MathemAttic

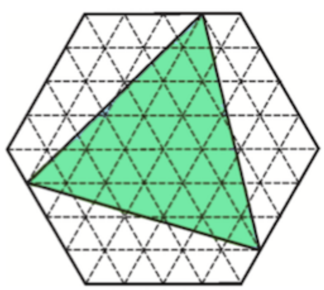
No. 5

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2019**.

MA21. An equilateral triangle is inscribed into a regular hexagon as shown below. If the area of the hexagon is 96, find the area of the inscribed triangle.



MA22. Do there exist three positive integers a, b and c for which both $a+b+c$ and abc are perfect squares? Justify your answer.

MA23. Integer numbers were placed in squares of a 4×4 grid so that the sum in each column and the sum in each row are all equal. Seven of the sixteen numbers are known as shown below, while the rest are hidden.

1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Show that it is possible to determine at least one of the missing numbers. Is it possible to determine more than one of the missing numbers?

MA24. You have 5 cards with numbers 3, 4, 5, 6 and 7 written on their backs. How many five digit numbers are divisible by 55 with the digits 3, 4, 5, 6, and 7 each appearing once in the number (the card with 6 cannot be rotated to be used as a 9)?

MA25.

1. Find four consecutive natural numbers such that the first is divisible by 3, the second is divisible by 5, the third is divisible by 7 and the fourth is divisible by 9.

2. Can you find 100 consecutive natural numbers such that the first is divisible by 3, the second is divisible by 5, ... the 100th is divisible by 201?

. . . .

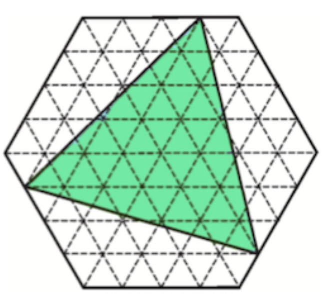
Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA21. Un triangle équilatéral est inscrit dans un hexagone régulier tel qu'indiqué ci-bas. Si la surface de l'hexagone est 96, déterminer la surface du triangle inscrit.



MA22. Existe-t-il un ensemble de trois entiers positifs a, b et c tels que $a + b + c$ et abc sont des carrés parfaits? Justifier votre réponse.

MA23. Des nombres entiers sont placés dans les cases d'une grille 4×4 de façon à ce que les sommes des entiers dans chaque colonne sont les mêmes et pareillement pour les rangées. Sept des seize nombres sont visibles et les autres ne le sont pas, comme indiqué ci-bas.

1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Prouver qu'il est possible de déterminer au moins un des nombres manquants. Est-ce possible d'en déterminer plus qu'un?

MA24. Vous disposez de cinq cartes, sur lesquelles sont inscrits les nombres 3, 4, 5, 6 et 7. Combien de nombres à cinq chiffres, divisibles par 55, peuvent être formés avec ces cartes, chacun des chiffres 3, 4, 5, 6 et 7 apparaissant une seule fois ? (Noter que la carte portant le chiffre 6 ne peut pas être réorientée pour lire 9.)

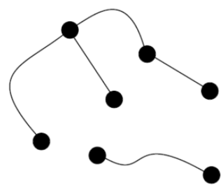
MA25.

1. Déterminer quatre entiers naturels consécutifs tels que le premier est divisible par 3, le deuxième est divisible par 5, le troisième est divisible par 7 et le quatrième est divisible par 9.
2. Pouvez-vous déterminer 100 entiers naturels consécutifs tels que le premier est divisible par 3, le deuxième est divisible par 5, $\ldots$ et le 100ième est divisible par 201?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(9), p. 364–366; and 44(10), p. 406–407.

CC341. The graph below shows 7 vertices (the dots) and 5 edges (the lines) connecting them. An edge here is defined to be a line that connects 2 vertices together. In other words, an edge cannot loop back and connect to the same vertex. Edges are allowed to cross each other, but the crossing of 2 edges does not create a new vertex. What is the least number of edges that could be added to the graph, in addition to the 5 already present, so that each of the 7 vertices has the same number of edges?



Originally Problem 23 from the 2018 Indiana State Math Contest.

We received 4 solutions, out of which we present the one by Richard Hess.

There are currently 5 edges drawn and one vertex already has three edges. Suppose that each of the 7 vertices has n edges. The total number of edges is then $7n/2$, which requires that n be even. The smallest posible n is then $n = 4$ and the total number of edges 14. Thus, 9 edges is the least number that must be added. This is achievable if we make the vertices those of a regular heptagon and connect each vertex to those that are one or two vertices away.

