SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4336. Proposed by Michel Bataille.

For non-negative integers $m$ and $n$, evaluate in closed form

$$\sum_{k=0}^{n} \sum_{j=0}^{m} (j+k+1)\binom{j+k}{j}.$$

We received 7 submissions, all correct. We present the solution by Paul Bracken, modified and enhanced by the editor.

Denote the given sum by $S$. Since both summations are finite, they can be interchanged to yield:

$$S = \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{j+k}{j} + \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{j+k}{k} + \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{j+k}{j} \tag{1}$$

The two formulae below are known for $m, n, r, k \in \mathbb{N} \cup \{0\}$.

\begin{align*}
(a) \quad \sum_{k=0}^{n} \binom{r+k}{r} &= \binom{r+k+1}{r+1}, \tag{2} \\
(b) \quad \sum_{k=0}^{m} \binom{n+k+1}{k+1} &= \frac{m(n+1)-1}{n+2}\binom{n+m+2}{n+1} + 1, \tag{3}
\end{align*}

From (1) – (3) we then have

$$S = \sum_{j=0}^{m} \binom{j+n+1}{j+1} + \sum_{k=1}^{n} \binom{k+m+1}{k+1} + \sum_{j=0}^{m} \binom{j+n+1}{j+1}$$

$$= \frac{m(n+1)-1}{n+2}\binom{m+n+2}{n+1} + 1 + \frac{n(m+1)-1}{m+2}\binom{m+n+2}{m+1} + 1$$

$$+ \frac{m+n+2}{n+1} - 1$$

$$= \left(\frac{m(n+1)-1}{n+2} + \frac{n(m+1)-1}{m+2} + 1\right)\binom{m+n+2}{n+1} + 1 \tag{4}$$

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Now by straightforward but tedious computations, we have

\[(m + 2)(n + 1)m - (m + 2) + (m + 1)(n + 2)n - (n + 2) + (m + 2)(n + 2)\]

\[= mn^2 + m^2n + m^2 + n^2 + 5mn + 3m + 3n\]

\[= mn(m + n + 3) + m(m + n + 3) + n(m + n + 3)\]

\[= (m + n + 3)(mn + m + n).\]  

(5)

Substitute (5) into (4) we finally obtain

\[S = \frac{(m + n + 3)(mn + m + n)}{(m + 2)(n + 2)} \cdot \left( \frac{m + n + 2}{n + 1} \right) + 1\]

\[= \frac{mn + m + n}{m + 2} \cdot \left( \frac{m + n + 3}{n + 2} \right) + 1.\]

4337. Proposed by Mihaela Berindeanu.

Let \(h_a, h_b, h_c\) be the altitudes from vertices \(A, B, C\), respectively, of triangle \(ABC\). Erect externally on its sides three rectangles \(ABB_1A_2, BCC_1B_2, CAA_1C_2\), whose widths are \(k\) times as long as the parallel altitudes; that is,

\[\frac{CC_1}{h_a} = \frac{AA_1}{h_b} = \frac{BB_1}{h_c} = k.\]

If \(X, Y, Z\) are the respective midpoints of the segments \(A_1A_2, B_1B_2, C_1C_2\), prove that the lines \(AX, BY, CZ\) are concurrent.

We received 8 solutions, all of which were correct. We present the solution by Ivko Dimitrić, who notes that this problem is very similar to the Crux problem 4258 by the same proposer, and the published solutions of that problem – in particular
solution 2 – are readily adapted to provide the solution of the present problem. The solution given here gives an alternative approach.

Let $O$ be the circumcenter of triangle $ABC$. Then

$$\angle OAC = \frac{1}{2}(180^\circ - \angle AOC) = 90^\circ - \angle B.$$ 

Let $E$ and $F$ be the feet of the altitudes from $B$ and $C$, respectively, and let $EK$ be the straight line segment in length and parallel to $CF$ that intersects $AB$ perpendicularly. Then the quadrilateral $KFCE$ is a parallelogram and $KF \parallel AC$ so that

$$\angle FKE = \angle KEA = 90^\circ - \angle A.$$ 

Moreover,

$$\angle KEB = \angle EHC = \angle A$$

since $AFHE$ is cyclic. Since

$$\tan A = \frac{h_b}{AE} = \frac{h_c}{AF},$$

we have

$$\frac{EB}{EK} = \frac{h_b}{h_c} = \frac{AE}{AF},$$

so $\triangle BEK$ is similar to $\triangle EAF$ (and to $\triangle BAC$) and consequently

$$\angle BKE = \angle AFE = \angle C.$$ 

Since $BCEF$ is cyclic we have

$$\angle BKF = \angle BKE - \angle FKE = \angle C - (90^\circ - \angle A) = 90^\circ - \angle B = \angle OAC,$$

