Introduction

Among the various angles of attack of a geometry problem, the use of complex numbers is sometimes chosen. The method, often very direct, involves clever calculations instead of auxiliary constructions or synthetic arguments and can produce elegant solutions. The purpose of this number and the next one is to show some examples of applications of this method. In this part I, we will focus on the frequent case when a triangle and its circumcircle are at the heart of the problem. In part II, we will consider and illustrate other applications.

For an introduction to the method with a selection of exercises, we refer the reader to [1]; for a more thorough treatise on the subject, a good reference is [2].

Complex numbers and circumcircle: a direct approach

We will consider problems for which we can suppose that the circumcircle $\Gamma$ of a given triangle $ABC$ is the unit circle. We will denote by the lower-case letter $m$ the affix of any point $M$. Thus, the affixes $a, b, c$ of the vertices $A, B, C$ satisfy $a\bar{a} = b\bar{b} = c\bar{c} = 1$ while the affix of the circumcentre $O$ is (conveniently) 0. We shall freely use the following lemma:

If $A, B$ are two distinct points of $\Gamma$ and $U, V$ two distinct points of the plane, then

(i) $UV$ is parallel to $AB$ if and only if $v - u = -ab(\overline{v} - \overline{w})$;

(ii) $UV$ is perpendicular to $AB$ if and only if $v - u = ab(\overline{v} - \overline{w})$.

Proof. (i) The line $UV$ is parallel to $AB$ if and only if $\frac{v - u}{b - a}$ is a real number, that is, if and only if

$$\frac{v - u}{b - a} = \frac{\overline{v} - \overline{w}}{\overline{b} - \overline{a}}.$$

Since $\overline{b} = \frac{1}{b}$ and $\overline{a} = \frac{1}{a}$, a simple calculation gives the condition $v - u = -ab(\overline{v} - \overline{w})$.

The proof of (ii) is similar, with $UV$ being perpendicular to $AB$ if and only if

$$\frac{v - u}{b - a} = -\frac{\overline{v} - \overline{w}}{\overline{b} - \overline{a}}.$$  \hfill $\Box$

With the help of this lemma, it is easy to obtain the equation of the line $AB$: $z + ab\bar{z} = a + b$ and the equation of the tangent to $\Gamma$ at $A$: $z + a^2\overline{z} = 2a$.
We are now ready to consider a first example, namely problem 11846 set in the *American Mathematical Monthly* in 2015. Here is the slightly modified statement:

Let $ABC$ be a triangle with no right angle, and let $B_1$ and $C_1$ be the points where the altitudes from $B$ and $C$ intersect the circumcircle. Let $X$ be a point of $\Gamma$, not diametrically opposite to $B$ or $C$, and let $B_2$ and $C_2$ denote the intersections of $XB_1$ with $AC$ and $XC_1$ with $AB$, respectively. Prove that the line $B_2C_2$ contains the orthocenter of $ABC$.

Recall that the orthocentre $H$ satisfies $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, hence its affix is $h = a + b + c$.

Since the line $BB_1$, whose equation is $z + bb_1\bar{z} = b + b_1$, passes through $H$, we have $$(a + b + c) + bb_1(a + b + c) = b + b_1$$
and so $b_1 = -\frac{ca}{b}$. It follows that the equation of $XB_1$ is

$$z - \frac{ca}{b}z = x - \frac{ca}{b}$$

(with the convention that $XB_1$ is the tangent to $\Gamma$ at $B_1$ if $X = B_1$). Since the equation of $AC$ is $z + ac\bar{z} = a + c$, we readily find $b_2 = \frac{hx - ca}{x + b}$.

