FOCUS ON...

No. 36

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Geometry with Complex Numbers (I)

Introduction

Among the various angles of attack of a geometry problem, the use of complex numbers is sometimes chosen. The method, often very direct, involves clever calculations instead of auxiliary constructions or synthetic arguments and can produce elegant solutions. The purpose of this number and the next one is to show some examples of applications of this method. In this part I, we will focus on the frequent case when a triangle and its circumcircle are at the heart of the problem. In part II, we will consider and illustrate other applications.

For an introduction to the method with a selection of exercises, we refer the reader to [1]; for a more thorough treatise on the subject, a good reference is [2].

Complex numbers and circumcircle: a direct approach

We will consider problems for which we can suppose that the circumcircle Γ of a given triangle ABC is the unit circle. We will denote by the lower-case letter m the affix of any point M. Thus, the affixes a, b, c of the vertices A, B, C satisfy $a\overline{a} = b\overline{b} = c\overline{c} = 1$ while the affix of the circumcentre O is (conveniently) 0. We shall freely use the following lemma:

> If A, B are two distinct points of Γ and U, V two distinct points of the plane, then
>
> (i) UV is parallel to AB if and only if $v - u = -ab(\overline{v} - \overline{u})$;
>
> (ii) UV is perpendicular to AB if and only if $v - u = ab(\overline{v} - \overline{u})$.

Proof. (i) The line UV is parallel to AB if and only if $\dfrac{v-u}{b-a}$ is a real number, that is, if and only if

$$\frac{v-u}{b-a} = \frac{\overline{v}-\overline{u}}{\overline{b}-\overline{a}}.$$

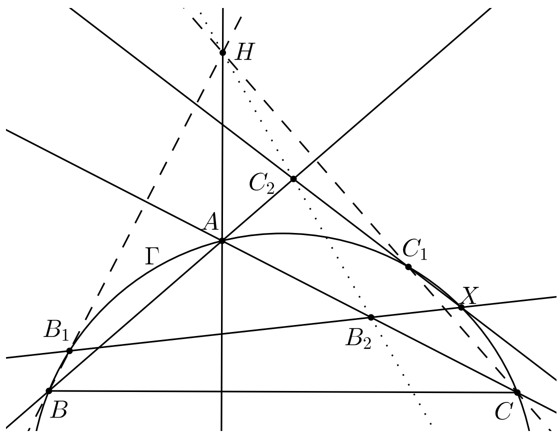
Since $\overline{b} = \frac{1}{b}$ and $\overline{a} = \frac{1}{a}$, a simple calculation gives the condition $v-u = -ab(\overline{v}-\overline{u})$.

The proof of *(ii)* is similar, with UV being perpendicular to AB if and only if $\dfrac{v-u}{b-a} = -\dfrac{\overline{v}-\overline{u}}{\overline{b}-\overline{a}}$. □

With the help of this lemma, it is easy to obtain the equation of the line AB: $z + ab\overline{z} = a + b$ and the equation of the tangent to Γ at A: $z + a^2\overline{z} = 2a$.

We are now ready to consider a first example, namely problem 11846 set in the *American Mathematical Monthly* in 2015. Here is the slightly modified statement:

> Let ABC be a triangle with no right angle, and let B_1 and C_1 be the points where the altitudes from B and C intersect the circumcircle. Let X be a point of Γ, not diametrically opposite to B or C, and let B_2 and C_2 denote the intersections of XB_1 with AC and XC_1 with AB, respectively. Prove that the line B_2C_2 contains the orthocenter of ABC.



Recall that the orthocentre H satisfies $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, hence its affix is $h = a + b + c$.

Since the line BB_1, whose equation is $z + bb_1\overline{z} = b + b_1$, passes through H, we have

$$(a+b+c) + bb_1\overline{(a+b+c)} = b + b_1$$

and so $b_1 = -\dfrac{ca}{b}$. It follows that the equation of XB_1 is

$$z - \frac{cax}{b}\overline{z} = x - \frac{ca}{b}$$

(with the convention that XB_1 is the tangent to Γ at B_1 if $X = B_1$). Since the equation of AC is $z + ac\overline{z} = a + c$, we readily find $b_2 = \dfrac{hx - ca}{x+b}$. Similarly, $c_2 = \dfrac{hx - ab}{x+c}$ and we deduce

$$h - b_2 = \frac{hb + ca}{x+b} \qquad \text{and} \qquad h - c_2 = \frac{hc + ab}{x+c}.$$

Let $\lambda = \dfrac{h - b_2}{h - c_2}$. Using $a\overline{a} = b\overline{b} = c\overline{c} = x\overline{x} = 1$, a straightforward calculation gives $\overline{\lambda} = \lambda$, hence λ is a real number. The collinearity of the points H, B_2, C_2 follows.