so $BK \parallel OA$. Further,

$$\overrightarrow{AX} = \frac{1}{2}(\overrightarrow{AA_1} + \overrightarrow{AA_2}) = \frac{k}{2}(\overrightarrow{h_b} + \overrightarrow{h_c}) = \frac{k}{2}\overrightarrow{BK}$$

and hence $AX \parallel BK \parallel OA$, meaning that $O, A, X$ are collinear. In the same manner, each triple $O, B, Y$ and $O, C, Z$ consists of collinear points, which implies that the lines $AX, BY$ and $CZ$ are concurrent at the circumcenter $O$, the isogonal conjugate of the orthocenter $H$.

4338. Proposed by Daniel Sitaru.

Prove that for any triangle $ABC$, we have

$$2 \sum \left| \cos \frac{A - B}{2} \right| \leq 3 + \sqrt{3 + 2 \sum \cos(A - B)}.$$ 

We received five submissions, all of which are correct. We present two different solutions.

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Solution 1, by Shuborno Das, modified and enhanced by the editor.

Note first that the absolute value sign in the statement is redundant since
\[-\frac{\pi}{2} < \frac{A - B}{2}, \frac{B - C}{2}, \frac{C - A}{2} < \frac{\pi}{2}.\]

Since
\[\cos(A - B) = 2\cos^2\frac{A - B}{2} - 1,\]
we have
\[2\cos^2\frac{A - B}{2} = 1 + \cos(A - B),\]
so
\[2\cos\frac{A - B}{2} = \sqrt{2 + 2\cos(A - B)}.\]

Similarly,
\[2\cos\frac{B - C}{2} = \sqrt{2 + 2\cos(B - C)} \quad (1)\]
and
\[2\cos\frac{C - A}{2} = \sqrt{2 + 2\cos(C - A)}.\]

By the Cauchy-Schwarz Inequality, we have
\[
\sum_{\text{cyc}} \sqrt{2 + 2\cos(A - B)} \leq \sqrt{\sum_{\text{cyc}} (1 + 1 + 1)(2 + 2\cos(A - B))}
= \sqrt{6 \sum_{\text{cyc}} (1 + \cos(A - B))}. \quad (2)
\]

It thus suffices to show that
\[
\sqrt{6 \sum_{\text{cyc}} (1 + \cos(A - B))} \leq 3 + \sqrt{3 + 2 \sum_{\text{cyc}} \cos(A - B)}
\]
or
\[
6 \sum_{\text{cyc}} (1 + \cos(A - B)) \leq 12 + 2 \sum_{\text{cyc}} \cos(A - B) + 6 \sqrt{3 + 2 \sum_{\text{cyc}} \cos(A - B)}
\]
or
\[
3 + 2 \sum_{\text{cyc}} \cos(A - B) \leq 3 \sqrt{3 + 2 \sum_{\text{cyc}} \cos(A - B)}. \quad (3)
\]

Let \(S = \sum_{\text{cyc}} \cos(A - B).\) Then (3) becomes
\[(2s + 3)^2 \leq 9(2s + 3) \iff 2s + 3 \leq 9 \iff s \leq 3,
\]
which is obviously true. From (1) – (3), the conclusion follows.
Solution 2, by Leonard Giugiuc.

Clearly, the absolute value sign in the question is redundant.

Let \( u = \cos A + i \sin A, \ v = \cos B + i \sin B, \) and \( w = \cos C + i \sin C. \)

Then \( |u| = |v| = |w| \) and
\[
|u + v| = |(\cos A + \cos B) + i(\sin A + \sin B)|
= 2 \left| \cos \frac{A + B}{2} - \cos \frac{A - B}{2} + i \sin \frac{A + B}{2} \cos \frac{A - B}{2} \right|
= 2 \left| \cos \frac{A - B}{2} \left( \sin \frac{C}{2} + i \cos \frac{C}{2} \right) \right|
= 2 \cos \frac{A - B}{2}.
\]

Similarly, \( |v + w| = 2 \cos \frac{B - C}{2} \) and \( |w + u| = 2 \cos \frac{C - A}{2}. \)

Also,
\[
|u + v + w| = \sqrt{(\cos A + \cos B + \cos C)^2 + (\sin A + \sin B + \sin C)^2}
= \sqrt{3 + 2 \sum_{\text{cyc}} \cos(A - B)}.
\]

Since
\[
|u + v| + |v + w| + |w + u| \leq |u| + |v| + |w| + |u + v + w|
\]

by Hlawka’s Inequality, we have
\[
2 \sum_{\text{cyc}} \cos \frac{A - B}{2} \leq 3 + \sqrt{3 + 2 \sum_{\text{cyc}} \cos(A - B),}
\]
completing the proof.