Similarly, $c_2 = \frac{hx - ab}{x + c}$ and we deduce

$$h - b_2 = \frac{hb + ca}{x + b} \quad \text{and} \quad h - c_2 = \frac{hc + ab}{x + c}.$$ 

Let $\lambda = \frac{h - b_2}{h - c_2}$. Using $a\bar{a} = b\bar{b} = c\bar{c} = x\bar{x} = 1$, a straightforward calculation gives $\overline{\lambda} = \lambda$, hence $\lambda$ is a real number. The collinearity of the points $H, B_2, C_2$ follows.
The efficiency and directness of the method are also noticeable in the following example proposed in the Mathematical Gazette in 2013:

Triangle $A'B'C'$ is the image of a given triangle $ABC$ after rotation through $180^\circ$ about a given point $P$ in its plane. Points $A'', B''$ and $C''$ are the reflections of $A'$ in $BC$, $B'$ in $CA$ and $C'$ in $AB$, respectively. Prove that

(i) the circumcentres of triangles $ABC, A''B''C''$ coincide;

(ii) triangles $ABC, A''B''C''$ are similar;

(iii) the orthocentre of triangle $A'B'C'$ lies on the circle $A''B''C''$.

Let $H$ and $H'$ be the orthocentres of $\Delta ABC$ and $\Delta A'B'C'$. Note that $H'$ is the image of $H$ under the rotation through $180^\circ$ about $P$.

Because the midpoint of $A'A''$ is on the line $BC$, we have

$$(a' + a'') + bc(a' + a'') = 2(b + c)$$

and because $A'A''$ is perpendicular to $BC$, we have

$$a'' - a' = bc(a'' - a').$$

Also, $a' = 2p - a$ expresses that $P$ is the midpoint of $AA'$. From these relations, we readily obtain:

$$a'' = b + c + bc - 2bc\overline{p},$$

that is, $a'' = \lambda \overline{a}$, where

$$\lambda = ab + bc + ca - 2abc\overline{p} = ab(\overline{\overline{a}} + \overline{\overline{b}} + \overline{\overline{c}} - 2\overline{p}) = ab(h - 2p) = -abc\overline{h}.$$

From the symmetry of $\lambda$ in $a, b, c$, we deduce $b'' = \lambda \overline{b}, c'' = \lambda \overline{c}$. Now, assuming that $h' \neq 0$, that is, $P$ is not the centre of the Euler circle of $\Delta ABC$, (i),(ii),(iii) result successively from

- $|a''| = |b''| = |e''| = |\lambda|$;
- $\frac{A''B''}{AB} = \frac{\lambda|b-a|}{|b-a|} = \lambda = \frac{B''C''}{BC} = \frac{C''A''}{CA}$;
- $|h'| = \lambda|\overline{p}| = -abc\overline{p} = |\lambda|$.

(When $P$ is the centre of the Euler circle of triangle $ABC$, we have that $A'' = B'' = C'' = O = H'$.)

We conclude the paragraph with a more difficult problem, problem 3585 [2013 : 414 ; 2014 : 399], of which two geometric solutions have been published.

Let $ABC$ be a triangle and let $F$ be a point that lies on the circumcircle of $ABC$. Further, let $H_a, H_b$ and $H_c$ denote projections of the
orthocenter onto sides $BC, AC$ and $AB$, respectively. The three circles $AH_aF, BH_bF$ and $CH_cF$ meet the three sides $BC, AC$ and $AB$ at points $A_1, B_1$ and $C_1$, respectively. Prove that $A_1, B_1$ and $C_1$ are collinear.

To ensure the existence of the circles $\gamma_a = (AH_aF), \gamma_b = (BH_bF), \gamma_c = (CH_cF)$, we assume that $F$ is different from $A, B, C$ and from the reflections of $H$ in the sidelines of $\Delta ABC$. We note that the triangle $AH_aA_1$ is right-angled at $H_a$ and inscribed in $\gamma_a$, hence $AA_1$ is a diameter of $\gamma_a$.