CC342. You are given the 5 points

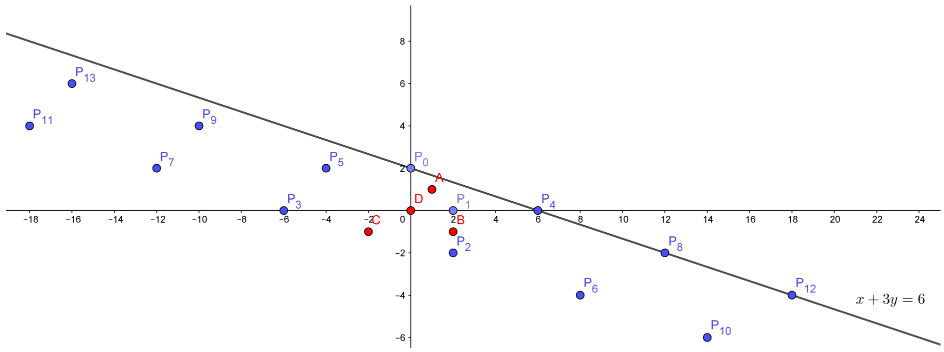
$$A = (1, 1), B = (2, -1), C = (-2, -1), D = (0, 0), P_0 = (0, 2).$$

P_1 equals the rotation of P_0 around A by 180°,
P_2 equals the rotation of P_1 around B by 180°,
P_3 equals the rotation of P_2 around C by 180°,
P_4 equals the rotation of P_3 around D by 180°,
P_5 equals the rotation of P_4 around A by 180°, and so on repeating this pattern.
If $P_{2016} = (a, b)$, then what is the value of $a + b$?

Originally Problem 13 from the 2016 Indiana State Math Contest.

We received 2 submissions, both correct and complete, and present both here.

Solution 1, by Andrea Fanchini.



P_{2016} is the $2016/4 = 504^{\text{th}}$ point on the line $x + 3y = 6$ after P_0. It follows that

$$x = 504 \cdot 6 = 3024, \quad y = \frac{6 - x}{3} = -1006.$$

Thus

$$P_{2016} = (3024, -1006)$$

and $a + b = 2018$.

Solution 2, by Ivko Dimitrić.

Denote the reflections in the points A, B, C and D by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, respectively. The rotation about a point by 180° is the reflection in that point. By the property of a mid-line in a triangle being half of the corresponding side in length, the composition $\mathcal{BA}$ of two point reflections is the translation by the vector $2\overrightarrow{AB} = 2\langle 1, -2\rangle$ and the composition $\mathcal{DC}$ is the translation by the vector $2\overrightarrow{CD} = 2\langle 2, 1\rangle$. Hence, the composition of these two translations, $\mathcal{DCBA}$, is the translation by the vector

$$2\overrightarrow{AB} + 2\overrightarrow{CD} = \langle 2, -4\rangle + \langle 4, 2\rangle = \langle 6, -2\rangle,$$

so

$$(\mathcal{DCBA})P = P + (6, -2)$$

and

$$(\mathcal{DCBA})^n P = P + n(6, -2),$$

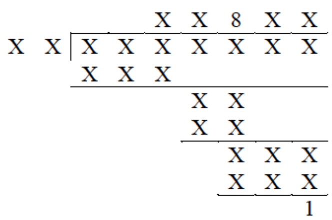
where addition of points and multiplication by a number is to be understood in vector sense, i. e. coordinate-wise. Then, since $2016 = 4 \cdot 504$, we have

$$P_{2016} = (\mathcal{DCBA})^{504} P_0 = (0, 2) + 504(6, -2) = (6 \cdot 504, -2 \cdot 503).$$

Finally,

$$a + b = 6 \cdot 504 - 2 \cdot 503 = 2018.$$

CC343. In the following long division problem, most of the digits (26 in fact) are hidden by the symbol X. What is the sum of all of the 26 hidden digits?



Originally Problem 17 from the 2017 Indiana State Math Contest.

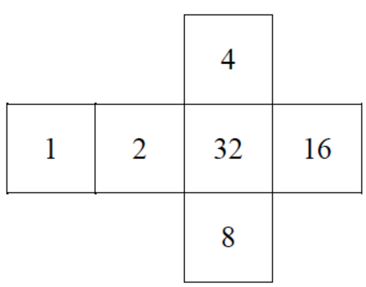
We received 4 solutions, out of which we present the one by Kathleen E. Lewis.

Since the divisor is a two digit number and 8 times the divisor is a two digit number, the divisor must be 10, 11, or 12. But some digit times the divisor gives a three digit number, so the divisor must be 12 and it must be multiplied by 9. From this, we can put together the entire division problem:

```
      90809
 12)1089709
    108
      97
      96
       109
       108
         1
```

Therefore the sum of the hidden digits is 114.

CC344. Take the pattern below and form a cube. Then take three of these exact same cubes and stack them one on top of another on a table so that exactly 13 numbers are visible. What is the greatest possible sum of these 13 visible numbers?



