TEACHING PROBLEMS

No.2

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The Marching Band Problem

A marching band is attempting to get organized for a performance in a local parade. Unfortunately, the band is having difficulty getting lined up correctly. When the members line up in twos, there is one person left over. When they line up in threes, there are two people left over. When they line up in fours, there are three people left over. When they line up in fives, there are four people left over. When they line up in sixes, there are five people left over. Finally, they discover that when they line up in sevens, they line up neatly with nobody left over. What is the smallest possible number of people in the marching band?

Some readers may look at the Marching Band Problem and say that is not a problem. However, if this type of question is unfamiliar, it ought to represent a problem. If it is a problem, take some time to at least make sure you understand the problem. Play with it (or ideally solve it) before reading further along here. On the other hand, what if the form of this question is familiar? Then you are encouraged to outline how you would solve it before challenging yourself to solve the question in at least one other way.

This problem invites a variety of approaches. Indeed this is one of the merits of this problem. Further, there is a teasing element to this problem as one of these ideas can lead us so far without necessarily being straightforward to bring to conclusion. For instance, my experience is that students sense (correctly too) a connection with the idea of a lowest common multiple but struggle with actually applying that idea in the solution. Something is amiss in their efforts as the remainders are nonzero, thus, seemingly a bit out of step with their understanding of multiples.

This problem has many interesting features that add to its value for teaching. A brief discussion of these features here precedes the overriding quality of multiple approaches to be discussed subsequently with various forms of solution.

- Various mathematical concepts can be brought into play in the discussion and solution of the problem. Among these are multiples, divisibility, and modular arithmetic. Generally the problem promotes application and development of number sense.
- The problem is easy to understand. Accessibility is not a concern, thus, encouraging engagement with the problem at many levels. Brute force and/or trial and error have a place here in terms of both understanding the question and motivating insight to enable elegance in solution.

- The presence of redundant information is valuable. Not everything stated in the problem is offering new information. Recognition of such redundancies is an underappreciated skill in mathematical problem solving.
- The problem lends itself to engagement as people do not quickly see the solution and hence, there is time to delve into the problem at the various levels.
- Extensions or variations of this problem are relatively easy to develop. This allows for differentiating within a classroom setting, or even allowing students to create their own challenges for sharing with peers.

Let us turn our attention to some of the ways of solving this problem.

The Last Digit Approach

The last digit of the number of band members must be 4 or 9, as there are four people left over when lined up in fives. However, the number of band members is odd as it is not divisible by 2. Hence, the final digit must be 9.

Brute force can result in people trying out all numbers ending in 9 until a result is found, and it will work. Rather, consider multiples of 7 that end in 9. Note that $7 \cdot 7$ results in a product ending in 9. However, checking 49 we find that it does not meet the requirement when grouped in threes.

No other multiple of 7 less than 70 ends in 9, and it follows that the next number to check is $17 \cdot 7 = 119$. Checking we find that 119 satisfies all of the conditions. The smallest number of people in the band is 119.

Readers who are learning about congruences through recent issues of *MathemAttic* may wish to convince themselves that the numbers $n = 0, 1, 2, 3, \ldots 9$ each produce a different remainder when considered as $7n \pmod{10}$. Of particular interest here is the fact that when $n = 7$, the result is $9 \pmod{10}$.

A note on redundant information before proceeding further

Note that any number that leaves a remainder of 3 when divided by 4 must also leave a remainder of 1 when divided by 2. Further, any number that leaves a remainder of 5 when divided by 6 must leave a remainder of 2 when divided by 3. The initial two conditions stated in the problem can be removed as they are satisfied by default, so to speak.

This fact will be applied in each of the following methods of solution as a given, thus, making the problem one that is reduced to four conditions rather than six.

The Lowest Common Multiple (LCM) Connection

Begin by noting that the addition of one band member would make the number of members divisible by 4, 5, and 6. So here is the leap: the number of people in the band must be 1 less than a number divisible by each of 4, 5, and 6.

The lowest common multiple of 2, 3, 4, 5, and 6 is 60. So the numbers to consider begin with 59 and there is only one question to answer, "Is this number divisible by 7?" The answer is no, so go up 60 and check 119 or $2 \cdot 60$ - 1. Aha! It works.

Applying Congruences

A solution using congruences and modular arithmetic is offered here. Readers may wish to use tables to verify some of the results along the process.