It follows that $\angle AFA_1 = 90^\circ$ so that $A_1$ is the point of intersection of $BC$ and the perpendicular to $AF$ at $F$. Since these two lines have respective equations

$$z + bc\bar{z} = b + c \quad \text{and} \quad z - af\bar{z} = f - a,$$

we obtain $a_1 = \frac{k}{bc + af}$ where $k = (ab + bc + ca)f - abc$. By circular permutation, we obtain $b_1 = \frac{k}{ca + bf}$ and $c_1 = \frac{k}{ab + cf}$. Now, $\frac{b_1}{a_1} = \frac{bc + af}{ca + bf}$ and,

$$\frac{b_1}{a_1} = \frac{b_1}{a_1} \cdot \frac{1}{ca + bf} = \frac{b_1}{a_1}.$$

Thus, $\frac{b_1}{a_1}$ is a real number and so $B_1$ is on the line $OA_1$. Similarly, $C_1$ is on $OA_1$ and the conclusion follows with an additional result: the line through $A_1, B_1, C_1$ passes through the centre $O$ of the circumcircle of $\Delta ABC$. 

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Complex numbers and circumcircle: when angle bisectors are involved

When the angle bisectors of $\Delta ABC$ play a central role in the problem, the above approach is not quite suitable, as it leads to complicated calculations. It seems preferable to adopt the following way: keeping the circumcircle $\Gamma$ as the unit circle, we suppose that the affixes of $A, B, C$ are $e^{i\alpha}, e^{i\beta}, e^{i\gamma}$, respectively, with $0 < \alpha < \beta < \gamma < 2\pi$. Denoting the incentre by $I$, the bisectors $IA, IB, IC$ then intersect $\Gamma$ again at $A', B', C'$ whose respective affixes are

$$e^{i(\beta+\gamma)/2}, -e^{i(\gamma+\alpha)/2}, e^{i(\alpha+\beta)/2}.$$ 

For sake of simplicity, we set

$$a = e^{i\alpha/2}, \quad b = -e^{i\beta/2}, \quad c = e^{i\gamma/2}$$

so that the affixes of $A, B, C, A', B', C'$ are $a^2, b^2, c^2$, $-bc, -ca, -ab$, respectively. Note that $A', B', C'$ are the respective circumcentres of the triangles $IBC, ICA, IAB$. From the equations $z - a^2bc = a^2 - bc$ and $z - ab^2c = b^2 - ca$ of the lines $AA'$ and $BB'$, we easily obtain the affix $-(ab + bc + ca)$ of $I$. In a similar way, the reader will obtain the respective affixes

$$ab + ca - bc, \quad bc + ba - ca, \quad ca + cb - ab$$

of the excenters $I_a, I_b, I_c$.

After these preliminaries, what about a starter and a main course? First consider problem 4268 [2017 : 303 ; 2018 : 312]:

Let $I$ be the incenter of the acute triangle $ABC$, and let the triangle’s internal angle bisectors intersect the circles $IBC, ICA, IAB$ again at $A_1, B_1, C_1$, respectively. Show that $IA_1 + IB_1 + IC_1 = \overrightarrow{0}$ if and only if $\Delta ABC$ is equilateral.

The solution is very short: the affixes of the vectors $\overrightarrow{IA'}, \overrightarrow{IB'}, \overrightarrow{IC'}$ are $ab + ac, \quad bc + ab, \quad ac + bc$, respectively, hence the affix of

$$\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 2(\overrightarrow{IA'} + \overrightarrow{IB'} + \overrightarrow{IC'})$$

is $4(ab + bc + ca)$. Thus, $\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = \overrightarrow{0}$ if and only if $ab + bc + ca = 0$, if and only if $I = O$, that is, if and only if $\Delta ABC$ is equilateral. Note that the hypothesis $ABC$ acute is not needed.

Less easy is problem 123 in 2015 issue 1 of Mathproblems, which can be found at [http://www.mathproblems-ks.org](http://www.mathproblems-ks.org). Here is the statement, a little extended:

Let $I_a, I_b, I_c$ be the excenters of a triangle $ABC$ and let $K_a$ be the point in which the perpendicular to $AB$ through $I_b$ meets the perpendicular to $AC$ through $I_c$. Similarly define $K_b$ and $K_c$. Prove that $K_a, K_b, K_c$ are the respective reflections of $I_a, I_b, I_c$ about the circumcentre $O$ of $\Delta ABC$ and that $A', B', C'$ are the midpoints of $K_bK_c, K_cK_a, K_aK_b$.
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$K_1, K_2, K_3$, respectively. Prove that the triangle $K_1 K_2 K_3$ (known as the hexil triangle) is similar to the pedal triangle of the inverse of the incenter in the circumcircle.