Suppose \( ABC \) is an acute-angled triangle, \( DEF \) is an orthic triangle of \( ABC, \) \( S \) is the symmedian point of \( ABC, \) \( G \) is the barycenter of \( DEF. \) If \( D \) is the foot of the altitude from \( A \) and \( K \) is the point of intersection of \( AS \) and \( FE, \) prove that \( D, G \) and \( K \) are collinear.

We received four submissions, all of which were correct, and feature the solution by Ivko Dimitrić.

We use barycentric coordinates based on \( \triangle ABC \) where the coordinates of the vertices are \( A(1 : 0 : 0), \ B(0 : 1 : 0) \) and \( C(0 : 0 : 1) \) and \( a, b, c \) denote the corresponding side lengths. The vertices of the orthic triangle are the feet of the altitudes of \( \triangle ABC. \) Hence, we get
\[
E \left( S_C : 0 : S_A \right) \quad \text{and} \quad F \left( S_B : S_A : 0 \right),
\]

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where
\[ S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}, \]
according to Conway’s triangle notation. Since \( S_C + S_A = b^2 \) and \( S_B + S_A = c^2 \), we get the respective normalized barycentric coordinates as
\[ E\left(\frac{S_C}{b^2}, 0, \frac{S_A}{b^2}\right) \quad \text{and} \quad F\left(\frac{S_B}{c^2}, \frac{S_A}{c^2}, 0\right). \]
Then the midpoint \( M \) of \( EF \) is
\[ M = \frac{1}{2} (E + F) = \left(\frac{b^2 S_B + c^2 S_C}{2b^2c^2}, \frac{S_A}{2c^2}, \frac{S_A}{2b^2}\right). \]
Since the symmedian point has homogeneous coordinates \( S = (a^2 : b^2 : c^2) \), the line through \( A \) and \( S \) has an equation \( c^2 y - b^2 z = 0 \). The coordinates of \( M \) clearly satisfy this equation, so \( M \) belongs to \( AS \) and, therefore, \( M \equiv K \). Since the centroid \( G \) of \( \triangle DEF \) belongs to the median from \( D \), namely \( DM = DK \), the points \( D, G, \) and \( K \) are collinear.

Editor’s Comments. In other words Dimitrić has shown that each symmedian of a triangle passes through the midpoint of a side of the orthic triangle. This result is an immediate consequence of the “bisection property” discussed by Michel Bataille at the start of his Crux article, “Characterizing a Symmedian” [Vol. 43(4), April 2017, 145-150], namely that the \( A \)-symmedian is the locus of the midpoints of the antiparallels to \( BC \) bounded by the lines \( AB \) and \( AC \). A very simple coordinate-free proof is given there.

4340. Proposed by Digby Smith.

Let \( a, b, c \) and \( d \) be positive real numbers such that
\[ a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}. \]
Show that
\[ a + b + c + d \geq \max \left\{ 4\sqrt{abcd}, \frac{4}{\sqrt{abcd}} \right\}. \]
We received 11 submissions of which all but two were correct and complete. We present a solution followed by a generalization.

Solution by Roy Barbara.

First, it suffices to prove that for any \( a > 0, b > 0, c > 0, \) and \( d > 0 \) that satisfy the hypothesis, \( a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \), the following inequality holds
\[ a + b + c + d \geq 4\sqrt{abcd}. \quad (1) \]
Indeed, positive numbers \( x, y, z, \) and \( t \) defined by
\[
x = \frac{1}{a}, \quad y = \frac{1}{b}, \quad z = \frac{1}{c} \quad \text{and} \quad t = \frac{1}{d}
\]
satisfy the hypothesis and consequently the inequality (1). However, inequality (1) for \( x, y, z, \) and \( t \) is
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq \frac{4}{\sqrt{abcd}},
\]
which proves the statement inequality.

Second, we establish inequality (1). Let
\[
S = a + b + c + d \quad \text{and} \quad T = abc + bcd + cda + dab.
\]

The following identity relates \( S \) and \( T \):
\[
S^3 - 16T = S(a + b - c - d) + 4(a + b)(c - d)^2 + 4(c + d)(a - b)^2.
\]
Therefore \( S^3 - 16T \geq 0 \). (Ed.: This inequality is known as Maclaurin’s inequality, as several solvers indicated.) The hypothesis, is equivalent to \( abcd = T/S \). Hence
\[
16abcd = \frac{16T}{S} \leq \frac{S^3}{S} = S^2
\]
and (1) follows.