The efficiency and directness of the method are also noticeable in the following example proposed in the *Mathematical Gazette* in 2013:

> Triangle $A'B'C'$ is the image of a given triangle ABC after rotation through 180° about a given point P in its plane. Points A'', B'' and C'' are the reflections of A' in BC, B' in CA and C' in AB, respectively. Prove that
>
> (i) the circumcentres of triangles $ABC, A''B''C''$ coincide;
>
> (ii) triangles $ABC, A''B''C''$ are similar;
>
> (iii) the orthocentre of triangle $A'B'C'$ lies on the circle $A''B''C''$.

Let H and H' be the orthocentres of ΔABC and $\Delta A'B'C'$. Note that H' is the image of H under the rotation through 180° about P.

Because the midpoint of $A'A''$ is on the line BC, we have

$$(a' + a'') + bc(\overline{a'} + \overline{a''}) = 2(b + c)$$

and because $A'A''$ is perpendicular to BC, we have

$$a'' - a' = bc(\overline{a''} - \overline{a'}).$$

Also, $a' = 2p - a$ expresses that P is the midpoint of AA'. From these relations, we readily obtain:

$$a'' = b + c + \overline{a}bc - 2bc\overline{p},$$

that is, $a'' = \lambda\overline{a}$, where

$$\lambda = ab + bc + ca - 2abc\overline{p} = abc(\overline{a} + \overline{b} + \overline{c} - 2\overline{p}) = abc\overline{(h - 2p)} = -abc\overline{h'}.$$

From the symmetry of λ in a, b, c, we deduce $b'' = \lambda\overline{b},\ c'' = \lambda\overline{c}$. Now, assuming that $h' \neq 0$, that is, P is not the centre of the Euler circle of ΔABC, (i),(ii),(iii) result successively from

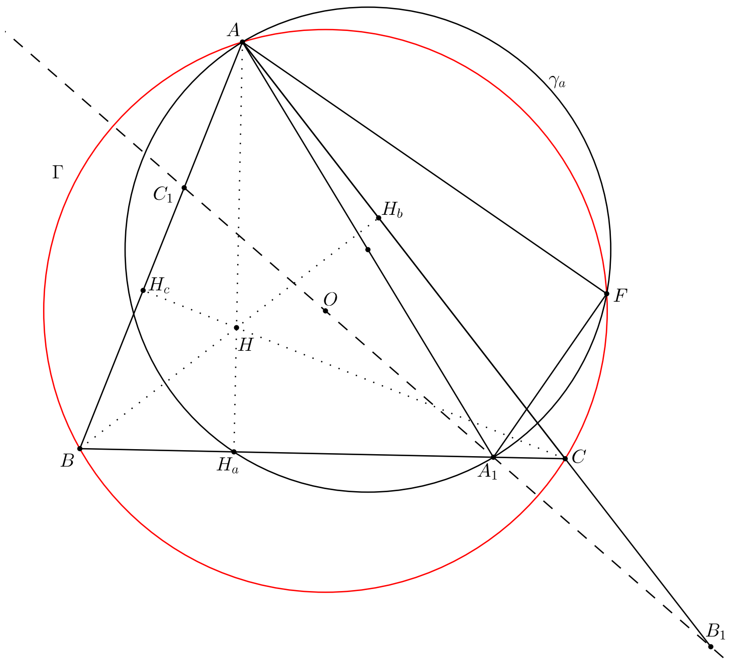
- $|a''| = |b''| = |c''|(= |\lambda|)$;
- $\dfrac{A''B''}{AB} = \dfrac{|\lambda||\overline{b} - \overline{a}|}{|b - a|} = |\lambda| = \dfrac{B''C''}{BC} = \dfrac{C''A''}{CA}$;
- $|h'| = |\overline{h'}| = |-abc\overline{h'}| = |\lambda|$.

(When P is the centre of the Euler circle of triangle ABC, we have that $A'' = B'' = C'' = O = H'$.)

We conclude the paragraph with a more difficult problem, problem **3585** [2013 : 414 ; 2014 : 399], of which two geometric solutions have been published.

> Let ABC be a triangle and let F be a point that lies on the circumcircle of ABC. Further, let H_a, H_b and H_c denote projections of the

orthocenter onto sides BC, AC and AB, respectively. The three circles AH_aF, BH_bF and CH_cF meet the three sides BC, AC and AB at points A_1, B_1 and C_1, respectively. Prove that A_1, B_1 and C_1 are collinear.



To ensure the existence of the circles $\gamma_a = (AH_aF), \gamma_b = (BH_bF), \gamma_c = (CH_cF)$, we assume that F is different from A, B, C and from the reflections of H in the sidelines of ΔABC. We note that the triangle AH_aA_1 is right-angled at H_a and inscribed in γ_a, hence AA_1 is a diameter of γ_a.