Originally Problem 16 from the 2015 Indiana State Math Contest.

We received 5 solutions, all of which were correct. We present the solution of Brad Meyer, modified by the editor.

To maximize the total sum, we maximize the sum found on each cube.

The top cube has five visible faces and one hidden face at the bottom. To maximize the sum, we select the face with the smallest number to be "hidden" on the bottom; the 1 face. This leaves visible faces 32, 16, 8, 4, and 2. Resulting in a sum of 62.

The middle cube has 4 visible faces and 2 hidden faces Ð the top and bottom. The cubes opposite face pairs are $[1, 32]$, $[2, 16]$, and $[4, 8]$. We select the pair of opposite faces with the smallest sum to be hidden on the top and bottom, $[4, 8]$. This leaves visible faces 32, 16, 2, and 1, Resulting in a sum of 51.

The same methodology used above for the middle cube holds for the bottom cube. Thus the total of the visible faces is $62 + 51 + 51 = 164$.

CC345. Your teacher asks you to write down five integers such that the median is one more than the mean, and the unique mode is one greater than the median. You then notice that the median is 10. What is the smallest possible integer that you could include in your list?

Originally Problem 17 from the 2015 Indiana State Math Contest.

We received 5 submissions, all of which were correct and complete. We present the solutions by Kathleen Lewis, Richard Hess, and Charles Justin Shi (done independently), combined by the editor.

Call the five integers a, b, c, d, and e in a non-decreasing order. Given that the median is 10, $c = 10$ and our mean and mode are 9 and 11, respectively. Our mode is unique and therefore occurs at least twice. Given that the median is 10 and our mode is 11, our mode can occur at most twice - else our median would be 11. Thus $d = e = 11$. As our mean is 9, we have that

$$\frac{a + b + 10 + 11 + 11}{5} = 9 \Rightarrow a + b = 13,$$

where $b \leq 9$. The smallest value of a is 4.

CC346. An ant paces along the x-axis at a constant rate of one unit per second. He begins at $x = 0$ and his path takes him one unit forward, then two back, then three forward, etc. How many times does the ant step on the point $x = 10$ in the first five minutes of his walk?

Originally Problem 10 of Game 3 from the 2015-16 Nova Scotia Math League.

We received 3 submissions to this problem, only one of which was correct. We present the solution by Ivko Dimitríc.

By a single move we mean a stretch that the ant paces in the same direction before turning to move in the opposite direction. We use notation $M_n^F(k)$ to denote the ant's nth move, which is a forward move ending up at number k on the x-axis. Similarly, $M_n^B(-k)$ means that on the nth move the ant moved back (to the left on the x-axis) and ended up at number $-k$. Since every subsequent move is one unit longer, the ant progresses 1 unit to the right on each move forward (an odd-numbered move) compared to the previous forward move and progresses 1 unit to the left on each even-numbered move:

$M_1^F(1),\ M_3^F(2),\ M_5^F(3),\ M_7^F(4),\ \ldots M_{2k-1}^F(k), \ldots$

$M_2^B(-1),\ M_4^B(-2),\ M_6^B(-3),\ M_8^B(-4),\ \ldots M_{2k}^B(-k), \ldots$

Therefore, after $19 = 2 \cdot 10 - 1$ moves the ant finally reaches the point $x = 10$ for the first time with $M_{19}^F(10)$, whereupon it turns back to make its next move $M_{20}^B(-10)$. The time to complete the first 20 moves is equal to the length traveled,

$$1 + 2 + 3 + \cdots + 20 = \frac{20 \cdot 21}{2} = 210$$

seconds and during this time the ant stepped on $x = 10$ only once. Then in each subsequent move the ant will step on the point 10 once each time as it moves forward or back. On the next four moves which last 21 seconds (F), 22 seconds (B), 23 seconds (F) and 24 seconds (B), the number 10 will be stepped on once during each move, thus four additional times. The total elapsed time to complete all 24 steps is

$$210 + 21 + 22 + 23 + 24 = 300 \text{ seconds} = 5 \text{ minutes},$$

during which time the ant stepped on the point $x = 10$ exactly 5 times.

CC347. Find the sum of all fractions p/q between 0 and 1 that have denominator 100 when expressed in lowest terms.

Originally Problem 9 of Game 1 from the 2017-18 Nova Scotia Math League.

We received 8 correct submissions. We present two of the solutions.

Solution 1, by Henry Ricardo.

The number of positive integers relatively prime to 100 is

$$\phi(100) = 100(1 - 1/2)(1 - 1/5) = 40,$$

where ϕ denotes Euler's totient function. Note that if a numerator $k \in \{1, 2, \ldots, 99\}$ is relatively prime to 100, then so is $100 - k$, thus we have $\phi(100)/2 = 20$ pairs $\{k, 100 - k\}$ of numerators, each pair adding to 100. Therefore the sum of all fractions p/q between 0 and 1 that have denominator 100 when expressed in lowest terms is $(20 \cdot 100)/100 = 20$.