**Generalization by Leonard Giugiuc, Ardak Mirzakhmedov, and Roy Barbara.** Let \( a_1, a_2, \ldots, \) and \( a_n \) be \( n \) strictly positive numbers that satisfy
\[
a_1 + \cdots + a_n = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.
\]

Then
\[
a_1 + \cdots + a_n \geq \max\{ n^{\frac{n}{n-2}}a_1 \ldots a_n, \frac{n}{\sqrt[n]{a_1 \ldots a_n}} \}. \tag{3}
\]

**Solution for generalized inequality.**

Maclaurin’s inequality is a refinement of the AM-GM inequality, which you can find at [https://en.wikipedia.org/wiki/Maclaurin%27s_inequality](https://en.wikipedia.org/wiki/Maclaurin%27s_inequality)

Due to Maclaurin’s inequality for \( n \) positive numbers,
\[
(a_1 + \cdots + a_n)^{n-1} \geq n^{n-2}(a_1 \ldots a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right),
\]

However, \( a_1 + \cdots + a_n = \frac{1}{a_1} + \cdots + \frac{1}{a_n} \), leading to
\[
(a_1 + \cdots + a_n)^{n-1} \geq n^{n-2}(a_1 \ldots a_n)(a_1 + \cdots + a_n),
\]
\[
(a_1 + \cdots + a_n)^{n-2} \geq n^{n-2}(a_1 \ldots a_n),
\]
\[
a_1 + \cdots + a_n \geq n \sqrt[n]{a_1 \ldots a_n}. \tag{4}
\]

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Denote by $x_i = 1/a_i$ for $i = 1,\ldots,n$. Then

$$x_1 + \cdots + x_n = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

and according to (4)

$$x_1 + \cdots + x_n \geq n \sqrt[n]{x_1 \cdots x_n}, \quad \text{or} \quad a_1 + \cdots + a_n \geq n \sqrt[n]{a_1 \cdots a_n}. \quad (5)$$

Based on (4) and (5), generalized inequality (3) follows.

**4341. Proposed by Daniel Sitaru and Leonard Giugiuc.**

Let $ABC$ be an arbitrary triangle. Show that

$$\sum_{\text{cyc}} \sin A(\cos A - |\cos B \cos C|) = \sin A \sin B \sin C.$$  

*Eleven correct solutions were received. Most of the solvers used the approach in the solution given here.*

From the expansion of

$$0 = \tan \pi = \tan(A + B + C),$$

we obtain

$$\tan A \tan B \tan C = \tan A + \tan B + \tan C,$$

whence

$$\sin A \sin B \sin C = \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B.$$  

Also,

$$2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C$$  
$$= \sin 2B + \sin 2C + 2 \sin A \cos A$$  
$$= 2 \sin(B + C) \cos(B - C) + 2 \sin A \cos A$$  
$$= 2 \sin A(\cos(B - C) - \cos(B + C))$$  
$$= 4 \sin A \sin B \sin C$$  
$$= 2(\sin A \sin B \sin C + \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B).$$

This yields the result for acute and right triangles. If, say, $\angle A > \pi/2$, then cosine of $A$ is negative and all the other trigonometric ratios are positive. Note that

$$\cos A + \cos B \cos C = -\cos(B + C) + \cos B \cos C = \sin B \sin C,$$

with analogous identities for other arrangements of $A, B, C$. 

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The left side of the equation is then
\[\begin{align*}
- \sin A \cos A - \sin A \cos B \cos C + \sin B \cos B \\
+ \sin B \cos C \cos A + \sin C \cos C + \sin C \cos A \cos B \\
= - \sin A (\cos A + \cos B \cos C) + \sin B (\cos B + \cos C \cos A) \\
+ \sin C (\sin C + \cos A \cos B) \\
= - \sin A \sin B \sin C + 2 \sin A \sin B \sin C \\
= \sin A \sin B \sin C.
\end{align*}\]

**4342. Proposed by Oai Thanh Dao and Leonard Giugiuc.**

In a convex quadrilateral $ABA_1C$, construct four similar triangles $ABC_1$, $A_1BC_2$, $ACB_1$ and $A_1CB_2$ as shown in the figure.

