha \le \frac{2}{n}$. This means that $\alpha \in (0, \frac{2}{n}]$. Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{2}\left(\frac{1-x}{n-2}\right)^{n-2}.$$

Since f is a continuous function on a closed and bounded interval, it must assume its maximum value. Differentiating f, we find

$$f'(x) = \frac{1}{2}\left(\frac{1-x}{n-2}\right)^{n-2} - \frac{x}{2}\left(\frac{1-x}{n-2}\right)^{n-3}$$

so that $f'(x) = 0$ only if $x = 1$ or $x = \frac{1}{n-1}$. Moreover, we have

$$f(0) = f(1) = 0 < \frac{1}{2(n-1)^{n-1}} = f\left(\frac{1}{n-1}\right)$$

meaning that f attains its maximum value at $x = \frac{1}{n-1}$. But as $0 < \frac{1}{n-1} < \frac{2}{n}$, the maximum value of f on $(0, \frac{2}{n}]$ is also $\frac{1}{2(n-1)^{n-1}}$. This in turn means that the greatest value of the quantity on the right hand side of (3) is $\frac{1}{2(n-1)^{n-1}}$. Combining all the inequalities, we get

$$\begin{aligned} F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) &\le \frac{x_1 x_2 \cdots x_n}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}} \\ &\le \frac{\alpha}{2}\left(\frac{1-\alpha}{n-2}\right)^{n-2} \\ &\le \frac{1}{2(n-1)^{n-1}} \end{aligned}$$

with equalities when $\alpha = \frac{1}{n-1}$. In other words, equalities occur when

$$y_1 = y_2 = \cdots = y_{n-2} = 0, \; y_{n-1} = y_n = \frac{1}{2}$$

and

$$x_1 = x_2 = \cdots = x_{n-2} = \frac{1}{n-1}, \; x_{n-1} = x_n = \frac{1}{2(n-1)}.$$

Also note that for $n = 2$, $\gamma = \frac{1}{2} = \frac{1}{2(2-1)^{2-1}}$. Therefore, the maximum value of F is $\frac{1}{2(n-1)^{n-1}}$, or equivalently, the smallest such value of γ is

$$\frac{1}{2(n-1)^{n-1}}$$

for all $n \ge 2$.

OC391. Let $x_1, x_2, x_3, \dots$ be a sequence of positive integers such that for every pair of positive integers (m, n) we have $x_{mn} \neq x_{m(n+1)}$. Prove that there exists a positive integer i such that $x_i \geq 2017$.

Originally Problem 6 of the 2017 Italy Mathematical Olympia.

We received 1 correct submission by Oliver Geupel which is presented here.

We prove that for every integer M there exists an index i such that $x_i \geq M$. The proof follows from two facts that we show below.

First, for every pair of integers i and j, $0 < i < j$, the following are equivalent:

(1) There are positive integers m and n such that $i = mn$ and $j = m(n+1)$.

(2) The difference $j - i$ is a divisor of j.

Indeed, under (1), the number $j - i = m$ is a divisor of $j = m(n+1)$, and (2) follows. Under (2), the difference $j - i$ is a divisor of $j - (j - i) = i$. Therefore, we choose the positive integers $m = j - i$ and $n = i/(j - i)$, such that (1) holds.

Secondly, we define a double sequence $(a_{k,\ell})$ with $k \in \mathbb{N}$ and $1 \leq \ell \leq k$ as follows:

$$a_{k,\ell} = \begin{cases} 1 & \text{if } (k,\ell) = (1,1) \\ a_{k-1,k-1}! & \text{if } 1 < k,\ \ell = 1 \\ a_{k-1,k-1}! + a_{k-1,\ell-1} & \text{if } 1 < k,\ 1 < \ell \leq k. \end{cases}$$

The first few terms of the double sequence are presented below:

$$\begin{pmatrix} a_{1,1} & & & \\ a_{2,1} & a_{2,2} & & \\ a_{3,1} & a_{3,2} & a_{3,3} & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\ & \cdots\cdots\cdots & & \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & 2 & & \\ 2 & 3 & 4 & \\ 24 & 26 & 27 & 28 \\ & \cdots\cdots\cdots & & \end{pmatrix}.$$

We establish the following property of the double sequence: for all positive integers k, i, and j, with $1 < k$ and $i < j \leq k$, the difference $a_{k,j} - a_{k,i}$ is a divisor of $a_{k,j}$.