It follows that $\angle AFA_1 = 90^\circ$ so that A_1 is the point of intersection of BC and the perpendicular to AF at F. Since these two lines have respective equations

$$z + bc\overline{z} = b + c \quad \text{and} \quad z - af\overline{z} = f - a,$$

we obtain $a_1 = \dfrac{k}{bc + af}$ where $k = (ab + bc + ca)f - abc$. By circular permutation, we obtain $b_1 = \dfrac{k}{ca + bf}$ and $c_1 = \dfrac{k}{ab + cf}$. Now, $\dfrac{b_1}{a_1} = \dfrac{bc + af}{ca + bf}$ and,

$$\frac{\overline{b_1}}{\overline{a_1}} = \frac{\frac{1}{bc} + \frac{1}{af}}{\frac{1}{ca} + \frac{1}{bf}} = \frac{b_1}{a_1}.$$

Thus, $\dfrac{b_1}{a_1}$ is a real number and so B_1 is on the line OA_1. Similarly, C_1 is on OA_1 and the conclusion follows with an additional result: the line through A_1, B_1, C_1 passes through the centre O of the circumcircle of ΔABC.

Complex numbers and circumcircle: when angle bisectors are involved

When the angle bisectors of ΔABC play a central role in the problem, the above approach is not quite suitable, as it leads to complicated calculations. It seems preferable to adopt the following way: keeping the circumcircle Γ as the unit circle, we suppose that the affixes of A, B, C are $e^{i\alpha}$, $e^{i\beta}$, $e^{i\gamma}$, respectively, with $0 < \alpha < \beta < \gamma < 2\pi$. Denoting the incentre by I, the bisectors IA, IB, IC then intersect Γ again at A', B', C' whose respective affixes are

$$e^{i(\beta+\gamma)/2},\ -e^{i(\gamma+\alpha)/2},\ e^{i(\alpha+\beta)/2}.$$

For sake of simplicity, we set

$$a = e^{i\alpha/2},\ b = -e^{i\beta/2},\ c = e^{i\gamma/2}$$

so that the affixes of A, B, C, A', B', C' are $a^2, b^2, c^2, -bc, -ca, -ab$, respectively. Note that A', B', C' are the respective circumcentres of the triangles IBC, ICA, IAB. From the equations $z - a^2bc\overline{z} = a^2 - bc$ and $z - ab^2c\overline{z} = b^2 - ca$ of the lines AA' and BB', we easily obtain the affix $-(ab+bc+ca)$ of I. In a similar way, the reader will obtain the respective affixes

$$ab + ca - bc,\ bc + ba - ca,\ ca + cb - ab$$

of the excenters I_a, I_b, I_c.

After these preliminaries, what about a starter and a main course? First consider problem **4268** [2017 : 303 ; 2018 : 312]:

> Let I be the incenter of the acute triangle ABC, and let the triangle's internal angle bisectors intersect the circles IBC, ICA, and IAB again at A_1, B_1, and C_1, respectively. Show that $\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = \overrightarrow{0}$ if and only if ΔABC is equilateral.

The solution is very short: the affixes of the vectors $\overrightarrow{IA'}, \overrightarrow{IB'}, \overrightarrow{IC'}$ are $ab + ac$, $bc + ab$, $ac + bc$, respectively, hence the affix of

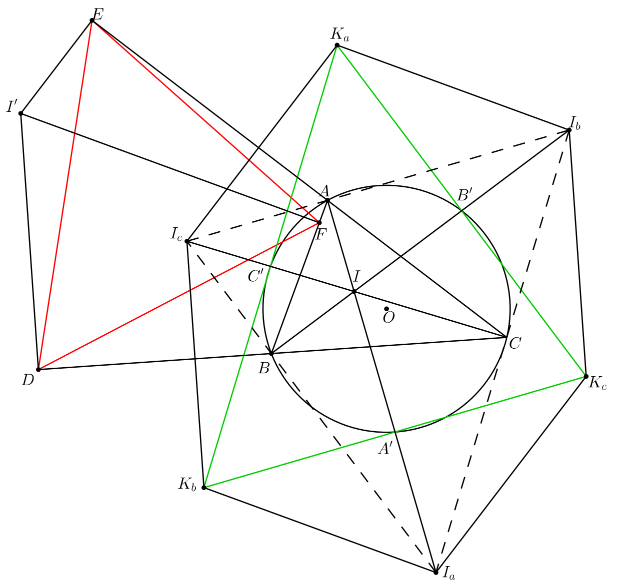
$$\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 2(\overrightarrow{IA'} + \overrightarrow{IB'} + \overrightarrow{IC'})$$

is $4(ab+bc+ca)$. Thus, $\overrightarrow{IA_1}+\overrightarrow{IB_1}+\overrightarrow{IC_1} = \overrightarrow{0}$ if and only if $ab+bc+ca = 0$, if and only if $I = O$, that is, if and only if ABC is equilateral. Note that the hypothesis ABC acute is not needed.