![Diagram of a quadrilateral with additional points and lines](image)

Show that $C_1C_2 = B_1B_2$ and that the directed angles satisfy
\[\angle(C_1C_2, B_1B_2) = 2\angle C_1BA.\]

*We received 6 submissions, all of which were correct, including fixing the typographical error in the problem’s statement (which had one unwanted subscript). We present a composite of the similar solutions by AN-anduud Problem Solving Group and Michel Bataille.*

Let us denote the pairs of equal oriented angles at $B$ and at $C$ by $\psi$; more precisely,
\[\psi = \angle C_1BA = \angle C_2BA_1 = \angle ACB_1 = \angle A_1CB_2.\]

Denote by $k$ the common ratio of the corresponding adjacent sides of the given similar triangles:
\[k = \frac{BA}{BC_1} = \frac{BA_1}{BC_2} = \frac{CA}{CB_1} = \frac{CA_1}{CB_2}.\]

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The spiral similarity about center $B$ with angle $\psi$ and ratio $k$ takes $C_1$ to $A$ and $C_2$ to $A_1$, while the spiral similarity about center $C$ with angle $-\psi$ and ratio $k$ takes $B_1$ to $A$ and $B_2$ to $A_1$.

Consequently, $C_1C_2 = B_1B_2$ (because they equal $\frac{1}{k}AA_1$). Moreover,

$$\psi = \angle(C_1C_2, AA_1) = \angle(AA_1, B_1B_2),$$

Therefore

$$\angle(C_1C_2, B_1B_2) = \angle(C_1C_2, AA_1) + \angle(AA_1, B_1B_2)$$

$$= 2\psi$$

$$= 2\angle C_1BA.$$

4343. Proposed by Mihaela Berindeanu.

Let $ABC$ be an acute triangle and $E$ be the center of the excircle tangent at $F$ and $G$ to the extended sides $AB$ and $AC$, respectively. If $GF \cap BE = \{B_1\}$, $FG \cap CE = \{C_1\}$ and $B'$ and $C'$ are feet of the altitudes from $B$, respectively $C$, show that $B_1C_1B'C'$ is a cyclic quadrilateral.

All of the 11 submissions we received were correct. Seven were essentially the same as our featured solution by Shuborno Das, and most solvers observed that the restriction to acute triangles is not required.

Note that

$$\angle CEB_1 = \angle CEB$$

$$= 180^\circ - \left(90^\circ - \frac{B}{2} + 90^\circ - \frac{C}{2}\right)$$

$$= 90^\circ - \frac{A}{2}.$$ 

Moreover, because $AF = AG$,

$$\angle CGB_1 = \frac{180^\circ - A}{2} = 90^\circ - \frac{A}{2}.$$ 

Thus $CGEB_1$ is cyclic.

Since $EG \perp CG$ it follows that $\angle EB_1C = 90^\circ$. Similarly we get $\angle EC_1B = 90^\circ$.

Hence, the points $B_1$ and $C_1$ lie on the circle whose diameter is $BC$. But $B'$ and $C'$ are the feet of the altitudes from $B$ and $C$ (that is, $BB' \perp B'C$ and $CC' \perp C'B$), so they also must lie on the same circle. In other words, for any given triangle $ABC$, acute or not, the quadrilateral $B_1C_1B'C'$ is inscribed in the circle whose diameter is $BC$, and we are done.
Proposed by Michel Bataille.

Let $n$ be a positive integer. Find all polynomials $p(x)$ with complex coefficients and degree less than $n$ such that $x^{2n} + x^n + p(x)$ has no simple root.

We received 4 correct solutions. We present 3 of them.

Solution 1, by the proposer.

The only polynomial is the constant $p(x) = \frac{1}{4}$.

Let $q(x) = x^{2n} + x^n + p(x)$ have $s$ distinct roots $r_1, r_2, \ldots, r_s$ with respective multiplicities $k_1, k_2, \ldots, k_s$, all exceeding 1. Then

$$2s \leq k_1 + k_2 + \cdots + k_s = 2n,$$

whence $s \leq n$. Suppose that

$$p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

and that $f(x) = 2nq(x) - xq'(x)$. Then $f(x)$ is a polynomial of degree $n$ equal to

$$nx^n + (n+1)a_{n-1} + (n+2)a_{n-2} + \cdots + (2n-1)a_1x + 2na_0.$$

Since each of the roots $r_i$ has multiplicity exceeding 1, it is a root of $q'(x)$ with multiplicity $k_i - 1$. Thus, each of the roots $r_i$ is a root of $f(x)$ with multiplicity at least $k_i - 1$, so that

$$2n - s = (k_1 + k_2 + \cdots + k_s) - s = (k_1 - 1) + (k_2 - 1) + \cdots + (k_s - 1) \leq n,$$

whence $s \geq n$. Therefore $n = s$, $k_1 = k_2 = \cdots = k_n = 2$, and

$$q(x) = \prod_{i=1}^{n} (x - r_i)^2 = (x^n + u_{n-1}x^{n-1} + \cdots + u_1x + u_0)^2,$$

for some complex coefficients $u_i$. Plugging this into the equation defining $q$ and noting that all the coefficients of powers of $x$ between $x^n$ and $x^{2n}$ vanish, we find that

$$2u_{n-1} = 0;$$
$$2u_{n-2} + u_{n-1}^2 = 0;$$
$$2u_{n-3} + 2u_{n-1}u_{n-2} = 0;$$
$$\vdots$$
$$2u_1 + 2u_2u_{n-1} + \cdots = 0;$$
$$\vdots$$
$$2u_0 + 2u_1u_{n-1} + \cdots = 1.$$

