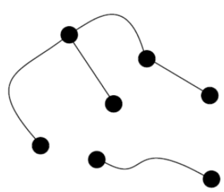
CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(9), p. 364–366; and 44(10), p. 406–407.

CC341. The graph below shows 7 vertices (the dots) and 5 edges (the lines) connecting them. An edge here is defined to be a line that connects 2 vertices together. In other words, an edge cannot loop back and connect to the same vertex. Edges are allowed to cross each other, but the crossing of 2 edges does not create a new vertex. What is the least number of edges that could be added to the graph, in addition to the 5 already present, so that each of the 7 vertices has the same number of edges?



Originally Problem 23 from the 2018 Indiana State Math Contest.

We received 4 solutions, out of which we present the one by Richard Hess.

There are currently 5 edges drawn and one vertex already has three edges. Suppose that each of the 7 vertices has n edges. The total number of edges is then $7n/2$, which requires that n be even. The smallest posible n is then $n = 4$ and the total number of edges 14. Thus, 9 edges is the least number that must be added. This is achievable if we make the vertices those of a regular heptagon and connect each vertex to those that are one or two vertices away.

CC342. You are given the 5 points

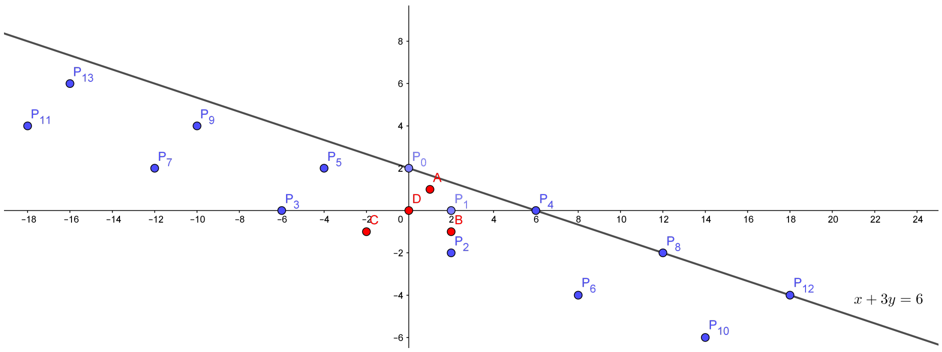
$$A = (1, 1), B = (2, -1), C = (-2, -1), D = (0, 0), P_0 = (0, 2).$$

P_1 equals the rotation of P_0 around A by 180°,
P_2 equals the rotation of P_1 around B by 180°,
P_3 equals the rotation of P_2 around C by 180°,
P_4 equals the rotation of P_3 around D by 180°,
P_5 equals the rotation of P_4 around A by 180°, and so on repeating this pattern.
If $P_{2016} = (a, b)$, then what is the value of $a + b$?

Originally Problem 13 from the 2016 Indiana State Math Contest.

We received 2 submissions, both correct and complete, and present both here.

Solution 1, by Andrea Fanchini.



P_{2016} is the $2016/4 = 504^{\text{th}}$ point on the line $x + 3y = 6$ after P_0. It follows that

$$x = 504 \cdot 6 = 3024, \quad y = \frac{6 - x}{3} = -1006.$$

Thus

$$P_{2016} = (3024, -1006)$$

and $a + b = 2018$.

Solution 2, by Ivko Dimitrić.

Denote the reflections in the points A, B, C and D by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, respectively. The rotation about a point by 180° is the reflection in that point. By the property of a mid-line in a triangle being half of the corresponding side in length, the composition $\mathcal{BA}$ of two point reflections is the translation by the vector $2\overrightarrow{AB} = 2\langle 1, -2\rangle$ and the composition $\mathcal{DC}$ is the translation by the vector $2\overrightarrow{CD} = 2\langle 2, 1\rangle$. Hence, the composition of these two translations, $\mathcal{DCBA}$, is the translation by the vector

$$2\overrightarrow{AB} + 2\overrightarrow{CD} = \langle 2, -4\rangle + \langle 4, 2\rangle = \langle 6, -2\rangle,$$

so

$$(\mathcal{DCBA})P = P + (6, -2)$$

and

$$(\mathcal{DCBA})^n P = P + n(6, -2),$$

where addition of points and multiplication by a number is to be understood in vector sense, i. e. coordinate-wise. Then, since $2016 = 4 \cdot 504$, we have

$$P_{2016} = (\mathcal{DCBA})^{504} P_0 = (0, 2) + 504(6, -2) = (6 \cdot 504, -2 \cdot 503).$$

Finally,

$$a + b = 6 \cdot 504 - 2 \cdot 503 = 2018.$$

CC343. In the following long division problem, most of the digits (26 in fact) are hidden by the symbol X. What is the sum of all of the 26 hidden digits?

```
          X X 8 X X
X X | X X X X X X X
      X X X
            X X
            X X
              X X X
              X X X
                  1
```

Originally Problem 17 from the 2017 Indiana State Math Contest.