Less easy is problem 123 in 2015 issue 1 of *Mathproblems*, which can be found at `http://www.mathproblems-ks.org`. Here is the statement, a little extended:

> Let I_a, I_b, I_c be the excenters of a triangle ABC and let K_a be the point in which the perpendicular to AB through I_b meets the perpendicular to AC through I_c. Similarly define K_b and K_c. Prove that K_a, K_b, K_c are the respective reflections of I_a, I_b, I_c about the circumcentre O of ΔABC and that A', B', C' are the midpoints of K_bK_c, K_cK_a,

K_aK_b, respectively. Prove that the triangle $K_aK_bK_c$ (known as the hexil triangle) is similar to the pedal triangle of the inverse of the incenter in the circumcircle.



For example, we show that K_a is the reflection of I_a about O and that C' is the midpoint of K_aK_b. The equations of the perpendiculars to AB through I_b and to AC through I_c are readily obtained:

$$z - a^2b^2\overline{z} = (b-a)\left(c + \frac{ab}{c}\right) \quad \text{and} \quad z - a^2c^2\overline{z} = (c-a)\left(b + \frac{ac}{b}\right).$$

Solving the system so formed shows that the affix of K_a is $bc - ab - ca$, clearly the opposite of the affix of I_a. The second assertion follows from

$$-ab = \frac{(bc - ab - ca) + (ca - bc - ab)}{2}.$$

Lastly, let $p = -(ab + bc + ca)$ be the affix of I. The affix of the inverse I' of I in Γ is $p' = \frac{1}{\overline{p}}$. The lines BC: $z + b^2c^2\overline{z} = b^2 + c^2$ and the perpendicular to BC through I': $z - \frac{1}{\overline{p}} = b^2c^2\left(\overline{z} - \frac{1}{p}\right)$ then yield the affix d of the projection D of I' onto BC

$$d = \frac{1}{2}\left(b^2 + c^2 + \frac{1}{\overline{p}} - \frac{b^2c^2}{p}\right).$$

Cyclically, we obtain the affixes e and f of the projections E and F of I' onto CA and AB

$$e = \frac{1}{2}\left(c^2 + a^2 + \frac{1}{\overline{p}} - \frac{c^2a^2}{p}\right) \quad \text{and} \quad f = \frac{1}{2}\left(a^2 + b^2 + \frac{1}{\overline{p}} - \frac{a^2b^2}{p}\right).$$

Simple calculations give

$$e - d = (a+b)(b+c)(c+a) \cdot \frac{b-a}{2p} \quad \text{and} \quad f - d = (a+b)(b+c)(c+a) \cdot \frac{c-a}{2p}$$

so that

$$\frac{e-d}{f-d} = \frac{b-a}{c-a}.$$

On the other hand, if p_a, p_b, p_c denote the affixes of I_a, I_b, I_c, then $\frac{p_b - p_a}{p_c - p_a} = \frac{c(b-a)}{b(c-a)}$ and so

$$\frac{\overline{p_b} - \overline{p_a}}{\overline{p_c} - \overline{p_a}} = \frac{\frac{1}{c}\left(\frac{1}{b} - \frac{1}{a}\right)}{\frac{1}{b}\left(\frac{1}{c} - \frac{1}{a}\right)} = \frac{b-a}{c-a}$$

Thus,

$$\frac{e-d}{f-d} = \frac{\overline{p_b} - \overline{p_a}}{\overline{p_c} - \overline{p_a}}, \tag{*}$$

meaning that ΔDEF and $\Delta I_aI_bI_c$ are (inversely) similar and so are ΔDEF and $\Delta K_aK_bK_c$. For the conclusion drawn from $(*)$, see part II or [2] p. 57-58.

The reader is invited to solve the two following exercises with the help of complex numbers.

Exercises

1. Given an acute triangle ABC, let O be its circumcenter, let M be the intersection of lines AO and BC, and let D be the other intersection of AO with the circumcircle of ABC. Let E be that point on AD such that M is the midpoint of ED. Let F be the point at which the perpendicular to AD at M meets AC. Prove that EF is perpendicular to AB. [Problem 11737 of the *American Mathematical Monthly*.]

2. Let H, I, O be the orthocenter, incentre, circumcentre of a triangle ABC and let J be the reflection of I about O. Prove that the line through the midpoints of JH and BC is parallel to AI.

References

[1] M. Bataille, Chapter 4 in *Géométrie plane, avec des nombres*, CMS/SMC Atom XV, 2015.

[2] Liang-shin Hahn, *Complex numbers and Geometry*, MAA, 1994.