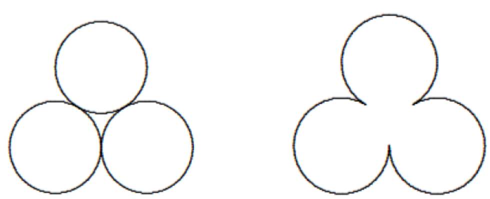
Solution 2, by Kathleen E. Lewis.

Since all of the fractions have the same denominator, we just need to add up the numerators and divide the sum by 100. The numerators we want are those that are relatively prime to 100, thus have no factor of 2 or 5. Using inclusion-exclusion, we start with all the positive integers less than or equal to 100, then subtract the multiples of 2 and the multiples of 5. Finally we need to add the multiples of 10, as they have been subtracted twice. So we get

$$\begin{aligned}&\frac{100\cdot 101}{2}-2\cdot\frac{50\cdot 51}{2}-5\cdot\frac{20\cdot 21}{2}+10\cdot\frac{10\cdot 11}{2}\\ &=\frac{100}{2}(101-51-21+11)\\ &=50\cdot 40=2000.\end{aligned}$$

Dividing by 100, we obtain 20 for the sum of the fractions.

CC348. Three circles with the same radius r are mutually tangent as shown on the left figure. The arcs in the middle are removed, making a trefoil (right figure). Determine the exact length of the trefoil in terms of r.



Originally from the 2017-18 Final Game of the Newfoundland and Labrador Teachers' Association Senior Math League.

We received 10 submissions, all correct. We present the solution provided by Richard Hess.

Each of the three circles has a circumference of $2\pi r$, so the total length before removal is $6\pi r$. The removed segments add to πr, giving the arc length of the trefoil as $5\pi r$.

CC349. Let O be the centre of equilateral triangle ABC (i.e. the unique point equidistant from each vertex). Another point P is selected uniformly at random in the interior of $\triangle ABC$. Find the probability that P is closer to O than it is to any of A, B or C.

Originally Problem 10 of Game 3 from the 2017-18 Nova Scotia Math League.

We received 3 submissions, all correct. We present the solution provided by Richard Hess.

The perpendicular bisectors of OA, OB and OC produce a regular hexagon inside the equilateral triangle. This hexagon has an area twice the sum of the three small triangular areas outside the hexagon but inside the large triangle. Thus P has a probability of $2/3$ of being closer to O than any of A, B or C.

CC350. A friend proposes the following guessing game: He chooses an integer between 1 and 100, inclusive, and you repeatedly try to guess his number. He tells you whether each incorrect guess is higher or lower than his chosen number, but you are allowed at most one high guess overall. You win the game when you guess his number correctly. You lose the game the instant you make a second high guess. What is the minimum number of guesses in which you can guarantee you will win the game?

Originally Problem 10 of Game 3 from the 2011-12 Nova Scotia Math League.

We received 3 submissions of which only one was correct. We present the solution by Richard Hess.

I can guess the number in 14 or fewer guesses if the number is 1 to 105 inclusive. My first guess is 14. If high, my next 13 guesses are 1 to 13. If low, my second guess is 27. If this is high, my next guesses are 15-26. If my second guess is low, my third guess is 39. If I continue to guess low, my next guesses are 50, 60, 69, 77, 84, 90, 95, 99, 102, 104, and 105. If any of these is a high guess, then the remaining guesses are enough to guess all possibilities from low to high in the gap between that guess and the prior low guess.

PROBLEM SOLVING VIGNETTES

No.5

Shawn Godin

Introducing Induction

Mathematical induction is a very powerful technique used to show that a certain property holds for infinitely many cases. It is based on a nice piece of logic where we show that if the property holds for a case then it must hold for the "next" case. This induction step then can be used recursively to show that if the property holds for $n = 1$, then it must hold for $n = 2$, but then it must hold for $n = 3$ and so on. We will examine induction in action by trying to prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

We may have come up with the expression $\frac{n(n+1)}{2}$ by playing with a few cases, realizing it must be a quadratic because the second differences would be constant (why?). To show it is true for all positive integers n, we will first show it is true for a specific case: $n = 1$. In this case our "sum" is just 1, while our expression gives $\frac{(1)(1+1)}{2} = 1$, which shows the expression is good in this case.

