PROBLEM SOLVING VIGNETTES

No.5

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Introducing Induction

Mathematical induction is a very powerful technique used to show that a certain property holds for infinitely many cases. It is based on a nice piece of logic where we show that if the property holds for a case then it must hold for the "next" case. This induction step then can be used recursively to show that if the property holds for $n = 1$, then it must hold for $n = 2$, but then it must hold for $n = 3$ and so on. We will examine induction in action by trying to prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

We may have come up with the expression $\frac{n(n+1)}{2}$ by playing with a few cases, realizing it must be a quadratic because the second differences would be constant (why?). To show it is true for all positive integers n, we will first show it is true for a specific case: $n = 1$. In this case our "sum" is just 1, while our expression gives $\frac{(1)(1+1)}{2} = 1$, which shows the expression is good in this case.

Next we will do our induction step, we will *assume* that the statement works for some $n = k$. We can do this, because we know there is at least one case where the expression gives the correct result, that is $n = 1$. So if the statement is true for $n = k$, then we must have

$$\sum_{i=1}^{k} i = 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \tag{1}$$

so then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= 1 + 2 + \cdots + k + (k+1) \\
&= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by our assumption (1)} \\
&= (k+1)\left[\frac{k}{2} + 1\right] \\
&= \frac{(k+1)(k+2)}{2} = \frac{(\boxed{k} + 1)(\boxed{k+1} + 1)}{2} && (2)
\end{aligned}$$

which is just our formula (1) with k replaced by $k+1$. This shows that if the formula holds for some positive integer $n=k$, then it holds for the next positive integer $n=k+1$. When we couple this with our demonstration that the formula works for $n=1$, it shows that the property is true for all positive integers n. $\square$

Some people prefer to refer to the thing that we are to prove as a proposition, $P(n)$, since the proposition is being made for various positive integers, n. In our case we would define

$$P(n) : 1+2+3+\cdots+n=\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$$

which is a *boolean* function. It is either *true* or *false* for any particular n.

Thus our proof goes by showing $P(1)$ is true, that is $1=\frac{(1)(1+1)}{2}$ then assuming $P(k)$ is true for some k (we can do this since we know it is true for at least $k=1$), which was (1). Then we try to show $P(k)\Rightarrow P(k+1)$, which we showed with (2).

Induction works like a row of dominoes. The induction step $P(k)\Rightarrow P(k+1)$, says "if one domino falls over it will knock the next one over". Then $P(1)$ says "knock over the first domino" and away it goes! Domino #1 knocks over domino #2, which knocks over domino #3, ... The difference is with a mathematical proposition we have infinitely many dominoes!

Take a moment and try your hand at a proof by induction. The statement below is similar to the one we just did. Prove

$$1^2+2^2+3^2+\cdots+n^2=\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}. \tag{3}$$

The proof of this will appear at the end of the column.

We will now examine another proof by induction. The problem below is problem 10b from the 2019 Euclid contest put on by the CEMC. You can check out past CEMC contests on their website `www.cemc.uwaterloo.ca`.

> *Carlos has* 14 *coins, numbered* 1 *to* 14*. Each coin has exactly one face called "heads". When flipped, coins* $1,2,3,\ldots,13,14$ *land heads with probabilities* $h_1,h_2,h_3,\ldots,h_{13},h_{14}$*, respectively. When Carlos flips each of the* 14 *coins exactly once, the probability that an even number of coins land heads is exactly* $\frac{1}{2}$*. Must there be a* k *between* 1 *and* 14*, inclusive, for which* $h_k=\frac{1}{2}$*? Prove your answer.*

We will actually prove the more general situation and use that to show that the specific case, with 14 coins, must then be true. Thus we will prove the following:

> *If we have* $2n$ *coins, with probabilities for flipping heads* $h_1,h_2,\ldots,h_{2n}$ *and the probability of flipping an even number of heads is* $\frac{1}{2}$ *then there exists some* k*, with* $1\le k\le 2n$*, such that* $h_k=\frac{1}{2}$*.*

Define $E(2n)$ and $O(2n)$ to be the probabilities of flipping an even number of heads and an odd number of heads, respectively, on a flip of $2n$ coins with probabilities of heads for each coin $h_1, h_2, \ldots, h_{2n}$. Then clearly $E(2n) + O(2n) = 1$ implies

$$\text{if } E(2n) = \frac{1}{2}, \text{ then } O(2n) = \frac{1}{2}$$

for all n since the events are mutually exclusive and there is no other possibility.