We use a proof by induction on $k \geq 2$ to show this property. The base case $k = 2$ is trivial, as $a_{2,1} = 1$ and $a_{2,2} = 2$. Let $k > 2$. Notice that

$$a_{k-1,1} < a_{k-1,2} < \cdots < a_{k-1,k-1}.$$

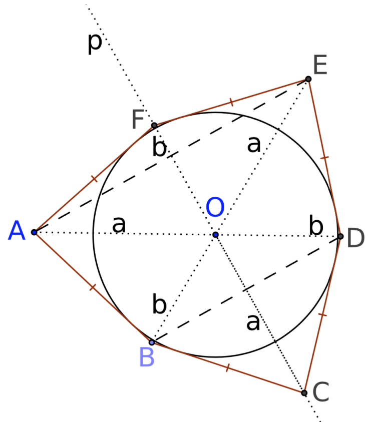
Hence, for $1 < j \leq k$, the difference $a_{k,j} - a_{k,1} = a_{k-1,j-1}$ is a divisor of $a_{k,j} = a_{k-1,k-1} + a_{k-1,j-1}$. For $1 < i < j \leq k$, by induction hypothesis, the difference $a_{k,j} - a_{k,i} = a_{k-1,j-1} - a_{k-1,i-1}$ divides $a_{k-1,j-1}$, which is a divisor of $a_{k,j} = a_{k-1,k-1}! + a_{k-1,j-1}$. This completes the induction.

Using the two facts outlined above, we can proceed to solve the problem. Consider M members, x_i's, of the given x-sequence with indices $a_{M,1}$, $a_{M,2}$, $\dots$, $a_{M,M}$ that are specified by the double a-sequence. These M members are distinct positive integers. Consequently, there exists an index i such that $x_i \geq M$.

OC392. In a convex hexagon $ABCDEF$ all sides are equal and also $AD = BE = CF$. Prove that a circle can be inscribed into this hexagon.

Originally Problem 4 of Grade 8 of the 2017 Moscow Math Olympiad.

We received 4 correct submissions. We present the solution by Oliver Geupel.



The triangles ABE and ADE, which have equal sides, are axially symmetric with respect to the perpendicular bisector p of the segment AE. Hence, the quadrilateral $ABDE$ is an isosceles trapezoid with $AE \parallel BD$. Its diagonals intersect in a point O on the line p. Let $OA = OE = a$ and $OB = OD = b$. Since $BC = CD$ and $EF = FA$, the points C and F also lie on p. Thus, the diagonals AD, BE, and CF are concurrent in O.

By an argument similar to that for $ABDE$, the quadrilateral $BCEF$ is an isosceles trapezoid. Therefore, $OC = a$ and $OF = b$. As a consequence, the triangles ABO, CBO, CDO, EDO, EFO, and AFO are congruent. This implies that the point O is equidistant from each of the six sides of the hexagon. Consequently, there is a circle centered at O that touches all the six sides.

OC393. The point O is the center of the circumcircle Ω of the acute triangle ABC. The circumcircle ω of the triangle AOC intersects the sides AB and BC again at the points E and F. Moreover, the line EF divides the area of the triangle ABC in half. Find $\angle B$.

Originally Problem 3 of Grade 10 of the 2017 Moscow Math Olympiad.

We received 5 correct submissions. We present two solutions.