Next we will do our induction step, we will *assume* that the statement works for some $n = k$. We can do this, because we know there is at least one case where the expression gives the correct result, that is $n = 1$. So if the statement is true for $n = k$, then we must have

$$\sum_{i=1}^{k} i = 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \qquad (1)$$

so then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= 1 + 2 + \cdots + k + (k+1) \\
&= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by our assumption (1)} \\
&= (k+1)\left[\frac{k}{2} + 1\right] \\
&= \frac{(k+1)(k+2)}{2} = \frac{(\boxed{k}+1)(\boxed{k+1}+1)}{2} && (2)
\end{aligned}$$

which is just our formula (1) with k replaced by $k+1$. This shows that if the formula holds for some positive integer $n = k$, then it holds for the next positive integer $n = k+1$. When we couple this with our demonstration that the formula works for $n = 1$, it shows that the property is true for all positive integers n. $\square$

Some people prefer to refer to the thing that we are to prove as a proposition, $P(n)$, since the proposition is being made for various positive integers, n. In our case we would define

$$P(n) : 1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

which is a *boolean* function. It is either *true* or *false* for any particular n.

Thus our proof goes by showing $P(1)$ is true, that is $1 = \frac{(1)(1+1)}{2}$ then assuming $P(k)$ is true for some k (we can do this since we know it is true for at least $k = 1$), which was (1). Then we try to show $P(k) \Rightarrow P(k+1)$, which we showed with (2).

Induction works like a row of dominoes. The induction step $P(k) \Rightarrow P(k+1)$, says "if one domino falls over it will knock the next one over". Then $P(1)$ says "knock over the first domino" and away it goes! Domino #1 knocks over domino #2, which knocks over domino #3, ... The difference is with a mathematical proposition we have infinitely many dominoes!

Take a moment and try your hand at a proof by induction. The statement below is similar to the one we just did. Prove

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}$$

The proof of this will appear at the end of the column.

We will now examine another proof by induction. The problem below is problem 10b from the 2019 Euclid contest put on by the CEMC. You can check out past CEMC contests on their website `www.cemc.uwaterloo.ca`.

> *Carlos has* 14 *coins, numbered* 1 *to* 14. *Each coin has exactly one face called "heads". When flipped, coins* $1, 2, 3, \ldots, 13, 14$ *land heads with probabilities* $h_1, h_2, h_3, \ldots, h_{13}, h_{14}$, *respectively. When Carlos flips each of the* 14 *coins exactly once, the probability that an even number of coins land heads is exactly* $\frac{1}{2}$. *Must there be a* k *between* 1 *and* 14, *inclusive, for which* $h_k = \frac{1}{2}$? *Prove your answer.*

We will actually prove the more general situation and use that to show that the specific case, with 14 coins, must then be true. Thus we will prove the following:

> *If we have* $2n$ *coins, with probabilities for flipping heads* $h_1, h_2, \ldots, h_{2n}$ *and the probability of flipping an even number of heads is* $\frac{1}{2}$ *then there exists some* k, *with* $1 \le k \le 2n$, *such that* $h_k = \frac{1}{2}$.

Define $E(2n)$ and $O(2n)$ to be the probabilities of flipping an even number of heads and an odd number of heads, respectively, on a flip of $2n$ coins with probabilities of heads for each coin $h_1, h_2, \ldots, h_{2n}$. Then clearly $E(2n) + O(2n) = 1$ implies

$$\text{if } E(2n) = \frac{1}{2}, \text{ then } O(2n) = \frac{1}{2}$$

for all n since the events are mutually exclusive and there is no other possibility.

Examining the case where $n = 1$, we have two coins with $E(2) = O(2) = \frac{1}{2}$. Hence, if we have an even number of heads we have both coins showing heads or neither coin showing heads. Thus

$$\begin{aligned} E(2) = \frac{1}{2} &\Rightarrow h_1h_2 + (1 - h_1)(1 - h_2) = \frac{1}{2} \\ &\Rightarrow 2h_1h_2 - h_1 - h_2 + 1 = \frac{1}{2} \\ &\Rightarrow 4h_1h_2 - 2h_1 - 2h_2 + 2 = 1 \\ &\Rightarrow 4h_1h_2 - 2h_1 - 2h_2 + 1 = 0 \\ &\Rightarrow (2h_1 - 1)(2h_2 - 1) = 0 \end{aligned}$$

hence $h_1 = \frac{1}{2}$ or $h_2 = \frac{1}{2}$ and our proposition is true for $n = 1$.

Now we will assume that the proposition is true for some $n = k$, that is we assume that if $E(2k) = \frac{1}{2}$, then at least one coin has probability of heads of $\frac{1}{2}$. It follows that if $E(2(k+1)) = \frac{1}{2}$, we will break our coins into two groups with probabilities $p_1, p_2, \ldots, p_{2k}$ and p_{2k+1}, p_{2k+2} and we will let $E(2k)$ be the probability of flipping an even number of heads with the first $2k$ coins.