Examining the case where $n = 1$, we have two coins with $E(2) = O(2) = \frac{1}{2}$. Hence, if we have an even number of heads we have both coins showing heads or neither coin showing heads. Thus

$$\begin{aligned}E(2) = \frac{1}{2} &\Rightarrow h_1h_2 + (1-h_1)(1-h_2) = \frac{1}{2}\\ &\Rightarrow 2h_1h_2 - h_1 - h_2 + 1 = \frac{1}{2}\\ &\Rightarrow 4h_1h_2 - 2h_1 - 2h_2 + 2 = 1\\ &\Rightarrow 4h_1h_2 - 2h_1 - 2h_2 + 1 = 0\\ &\Rightarrow (2h_1 - 1)(2h_2 - 1) = 0\end{aligned}$$

hence $h_1 = \frac{1}{2}$ or $h_2 = \frac{1}{2}$ and our proposition is true for $n = 1$.

Now we will assume that the proposition is true for some $n = k$, that is we assume that if $E(2k) = \frac{1}{2}$, then at least one coin has probability of heads of $\frac{1}{2}$. It follows that if $E(2(k+1)) = \frac{1}{2}$, we will break our coins into two groups with probabilities $p_1, p_2, \ldots, p_{2k}$ and p_{2k+1}, p_{2k+2} and we will let $E(2k)$ be the probability of flipping an even number of heads with the first $2k$ coins.

To have an even number of coins showing heads, we have either both or neither coins $2k+1$ and $2k+2$ are heads and an even number of the first $2k$ coins are heads; or exactly one of the coins $2k+1$ and $2k+2$ is heads and an odd number of the first $2k$ coins are heads. Since $E(2(k+1)) = \frac{1}{2}$, we have

$$\begin{aligned}&(p_{2k+1}p_{2k+2} + (1-p_{2k+1})(1-p_{2k+2}))E(2k)\\ &\qquad + (p_{2k+1}(1-p_{2k+2}) + (1-p_{2k+1})p_{2k+2})O(2k) = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}&(4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 2)E(2k)\\ &\qquad + (2p_{2k+1} + 2p_{2k+2} - 4p_{2k+1}p_{2k+2})O(2k) = 1,\end{aligned}$$

$$\begin{aligned}&(4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 1 + 1)E(2k)\\ &\qquad - (4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 1 - 1)O(2k) = 1,\end{aligned}$$

$$(2p_{2k+1} - 1)(2p_{2k+2} - 1)(E(2k) - O(2k)) + E(2k) + O(2k) = 1,$$

but since $E(2k) + O(2k) = 1$ we have

$$(2p_{2k+1} - 1)(2p_{2k+2} - 1)(E(2k) - O(2k)) = 0$$

thus

$$\begin{aligned} 2p_{2k+1} - 1 = 0 \Rightarrow p_{2k+1} &= \frac{1}{2} \quad \text{or,} \\ 2p_{2k+2} - 1 = 0 \Rightarrow p_{2k+2} &= \frac{1}{2} \quad \text{or,} \\ E(2k) - O(2k) = 0 \Rightarrow E(2k) = O(2k) \Rightarrow E(2k) = O(2k) &= \frac{1}{2}. \end{aligned}$$

So either the probability of at least one of coins $2k+1$ and $2k+2$ have probability of $\frac{1}{2}$ or the probability of there being an even number of heads in the first $2k$ coins is $\frac{1}{2}$. But by our assumption if any set of $2k$ coins has $E(2k) = \frac{1}{2}$ then at least one of the coins has probability of $\frac{1}{2}$, which shows it works for $n = k+1$ and hence the property is true for all positive integers n and hence must be true for $n = 7$, which is the original problem. □

Our proof showed that for any even number of coins, if the probability of flipping an even number of heads is $\frac{1}{2}$ then at least one coin has probability of heads being $\frac{1}{2}$. It turns out that we were being too specific, and it can be shown that the property works when we have an odd number of coins as well. If you look at the official solution from the CEMC, they use a method of descent that is similar to induction. As an exercise, you may want to prove that the property holds no matter how many coins we have to start. The proof should be a bit easier than the one we presented.

Mathematical induction is a useful technique to add to your problem solving toolkit. Look for places that you can use it!

Ok, here is the proof of (3), namely that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since

$$\frac{(1)((1)+1)(2(1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1 = 1^2$$

the statement is true for $n = 1$.

Let's assume the statement is true for $n = k$, that is

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}. \tag{4}$$

Then, we must have

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \sum_{i=1}^{k+1} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{by (4)} \\
&= \frac{k+1}{6}(k(2k+1) + 6(k+1)) \\
&= \frac{k+1}{6}(2k^2 + 7k + 6) \\
&= \frac{k+1}{6}(k+2)(2k+3) \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}
$$

which means the statement is true for $n = k+1$ and hence the statement is true for all positive integers by induction. □