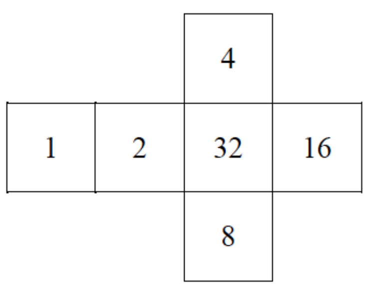
We received 4 solutions, out of which we present the one by Kathleen E. Lewis.

Since the divisor is a two digit number and 8 times the divisor is a two digit number, the divisor must be 10, 11, or 12. But some digit times the divisor gives a three digit number, so the divisor must be 12 and it must be multiplied by 9. From this, we can put together the entire division problem:

```
     90809
12)1089709
     108
      97
      96
     109
     108
       1
```

Therefore the sum of the hidden digits is 114.

CC344. Take the pattern below and form a cube. Then take three of these exact same cubes and stack them one on top of another on a table so that exactly 13 numbers are visible. What is the greatest possible sum of these 13 visible numbers?



Originally Problem 16 from the 2015 Indiana State Math Contest.

We received 5 solutions, all of which were correct. We present the solution of Brad Meyer, modified by the editor.

To maximize the total sum, we maximize the sum found on each cube.

The top cube has five visible faces and one hidden face at the bottom. To maximize the sum, we select the face with the smallest number to be "hidden" on the bottom; the 1 face. This leaves visible faces 32, 16, 8, 4, and 2. Resulting in a sum of 62.

The middle cube has 4 visible faces and 2 hidden faces Ð the top and bottom. The cubes opposite face pairs are $[1, 32]$, $[2, 16]$, and $[4, 8]$. We select the pair of opposite faces with the smallest sum to be hidden on the top and bottom, $[4, 8]$. This leaves visible faces 32, 16, 2, and 1, Resulting in a sum of 51.

The same methodology used above for the middle cube holds for the bottom cube. Thus the total of the visible faces is $62 + 51 + 51 = 164$.

CC345. Your teacher asks you to write down five integers such that the median is one more than the mean, and the unique mode is one greater than the median. You then notice that the median is 10. What is the smallest possible integer that you could include in your list?

Originally Problem 17 from the 2015 Indiana State Math Contest.

We received 5 submissions, all of which were correct and complete. We present the solutions by Kathleen Lewis, Richard Hess, and Charles Justin Shi (done independently), combined by the editor.

Call the five integers a, b, c, d, and e in a non-decreasing order. Given that the median is 10, $c = 10$ and our mean and mode are 9 and 11, respectively. Our mode is unique and therefore occurs at least twice. Given that the median is 10 and our mode is 11, our mode can occur at most twice - else our median would be 11. Thus $d = e = 11$. As our mean is 9, we have that

$$\frac{a + b + 10 + 11 + 11}{5} = 9 \Rightarrow a + b = 13,$$

where $b \leq 9$. The smallest value of a is 4.

CC346. An ant paces along the x-axis at a constant rate of one unit per second. He begins at $x = 0$ and his path takes him one unit forward, then two back, then three forward, etc. How many times does the ant step on the point $x = 10$ in the first five minutes of his walk?

Originally Problem 10 of Game 3 from the 2015-16 Nova Scotia Math League.

We received 3 submissions to this problem, only one of which was correct. We present the solution by Ivko Dimitríc.

By a single move we mean a stretch that the ant paces in the same direction before turning to move in the opposite direction. We use notation $M_n^F(k)$ to denote the ant's nth move, which is a forward move ending up at number k on the x-axis. Similarly, $M_n^B(-k)$ means that on the nth move the ant moved back (to the left on the x-axis) and ended up at number $-k$. Since every subsequent move is one unit longer, the ant progresses 1 unit to the right on each move forward (an odd-numbered move) compared to the previous forward move and progresses 1 unit to the left on each even-numbered move:

$M_1^F(1),\ M_3^F(2),\ M_5^F(3),\ M_7^F(4),\ \ldots M_{2k-1}^F(k), \ldots$

$M_2^B(-1),\ M_4^B(-2),\ M_6^B(-3),\ M_8^B(-4),\ \ldots M_{2k}^B(-k), \ldots$

Therefore, after $19 = 2 \cdot 10 - 1$ moves the ant finally reaches the point $x = 10$ for the first time with $M_{19}^F(10)$, whereupon it turns back to make its next move $M_{20}^B(-10)$. The time to complete the first 20 moves is equal to the length traveled,

$$1 + 2 + 3 + \cdots + 20 = \frac{20 \cdot 21}{2} = 210$$

seconds and during this time the ant stepped on $x = 10$ only once. Then in each subsequent move the ant will step on the point 10 once each time as it moves forward or back. On the next four moves which last 21 seconds (F), 22 seconds (B), 23 seconds (F) and 24 seconds (B), the number 10 will be stepped on once during each move, thus four additional times. The total elapsed time to complete all 24 steps is

$$210 + 21 + 22 + 23 + 24 = 300 \text{ seconds} = 5 \text{ minutes},$$

during which time the ant stepped on the point $x = 10$ exactly 5 times.