To have an even number of coins showing heads, we have either both or neither coins $2k + 1$ and $2k + 2$ are heads and an even number of the first $2k$ coins are heads; or exactly one of the coins $2k + 1$ and $2k + 2$ is heads and an odd number of the first $2k$ coins are heads. Since $E(2(k + 1)) = \frac{1}{2}$, we have

$$\begin{aligned} &(p_{2k+1}p_{2k+2} + (1 - p_{2k+1})(1 - p_{2k+2}))E(2k) \\ &\qquad + (p_{2k+1}(1 - p_{2k+2}) + (1 - p_{2k+1})p_{2k+2})O(2k) = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} &(4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 2)E(2k) \\ &\qquad + (2p_{2k+1} + 2p_{2k+2} - 4p_{2k+1}p_{2k+2})O(2k) = 1, \end{aligned}$$

$$\begin{aligned} &(4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 1 + 1)E(2k) \\ &\qquad - (4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 1 - 1)O(2k) = 1, \end{aligned}$$

$$(2p_{2k+1} - 1)(2p_{2k+2} - 1)(E(2k) - O(2k)) + E(2k) + O(2k) = 1,$$

but since $E(2k) + O(2k) = 1$ we have

$$(2p_{2k+1} - 1)(2p_{2k+2} - 1)(E(2k) - O(2k)) = 0$$

thus

$$\begin{aligned} 2p_{2k+1} - 1 = 0 &\Rightarrow p_{2k+1} = \frac{1}{2} \quad \text{or,} \\ 2p_{2k+2} - 1 = 0 &\Rightarrow p_{2k+2} = \frac{1}{2} \quad \text{or,} \\ E(2k) - O(2k) = 0 &\Rightarrow E(2k) = O(2k) \Rightarrow E(2k) = O(2k) = \frac{1}{2}. \end{aligned}$$

So either the probability of at least one of coins $2k+1$ and $2k+2$ have probability of $\frac{1}{2}$ or the probability of there being an even number of heads in the first $2k$ coins is $\frac{1}{2}$. But by our assumption if any set of $2k$ coins has $E(2k) = \frac{1}{2}$ then at least one of the coins has probability of $\frac{1}{2}$, which shows it works for $n = k+1$ and hence the property is true for all positive integers n and hence must be true for $n = 7$, which is the original problem. □

Our proof showed that for any even number of coins, if the probability of flipping an even number of heads is $\frac{1}{2}$ then at least one coin has probability of heads being $\frac{1}{2}$. It turns out that we were being too specific, and it can be shown that the property works when we have an odd number of coins as well. If you look at the official solution from the CEMC, they use a method of descent that is similar to induction. As an exercise, you may want to prove that the property holds no matter how many coins we have to start. The proof should be a bit easier than the one we presented.

Mathematical induction is a useful technique to add to your problem solving toolkit. Look for places that you can use it!

Ok, here is the proof of (3), namely that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since

$$\frac{(1)((1)+1)(2(1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1 = 1^2$$

the statement is true for $n = 1$.

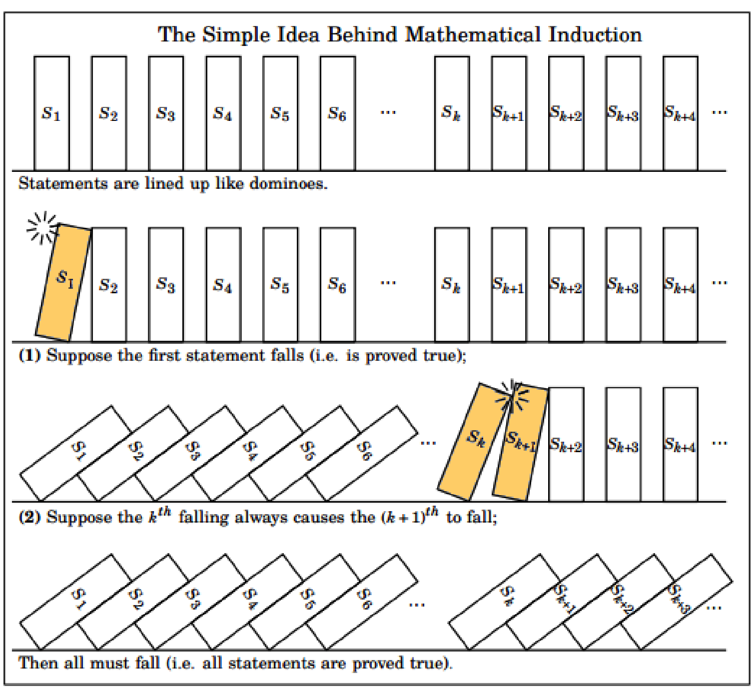
Let's assume the statement is true for $n = k$, that is