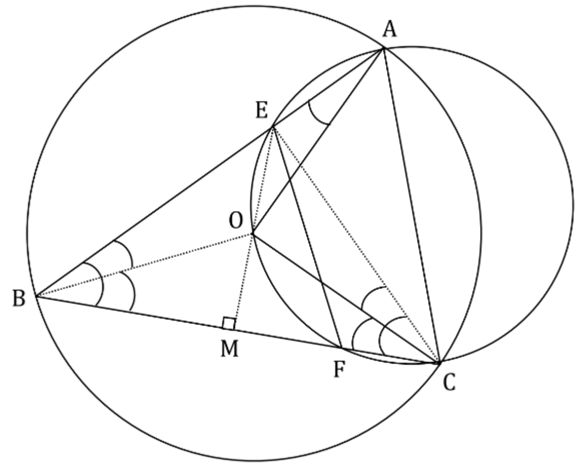
Solution 1, by Charles Justin Shi.

Since EF divides the area of triangle ABC in half, we have $2[BFE] = [ABC]$. Also, since $ACFE$ is a cyclic quadrilateral, we have $\angle BAC = 180^\circ - \angle CFE = \angle BFE$, and $\angle ACB = 180^\circ - \angle AEF = \angle BEF$. Therefore, triangles BAC and

BFE are similar, and

$$\frac{BC}{BE} = \sqrt{\frac{[ABC]}{[BFE]}} = \sqrt{2}.$$

Since O is the center of the circumcircle of the triangle ABC, it follows that OA, OB, and OC are radii of the circumcircle, and $OA = OB = OC$. This implies that triangles OAB and OBC are isosceles triangles with $\angle OBC = \angle OCB$ and $\angle OAB = \angle OBA$. In cyclic quadilateral $ACFE$, $\angle OCE$ and $\angle OAE$ subtend the same arc OE, hence, $\angle OCE = \angle OAE$. Then $\angle OCE = \angle OAE = \angle OAB = \angle OBA$, and $\angle EBC = \angle ECB$. This implies that the triangle EBC is isosceles with $BE = BC$.



Since EBC is an isosceles triangle, a median from vertex E to BC is also the altitude of the triangle. Let M be the midpoint of BC. Then $\angle EMB = 90°$, and triangle EMB is a right triangle. Using the previous result, $BC = \sqrt{2}BE$, it follows that

$$\cos(\angle ABC) = \frac{BM}{BE} = \frac{BC}{2BE} = \frac{\sqrt{2}}{2},$$

and $\angle ABC = 45°$.

Solution 2, by Andrea Fanchini.

We use barycentric coordinates with reference to the triangle ABC. The circumcircle ω of the triangle AOC, is described by

$$\omega : a^2yz + b^2zx + c^2xy - (x + y + z)S_By = 0.$$

Then the points E and F are

$$E = \omega \cap AB = (S_B : S_A : 0) \quad \text{and} \quad F = \omega \cap BC = (0 : S_C : S_B).$$

Therefore the area of triangle BEF is

$$[BEF] = \frac{[ABC]}{a^2c^2} \times \begin{vmatrix} 0 & 1 & 0 \\ S_B & S_A & 0 \\ 0 & S_C & S_B \end{vmatrix} = [ABC] \times \frac{S_B^2}{a^2c^2} = [ABC] \times \cos^2(\angle ABC).$$

However, $[BEF] = \frac{1}{2}[ABC]$, then $\cos(\angle ABC) = \sqrt{2}/2$, and $\angle ABC = 45^\circ$.

OC394. In Chicago, there are 36 criminal gangs, some of which are at war with each other. Each gangster belongs to several gangs and every pair of gangsters belongs to a different set of gangs. It is known that no gangster is a member of two gangs that are at war with each other. Furthermore, each gang that some gangster does not belong to is at war with some gang he does belong to. What is the largest possible number of gangsters in Chicago?

Originally Problem 6 of Grade 10 of the 2017 Moscow Math Olympiad.

We received 1 correct submission by Oliver Geupel presented here.

The answer is $3^{12} = 531,441$.

We establish this result using graph theory. A graph can be defined as follows. A node is assigned for each gang and two nodes are joined by an edge if the corresponding gangs are at war. The gangs that some gangster belongs to define a set of nodes. Such a set has the following two crucial properties. First, it is independent. Equivalently, there are no edges between any two nodes of the set, because a gangster does not belong to two gangs that are at war. Second, it is a maximal independent (MI) set, since it is not properly contained in any other independent set. The second property follows from the fact that each gang that some gangster does not belong to is at war with some gang he does belong to. The question asks for the maximum number of MI sets that are possible in a graph with 36 nodes.