Thus $q(x) = (x^n + \frac{1}{4})^2$ and $p(x) = \frac{1}{4}$ is the unique solution to the problem.

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Solution 2, AN-anduud Problems Solving Group.

Using the notation and argument of the first solution, we have that
\[ q(x) = x^{2n} + x^n + p(x) = (u(x))^2 \]
for some monic polynomial \( u(x) \) of degree \( n \). Then
\[ p(x) - \frac{1}{4} = u(x)^2 - \left(x^n + \frac{1}{2}\right)^2 = \left(u(x) - x^n - \frac{1}{2}\right)\left(u(x) + x^n + \frac{1}{2}\right). \]

Either \( p(x) = 1/4 \) and \( u(x) = x^n + 1/2 \) or the polynomials on both sides of the equation are nontrivial. But the latter possibility is precluded by the fact that the degree of \( p(x) - 1/4 \) is less than \( n \) while that of \( u(x) + x^n + 1/2 \) is equal to \( n \).

Solution 3, by Leonard Ciugiuc and Ramanujan Srijari (done independently).

With the notation of the previous solutions and from the fact that \( s \leq n \), we note
that the coefficients of the powers \( x^{2n-s+1} \) up to \( x^{2n-1} \) in \( q(x) \) vanish, along with the corresponding symmetric functions of the roots. From Newton’s formulae for the powers of the roots, we obtain the equation
\[ k_1 + k_2 + \cdots + k_s = 2n \]
as well as the equations in the system
\[ S = \{k_1 r_1^t + k_2 r_2^t + \cdots + k_s r_s^t = 0 : 1 \leq t \leq s - 1 \leq n - 1 < 2n - n\}, \]
for integers \( k_1, k_2, \ldots, k_s \) in terms of distinct complex numbers \( r_1, r_2, \ldots, r_s \).

If, say, \( r_s = 0 \), then \( S \) becomes a system of \( s - 1 \) equations in \( s - 1 \) unknowns with a nonzero Vandermonde determinant of its coefficients, so it has the unique solution \( k_1 = k_2 = \cdots = k_{s-1} = 0 \). But this forces \( k_s = 2n \) and \( q(x) = x^{2n} \), a palpable falsehood.

Thus, all the \( r_i \) are nonzero. If \( s < n \), we can append to \( S \) the equation
\[ k_1 r_1^t + k_2 r_2^t + \cdots + k_s r_s^t = 0 \]
and obtain the solution \( k_1 = k_2 = \cdots = k_s = 0 \), which is again false. Therefore \( s = n, k_1 = k_2 = \cdots = k_n = 2 \), and \( x^{2n} + x^n + p(x) \) is the square of a polynomial of degree \( n \). Since the roots of this polynomial satisfy
\[ r_1^t + r_2^t + \cdots + r_n^t = 0 \]
for \( 1 \leq t \leq n - 1 \), we must have that
\[ x^{2n} + x^n + p(x) = (x^n + c)^2 \]
for some constant \( c \). A check of the coefficient of \( x^n \) reveals that \( c = \frac{1}{4} \).
Let $ABC$ be a triangle with $AB < BC$ and incenter $I$. Let $F$ be the midpoint of $AC$. Suppose that the $C$-excircle is tangent to $AB$ at $E$. Prove that the points $E, I$ and $F$ are collinear if and only if $\angle BAC = 90^\circ$.

We received 16 submissions, all of which were correct, and will feature two of them.

Solution 1, by Ivo Dimitrić (typical of the five submissions that used barycentric coordinates).