CC347. Find the sum of all fractions p/q between 0 and 1 that have denominator 100 when expressed in lowest terms.

Originally Problem 9 of Game 1 from the 2017-18 Nova Scotia Math League.

We received 8 correct submissions. We present two of the solutions.

Solution 1, by Henry Ricardo.

The number of positive integers relatively prime to 100 is

$$\phi(100) = 100(1 - 1/2)(1 - 1/5) = 40,$$

where ϕ denotes Euler's totient function. Note that if a numerator $k \in \{1, 2, \ldots, 99\}$ is relatively prime to 100, then so is $100 - k$, thus we have $\phi(100)/2 = 20$ pairs $\{k, 100 - k\}$ of numerators, each pair adding to 100. Therefore the sum of all fractions p/q between 0 and 1 that have denominator 100 when expressed in lowest terms is $(20 \cdot 100)/100 = 20$.

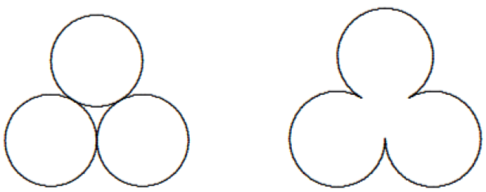
Solution 2, by Kathleen E. Lewis.

Since all of the fractions have the same denominator, we just need to add up the numerators and divide the sum by 100. The numerators we want are those that are relatively prime to 100, thus have no factor of 2 or 5. Using inclusion-exclusion, we start with all the positive integers less than or equal to 100, then subtract the multiples of 2 and the multiples of 5. Finally we need to add the multiples of 10, as they have been subtracted twice. So we get

$$\begin{aligned}&\frac{100\cdot 101}{2}-2\cdot\frac{50\cdot 51}{2}-5\cdot\frac{20\cdot 21}{2}+10\cdot\frac{10\cdot 11}{2}\\&=\frac{100}{2}(101-51-21+11)\\&=50\cdot 40=2000.\end{aligned}$$

Dividing by 100, we obtain 20 for the sum of the fractions.

CC348. Three circles with the same radius r are mutually tangent as shown on the left figure. The arcs in the middle are removed, making a trefoil (right figure). Determine the exact length of the trefoil in terms of r.



Originally from the 2017-18 Final Game of the Newfoundland and Labrador Teachers' Association Senior Math League.

We received 10 submissions, all correct. We present the solution provided by Richard Hess.

Each of the three circles has a circumference of $2\pi r$, so the total length before removal is $6\pi r$. The removed segments add to πr, giving the arc length of the trefoil as $5\pi r$.

CC349. Let O be the centre of equilateral triangle ABC (i.e. the unique point equidistant from each vertex). Another point P is selected uniformly at random in the interior of $\triangle ABC$. Find the probability that P is closer to O than it is to any of A, B or C.

Originally Problem 10 of Game 3 from the 2017-18 Nova Scotia Math League.

We received 3 submissions, all correct. We present the solution provided by Richard Hess.

The perpendicular bisectors of OA, OB and OC produce a regular hexagon inside the equilateral triangle. This hexagon has an area twice the sum of the three small triangular areas outside the hexagon but inside the large triangle. Thus P has a probability of 2/3 of being closer to O than any of A, B or C.

CC350. A friend proposes the following guessing game: He chooses an integer between 1 and 100, inclusive, and you repeatedly try to guess his number. He tells you whether each incorrect guess is higher or lower than his chosen number, but you are allowed at most one high guess overall. You win the game when you guess his number correctly. You lose the game the instant you make a second high guess. What is the minimum number of guesses in which you can guarantee you will win the game?

Originally Problem 10 of Game 3 from the 2011-12 Nova Scotia Math League.

We received 3 submissions of which only one was correct. We present the solution by Richard Hess.

I can guess the number in 14 or fewer guesses if the number is 1 to 105 inclusive. My first guess is 14. If high, my next 13 guesses are 1 to 13. If low, my second guess is 27. If this is high, my next guesses are 15-26. If my second guess is low, my third guess is 39. If I continue to guess low, my next guesses are 50, 60, 69, 77, 84, 90, 95, 99, 102, 104, and 105. If any of these is a high guess, then the remaining guesses are enough to guess all possibilities from low to high in the gap between that guess and the prior low guess.