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}. \qquad (4)$$

Then, we must have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \sum_{i=1}^{k+1} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{by (4)} \\
&= \frac{k+1}{6}(k(2k+1) + 6(k+1)) \\
&= \frac{k+1}{6}(2k^2 + 7k + 6) \\
&= \frac{k+1}{6}(k+2)(2k+3) \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}$$

which means the statement is true for $n = k+1$ and hence the statement is true for all positive integers by induction. □

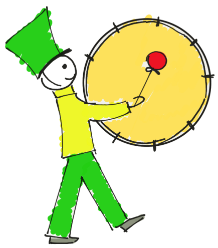


TEACHING PROBLEMS

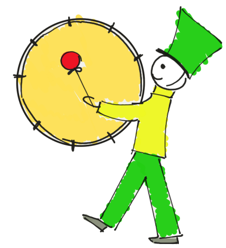
No.2

John McLoughlin

The Marching Band Problem



A marching band is attempting to get organized for a performance in a local parade. Unfortunately, the band is having difficulty getting lined up correctly. When the members line up in twos, there is one person left over. When they line up in threes, there are two people left over. When they line up in fours, there are three people left over. When they line up in fives, there are four people left over. When they line up in sixes, there are five people left over. Finally, they discover that when they line up in sevens, they line up neatly with nobody left over. What is the smallest possible number of people in the marching band?



Some readers may look at the Marching Band Problem and say that is not a problem. However, if this type of question is unfamiliar, it ought to represent a problem. If it is a problem, take some time to at least make sure you understand the problem. Play with it (or ideally solve it) before reading further along here. On the other hand, what if the form of this question is familiar? Then you are encouraged to outline how you would solve it before challenging yourself to solve the question in at least one other way.

This problem invites a variety of approaches. Indeed this is one of the merits of this problem. Further, there is a teasing element to this problem as one of these ideas can lead us so far without necessarily being straightforward to bring to conclusion. For instance, my experience is that students sense (correctly too) a connection with the idea of a lowest common multiple but struggle with actually applying that idea in the solution. Something is amiss in their efforts as the remainders are nonzero, thus, seemingly a bit out of step with their understanding of multiples.

This problem has many interesting features that add to its value for teaching. A brief discussion of these features here precedes the overriding quality of multiple approaches to be discussed subsequently with various forms of solution.

- Various mathematical concepts can be brought into play in the discussion and solution of the problem. Among these are multiples, divisibility, and modular arithmetic. Generally the problem promotes application and development of number sense.
- The problem is easy to understand. Accessibility is not a concern, thus, encouraging engagement with the problem at many levels. Brute force and/or trial and error have a place here in terms of both understanding the question and motivating insight to enable elegance in solution.

- The presence of redundant information is valuable. Not everything stated in the problem is offering new information. Recognition of such redundancies is an underappreciated skill in mathematical problem solving.
- The problem lends itself to engagement as people do not quickly see the solution and hence, there is time to delve into the problem at the various levels.
- Extensions or variations of this problem are relatively easy to develop. This allows for differentiating within a classroom setting, or even allowing students to create their own challenges for sharing with peers.

Let us turn our attention to some of the ways of solving this problem.

The Last Digit Approach

The last digit of the number of band members must be 4 or 9, as there are four people left over when lined up in fives. However, the number of band members is odd as it is not divisible by 2. Hence, the final digit must be 9.

Brute force can result in people trying out all numbers ending in 9 until a result is found, and it will work. Rather, consider multiples of 7 that end in 9. Note that $7 \cdot 7$ results in a product ending in 9. However, checking 49 we find that it does not meet the requirement when grouped in threes.

No other multiple of 7 less than 70 ends in 9, and it follows that the next number to check is $17 \cdot 7 = 119$. Checking we find that 119 satisfies all of the conditions. The smallest number of people in the band is 119.

Readers who are learning about congruences through recent issues of *MathemAttic* may wish to convince themselves that the numbers $n = 0, 1, 2, 3, \ldots 9$ each produce a different remainder when considered as $7n \pmod{10}$. Of particular interest here is the fact that when $n = 7$, the result is $9 \pmod{10}$.

A note on redundant information before proceeding further

Note that any number that leaves a remainder of 3 when divided by 4 must also leave a remainder of 1 when divided by 2. Further, any number that leaves a remainder of 5 when divided by 6 must leave a remainder of 2 when divided by 3. The initial two conditions stated in the problem can be removed as they are satisfied by default, so to speak.

This fact will be applied in each of the following methods of solution as a given, thus, making the problem one that is reduced to four conditions rather than six.

The Lowest Common Multiple (LCM) Connection

Begin by noting that the addition of one band member would make the number of members divisible by 4, 5, and 6. So here is the leap: the number of people in the band must be 1 less than a number divisible by each of 4, 5, and 6.