We need to find a value n that satisfies four congruences:

$$n \equiv 3 \pmod 4; n \equiv 4 \pmod 5; n \equiv 5 \pmod 6; n \equiv 0 \pmod 7$$

Since $n \equiv 0 \pmod 7$, we can represent $n = 7t$ for some integer t.

Continuing we write $7t \equiv 5 \pmod 6$. Removing $6t$ will not change the remainder, and hence, we have $t \equiv 5 \pmod 6$. Therefore, $t = 6k + 5$ giving

$$n = 7t = 7(6k + 5) = 42k + 35.$$

Now it follows that $42k + 35 \equiv 4 \pmod 5$. Simplifying gives $2k \equiv 4 \pmod 5$. Therefore $k \equiv 2 \pmod 5$ and $k = 5m + 2$. Substituting, we get

$$n = 42k + 35 = 42(5m + 2) + 35 = 210m + 119.$$

Finally, we require that $210m + 119 \equiv 3 \pmod 4$ giving $2m + 3 \equiv 3 \pmod 4$ or $2m \equiv 0 \pmod 4$.

This final congruence is actually more difficult in that 2 (the coefficient of m) and the modulus of 4 share a common factor other than 1, and hence we have a situation that will not have a unique solution. Let us write a table here to see this fact.

m	0	1	2	3
$2m \pmod 4$	0	2	0	2

This gives us two solutions, in that $m \equiv 0$ or $2 \pmod 4$. Prior to substituting these separately, we can make an observation that any number that leaves a remainder of 0 or 2 upon division by 4 is an even number. All even numbers are solutions, and they can be represented as being $0 \pmod 2$. So we can write $m = 2b$ and then substitute into $n = 210m + 119$ giving $n = 420b + 119$.

The solution to the original set of 4 congruences is given by $n \equiv 119 \pmod{420}$. The smallest positive integer that satisfies the congruence is 119 or the smallest number of people in the band given the conditions.

Note that the number 420 is not appearing accidentally either as it represents the LCM of 4, 5, 6, and 7. So theoretically the same conditions would apply if we increased the band size by 420 or 840 or 1260 or any multiple of 420. For example, the second smallest possible band size would be 539 people.

Concluding Comments

Over the years this problem has been a rich example for me in work with teachers, as few are familiar with the problem but they can all solve it. Discussion of the problem or submissions of solutions have offered many correct answers with comments like "there must be an easier way" as partially (in)complete ideas have led to them checking every number that ends in 9, or perhaps all multiples of 7 that are odd. The appreciation of the insights shared above is enhanced through prior experience with the problem. At a secondary level (or with secondary teachers), the approach using congruences may be considered as a source of enrichment. Congruences have been featured in vignettes #3 and #4 in *MathemAttic*, and it is hoped that the inclusion of the application here will add to the growing appreciation of the value of modular arithmetic.

Prior to closing, a few suggestions are shared here. If this problem is too big for starters, it may be that reducing the number of conditions would be practical to consider. My first exposure to a band number problem involved having one person left over each time the groups were formed, but then the trivial band size of 1 had to be accounted for with a note mentioning there was more than one person in the band. Then it was thought that a requirement of exact groups for some number would take care of that. So it may be that there is one person left over when grouping in threes, fours, and fives, but no one left over when grouped in sixes. Of course, there is a problem as that final statement contradicts others as the number had to be a multiple of 6 but not a multiple of 3. Hence, going to a prime number like 7 as a factor makes it mathematically sound. Finally, it seems to be a richer problem when each group falls one short of being exact as in two left over in groups of three, three left over in groups of four and so on. There is not that immediate sense that a particular number will obviously work (like 1 if there is always one left over). The combination of these ideas exemplifies how a problem can be adapted to make another, and in this case, a better problem in my opinion.

This issue of *Teaching Problems* closes with a couple of variations that may be considered to ensure the concepts at hand are understood. Readers are encouraged to use a blend of methods in their solutions.

1. A marching band has 1 person left over when it lines up in twos, threes, fours, fives, or sixes. What is the smallest number of people in this band if it can line up with no people left over when arranged in rows of seven?

2. What is the smallest number that leaves a remainder of 1 when divided by 4, a remainder of 2 when divided by 5, a remainder of 4 when divided by 7, and no remainder when divided by 9?

3. The bandmaster claims that the band had one player left over when they tried to line up by twos, two when they tried to line up by threes, three when they tried to line up by fours, four when they tried to line up by fives, but successfully lined up by sixes. Why are you suspicious?