We will see that that the assumption $AB < BC$ is superfluous. We use barycentric coordinates based on $\triangle ABC$ where $A(1 : 0 : 0)$, $B(0 : 1 : 0)$ and $C(0 : 0 : 1)$ are the coordinates of the vertices and $a, b, c$ denote the corresponding side lengths. As usual, $s = \frac{1}{2}(a + b + c)$ is the semi-perimeter. Then $F(1 : 0 : 1)$ and $I(a : b : c)$. Since $BE = s - a$ and $AE = s - b$, it follows that $E(s - a : s - b : 0)$. Hence, $F, I$ and $E$ are collinear if and only if

$$\begin{vmatrix}
1 & 0 & 1 \\
a & b & c \\
s - a & s - b & 0
\end{vmatrix} = 0.$$

Expanding this determinant gives

$$-c(s - b) + a(s - b) - b(s - a) = 0$$

or

$$bc - (-a + b + c)s = 0.$$

This, in turn, is equivalent to

$$0 = (-a + b + c)(a + b + c) - 2bc = (b + c)^2 - a^2 - 2bc = b^2 + c^2 - a^2,$$

which reduces to the Pythagorean condition $b^2 + c^2 = a^2$. In other words, starting with an arbitrary triangle $ABC$, we have that $F, I$ and $E$ are collinear if and only if $\triangle ABC$ is a right-angled triangle with $\angle BAC = 90^\circ$.

Solution 2, by AN-anduud Problem Solving Group, with some details supplied by the editor.

Let $\sigma$ be the dilatation with center $C$ that takes the $C$-excircle to the incircle. Side $AB$ of $\triangle ABC$, which is tangent to the excircle at $E$, is taken by $\sigma$ to a line segment $A'B'$ (with $A'$ on $AC$ and $B'$ on $BC$) that is tangent to the incircle at a point $E'$. If $D$ is the point where the incircle is tangent to $AB$, then $DE'$ is a diameter of the incircle.

Assume now that $\angle BAC = 90^\circ$. Then $\triangle EDE'$ and $\triangle EAC$ are homothetic right triangles that share the angle at $E$; because $I$ and $F$ are the midpoints of the

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parallel sides opposite $E$, namely $DE'$ and $AC$, we deduce that $E, I,$ and $F$ are collinear.

Conversely, we assume that $E, I,$ and $F$ are collinear. Consider the dilatation with center $E$ that takes $I$ to $F$. It takes $E'$ to a point $E''$ on the line $EE'$ and $D$ to a point $D'$ on $AB$ such that $E''D' \perp BD'$ and $F$ is the midpoint of $D'E''$. This forces $D'$ to coincide with $A$ and $E''$ with $C$ as follows: as a moving point $X$ slides along the line $AB$, the locus of points $Y$ for which $F$ is the midpoint of $XY$ is the line through $C$ that is parallel to $AB$, and that line intersects $EE'$ in the unique point $C$. We conclude that $\angle BAC = \angle BD'E'' = 90^\circ$, as desired.

4346. Proposed by Daniel Sitaru.

Find all $x, y, z \in (0, \infty)$ such that

\[
\begin{align*}
64(x + y + z)^2 &= 27(x^2 + 1)(y^2 + 1)(z^2 + 1), \\
x + y + z &= xyz.
\end{align*}
\]

We received 17 correct solutions, along with 1 incorrect solution. Eight solutions proceeded as in Solution 1; they all followed the same strategy, some depending on the Hermite-Hadamard inequality. We present 3 solutions in total.

Solution 1, by Paul Bracken.

Let

\[(x, y, z) = (\tan A, \tan B, \tan C),\]

where $0 < A, B, C < \pi/2$. Then the two equations become

\[\tan A + \tan B + \tan C = \tan A \tan B \tan C\]
and

\[ 64(\tan A \tan B \tan C)^2 = 27(\sec^2 A)(\sec^2 B)(\sec^2 C). \]

These are equivalent to \( A + B + C = \pi \) (expand \( \tan(A + B + C) \)) and

\[ \sin^2 A \sin^2 B \sin^2 C = \frac{27}{64}. \]

Since \( 2 \ln \sin t \) is a strictly concave function of \( t \) on \((0, \pi/2)\), by Jensen’s inequality we get

\[ \ln \sin^2 A + \ln \sin^2 B + \ln \sin^2 C \leq 3 \ln \sin^2 \left( \frac{A + B + C}{3} \right) = 3 \ln \sin^2 \left( \frac{\pi}{3} \right). \]

Hence

\[ \sin^2 A \sin^2 B \sin^2 C \leq \left( \frac{3}{4} \right)^3 = \frac{27}{64}, \]

with equality if and only if \( A = B = C = \pi/3 \).

Therefore the equations are satisfied if and only if \((x, y, z) = (\sqrt{3}, \sqrt{3}, \sqrt{3})\).