We prove by induction over the number of nodes that the maximum number of MI sets that are possible in a graph with $3k$ nodes is 3^k.

The base case $k = 1$ is obvious. The complete graph with 3 nodes, i.e. the triangle, has exactly 3 MI sets, specifically its 3 nodes. This graph is generated by 3 gangs that are at war. The maximum number of gangsters is 3, one gangster per gang.

Let G be a graph with $3k \geq 6$ nodes, and let N be the set of its nodes. Assume that the maximum number of MI sets that are possible in a graph with $3j$ nodes is 3^j, for any $1 \leq j \leq k-1$. Split the set of nodes of G into a disjoint union of sets U and V with $3(k-1)$ and 3 elements, respectively. Let $[U]$ and $[V]$ be the induced subgraphs formed from U and V, respectively.

For any MI set M of G, the sets $M \cap U$ and $M \cap V$ are MI sets of $[U]$ and $[V]$, respectively. Let $m(G)$ be the number of MI sets in the graph G. We have obtained that

$$m(G) \leq m([U])m([V]).$$

Hence, by induction,

$$m(G) \leq 3^{k-1} \cdot 3 = 3^k.$$

The equality holds when there are no edges between the nodes U and V, V is the complete graph on 3 nodes, and U is a union of $k-1$ disjoint complete graphs on 3 nodes, i.e. $k-1$ disjoint triangles. This completes the proof and shows that the maximum number of gangsters in a city with 36 gangs is 3^{12}.

This problem is discussed and solved on graphs in two articles:

(1) J. W. Moon and L. Moser, On cliques in graphs, Israel J. Math. 3 (1965) 23-28

(2) V. Vatter, Maximal independent sets and separating covers, Amer. Math. Monthly 118 (2011) 418-423.

Specifically, it is shown that the maximum number of possible MI sets in a graph with $3k$ nodes is 3^k, with $3k+1$ nodes is $4 \cdot 3^{k-1}$, and with $3k+2$ nodes is $2 \cdot 3^k$.

OC395. Let $A_1, A_2, \ldots, A_k \in \mathcal{M}_n(\mathbb{R})$ be symmetric matrices. Prove that the following statements are equivalent:

(a) $\det(A_1^2 + A_2^2 + \cdots + A_k^2) = 0$;

(b) for all matrices $B_1, B_2, \ldots, B_k \in \mathcal{M}_n(\mathbb{R})$ it holds

$$\det(A_1B_1 + A_2B_2 + \cdots + A_kB_k) = 0.$$

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We received 1 correct submission by Oliver Geupel presented here.

First we prove the implication "(a) $\Rightarrow$ (b)". By the hypothesis (a), there exists a nonzero row vector $v \in \mathbb{R}^{1\times n}$ such that $v(A_1^2 + A_2^2 + \cdots + A_k^2) = o$, where $o = (0, \ldots, 0) \in \mathbb{R}^{1\times n}$. Since the matrices $A_1, A_2, \ldots, A_k$ are symmetric, it follows

$$\begin{aligned} |vA_1|^2 + |vA_2|^2 + \cdots + |vA_k|^2 &= vA_1A_1^Tv^T + vA_2A_2^Tv^T + \cdots + vA_kA_k^Tv^T \\ &= v(A_1^2 + A_2^2 + \cdots + A_k^2)v^T = 0. \end{aligned}$$

Therefore, $vA_1 = vA_2 = \cdots = vA_k = o$ and, for all $B_1, B_2, \ldots, B_k \in \mathcal{M}_n(\mathbb{R})$

$$v(A_1B_1 + A_2B_2 + \cdots + A_kB_k) = (vA_1)B_1 + (vA_2)B_2 + \cdots + (vA_k)B_k = o.$$

The conclusion (b), follows.

Finally, for the implication "(b) $\Rightarrow$ (a)", it is enough to put $B_1 = A_1$, $B_2 = A_2$, $\ldots$, $B_k = A_k$.