For example, we show that $K_1$ is the reflection of $I_a$ about $O$ and that $C'$ is the midpoint of $K_1 K_2$. The equations of the perpendiculars to $AB$ through $I_b$ and to $AC$ through $I_c$ are readily obtained:

$$z - a^2b^2 = (b - a) \left( c + \frac{ab}{c} \right) \quad \text{and} \quad z - a^2c^2 = (c - a) \left( b + \frac{ac}{b} \right).$$

Solving the system so formed shows that the affix of $K_1$ is $bc - ab - ca$, clearly the opposite of the affix of $I_a$. The second assertion follows from

$$-ab = \frac{(bc - ab - ca) + (ca - bc - ab)}{2}.$$

Lastly, let $p = -(ab + bc + ca)$ be the affix of $I$. The affix of the inverse $I'$ of $I$ in $\Gamma$ is $p' = \frac{1}{p}$. The lines $BC$: $z + b^2c^2 = b^2 + c^2$ and the perpendicular to $BC$ through $I'$: $z - \frac{1}{p} = b^2c^2 \left( \frac{1}{p} \right)$ then yield the affix $d$ of the projection $D$ of $I'$ onto $BC$

$$d = \frac{1}{2} \left( b^2 + c^2 + \frac{1}{p} - \frac{b^2c^2}{p} \right).$$

Cyclically, we obtain the affixes $e$ and $f$ of the projections $E$ and $F$ of $I'$ onto $CA$ and $AB$

$$e = \frac{1}{2} \left( c^2 + a^2 + \frac{1}{p} - \frac{c^2a^2}{p} \right) \quad \text{and} \quad f = \frac{1}{2} \left( a^2 + b^2 + \frac{1}{p} - \frac{a^2b^2}{p} \right).$$

Simple calculations give
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\[ e - d = (a + b)(b + c)(c + a) \cdot \frac{b - a}{2p} \quad \text{and} \quad f - d = (a + b)(b + c)(c + a) \cdot \frac{c - a}{2p} \]

so that

\[ \frac{e - d}{f - d} = \frac{b - a}{c - a}. \]

On the other hand, if \( p_a, p_b, p_c \) denote the affixes of \( I_a, I_b, I_c \), then

\[ \frac{p_b - p_a}{p_c - p_a} = \frac{c(b - a)}{b(c - a)} \]

and so

\[ \frac{p_b - p_a}{p_c - p_a} = \frac{1}{b} \left( \frac{1}{c} - \frac{1}{a} \right) = \frac{b - a}{c - a}. \]

Thus,

\[ \frac{e - d}{f - d} = \frac{p_b - p_a}{p_c - p_a} \quad (*) \]

meaning that \( \triangle DEF \) and \( \triangle I_aI_bI_c \) are (inversely) similar and so are \( \triangle DEF \) and \( \triangle K_aK_bK_c \). For the conclusion drawn from (*), see part II or [2] p. 57-58.

The reader is invited to solve the two following exercises with the help of complex numbers.

**Exercises**

1. Given an acute triangle \( ABC \), let \( O \) be its circumcenter, let \( M \) be the intersection of lines \( AO \) and \( BC \), and let \( D \) be the other intersection of \( AO \) with the circumcircle of \( ABC \). Let \( E \) be that point on \( AD \) such that \( M \) is the midpoint of \( ED \). Let \( F \) be the point at which the perpendicular to \( AD \) at \( M \) meets \( AC \). Prove that \( EF \) is perpendicular to \( AB \). [Problem 11737 of the *American Mathematical Monthly*.]

2. Let \( H, I, O \) be the orthocenter, incentre, circumcentre of a triangle \( ABC \) and let \( J \) be the reflection of \( I \) about \( O \). Prove that the line through the midpoints of \( JH \) and \( BC \) is parallel to \( AI \).

**References**