The lowest common multiple of 2, 3, 4, 5, and 6 is 60. So the numbers to consider begin with 59 and there is only one question to answer, "Is this number divisible by 7?" The answer is no, so go up 60 and check 119 or $2 \cdot 60$ - 1. Aha! It works.

Applying Congruences

A solution using congruences and modular arithmetic is offered here. Readers may wish to use tables to verify some of the results along the process.

We need to find a value n that satisfies four congruences:

$$n \equiv 3 \pmod 4; n \equiv 4 \pmod 5; n \equiv 5 \pmod 6; n \equiv 0 \pmod 7$$

Since $n \equiv 0 \pmod 7$, we can represent $n = 7t$ for some integer t.

Continuing we write $7t \equiv 5 \pmod 6$. Removing $6t$ will not change the remainder, and hence, we have $t \equiv 5 \pmod 6$. Therefore, $t = 6k + 5$ giving

$$n = 7t = 7(6k + 5) = 42k + 35.$$

Now it follows that $42k + 35 \equiv 4 \pmod 5$. Simplifying gives $2k \equiv 4 \pmod 5$. Therefore $k \equiv 2 \pmod 5$ and $k = 5m + 2$. Substituting, we get

$$n = 42k + 35 = 42(5m + 2) + 35 = 210m + 119.$$

Finally, we require that $210m + 119 \equiv 3 \pmod 4$ giving $2m + 3 \equiv 3 \pmod 4$ or $2m \equiv 0 \pmod 4$.

This final congruence is actually more difficult in that 2 (the coefficient of m) and the modulus of 4 share a common factor other than 1, and hence we have a situation that will not have a unique solution. Let us write a table here to see this fact.

m	0	1	2	3
$2m \pmod 4$	0	2	0	2

This gives us two solutions, in that $m \equiv 0$ or $2 \pmod 4$. Prior to substituting these separately, we can make an observation that any number that leaves a remainder of 0 or 2 upon division by 4 is an even number. All even numbers are solutions, and they can be represented as being $0 \pmod 2$. So we can write $m = 2b$ and then substitute into $n = 210m + 119$ giving $n = 420b + 119$.

The solution to the original set of 4 congruences is given by $n \equiv 119 \pmod{420}$. The smallest positive integer that satisfies the congruence is 119 or the smallest number of people in the band given the conditions.

Note that the number 420 is not appearing accidentally either as it represents the LCM of 4, 5, 6, and 7. So theoretically the same conditions would apply if we increased the band size by 420 or 840 or 1260 or any multiple of 420. For example, the second smallest possible band size would be 539 people.

Concluding Comments

Over the years this problem has been a rich example for me in work with teachers, as few are familiar with the problem but they can all solve it. Discussion of the problem or submissions of solutions have offered many correct answers with comments like "there must be an easier way" as partially (in)complete ideas have led to them checking every number that ends in 9, or perhaps all multiples of 7 that are odd. The appreciation of the insights shared above is enhanced through prior experience with the problem. At a secondary level (or with secondary teachers), the approach using congruences may be considered as a source of enrichment. Congruences have been featured in vignettes #3 and #4 in *MathemAttic*, and it is hoped that the inclusion of the application here will add to the growing appreciation of the value of modular arithmetic.

Prior to closing, a few suggestions are shared here. If this problem is too big for starters, it may be that reducing the number of conditions would be practical to consider. My first exposure to a band number problem involved having one person left over each time the groups were formed, but then the trivial band size of 1 had to be accounted for with a note mentioning there was more than one person in the band. Then it was thought that a requirement of exact groups for some number would take care of that. So it may be that there is one person left over when grouping in threes, fours, and fives, but no one left over when grouped in sixes. Of course, there is a problem as that final statement contradicts others as the number had to be a multiple of 6 but not a multiple of 3. Hence, going to a prime number like 7 as a factor makes it mathematically sound. Finally, it seems to be a richer problem when each group falls one short of being exact as in two left over in groups of three, three left over in groups of four and so on. There is not that immediate sense that a particular number will obviously work (like 1 if there is always one left over). The combination of these ideas exemplifies how a problem can be adapted to make another, and in this case, a better problem in my opinion.

This issue of *Teaching Problems* closes with a couple of variations that may be considered to ensure the concepts at hand are understood. Readers are encouraged to use a blend of methods in their solutions.

1. A marching band has 1 person left over when it lines up in twos, threes, fours, fives, or sixes. What is the smallest number of people in this band if it can line up with no people left over when arranged in rows of seven?

2. What is the smallest number that leaves a remainder of 1 when divided by 4, a remainder of 2 when divided by 5, a remainder of 4 when divided by 7, and no remainder when divided by 9?

3. The bandmaster claims that the band had one player left over when they tried to line up by twos, two when they tried to line up by threes, three when they tried to line up by fours, four when they tried to line up by fives, but successfully lined up by sixes. Why are you suspicious?