**Solution 2, by Nghia Doan.**

Let

\[ p = x + y + z = xyz \quad \text{and} \quad q = xy + yz + zx. \]

Since

\[ x^2 + y^2 + z^2 = p^2 - 2q \]

and

\[ x^2y^2 + y^2z^2 + z^2x^2 = q^2 - 2p^2, \]

the first equation becomes

\[ 64p^2 = 27[p^2 + (q^2 - 2p^2)] + (p^2 - 2q) + 1 = 27(q - 1)^2. \]

Since

\[ q = \left( \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} \right)(xy + yz + zx) \geq 9, \]

then \( 8p = 3\sqrt{3}(q - 1) \).

By the AM-GM inequality,

\[ p^3 = (x + y + z)^3 \geq 27xyz = p, \]

so that \( p \geq 3\sqrt{3} \), with equality if and only if \( x = y = z = \sqrt{3} \).

On the other hand,

\[ (xy + yz + zx)^2 \geq 3((xy)(yz) + (yz)(zx) + (zx)(xy)) = 3xyz(x + y + z) = 3p^2, \]

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so that \( q \geq \sqrt{3}p \) with equality if and only if \( xy = yz = zx \). Hence
\[
8p \geq 3\sqrt{3}(\sqrt{3}p - 1) = 9p - 3\sqrt{3},
\]
so that \( p \leq 3\sqrt{3} \).

It follows that the only solution of the given system of equations is \( x = y = z = \sqrt{3} \).

**Solution 3, by Madhav Modak.**

From the previous solution, we have that
\[
p^2 = (27/64)(q - 1)^2, \quad p^2 \geq 3q \quad \text{and} \quad q^2 \geq 3p^2.
\]

Substituting for \( p^2 \) in these two inequalities yield respectively
\[
0 \leq 9q^2 - 82q + 9 = (9q - 1)(q - 9)
\]
and
\[
0 \leq -(17q^2 - 162q + 81) = (9 - q)(17q - 9).
\]

The only value of the pair \( (p, q) \) that allows both inequalities to hold is \( (\sqrt{3}, 9) \) and this in turn forces \( x = y = z = 3\sqrt{3} \) as the unique solution of the system.

**Editor’s comment.** Some solvers tackled the problem by examining when the polynomial \( t^3 - pt^2 + qt - p \) has three real roots with \( 64p^2 = 27(q - 1)^2 \). This led to some straightforward technical gymnastics that are not sufficiently edifying to reproduce here.

4347. **Proposed by J. Chris Fisher.**

Given a cyclic quadrilateral \( ABCD \) with diameter \( AC \) (and, therefore, right angles at \( B \) and \( D \)), let \( P \) be an arbitrary point on the line \( BD \) and \( Q \) a point on \( AP \). Let the line perpendicular to \( AP \) at \( P \) intersect \( CB \) at \( R \) and \( CD \) at \( S \). Finally, let \( E \) be the point where the line from \( R \) perpendicular to \( SQ \) meets \( AP \). Prove that \( P \) is the midpoint of \( AE \) if and only if \( CQ \) is parallel to \( BD \).
Comment by the proposer: This is a slightly generalized restatement of OC266 [2017:137-138], Problem 5 on the 2014 India National Olympiad. The solution featured recently in Crux was a long and uninformative algebraic verification; in particular, it failed to explain why the triangle had to be acute (it probably didn’t), and it hid the true nature of the problem.

One of the five submissions we received was incomplete; the other four were correct and complete. We feature the solution by Michel Bataille.

Let $\gamma$ be the circle with diameter $AS$. Since $AP \perp PS$ and $AD \perp DS$, the points $P$ and $D$ are on $\gamma$ and it follows that

$$\angle(DP, DA) = \angle(SP, SA);$$

that is,

$$\angle(DB, DA) = \angle(SR, SA).$$

But

$$\angle(DB, DA) = \angle(CB, CA)$$

(since $A, B, C, D$ are concyclic), hence

$$\angle(CB, CA) = \angle(SR, SA);$$

that is,

$$\angle(CR, CA) = \angle(SR, SA)$$

and so $C, R, A, S$ are concyclic on a circle that we denote by $\gamma$. (They cannot be collinear since otherwise $B$ would be on $CA$.) Assuming that $Q \neq P$, we also observe that $E$ is the orthocentre of $\triangleQRS$ (since $QP \perp RS$ and $RE \perp QS$).
Now, suppose that $P$ is the midpoint of $AE$. Then $A$ is the reflection of $E$ in $RS$, which implies that $A$ is on the circumcircle of $\triangle QRS$, and therefore this circumcircle coincides with $\gamma'$. We deduce that
\[
\angle(CD,CQ) = \angle(CS,CQ) = \angle(AS,AQ) = \angle(AS,AP).
\]
Since
\[
\angle(AS,AP) = \angle(DS,DP)
\]
($A,D,S,P$ being concyclic) and
\[
\angle(DS,DP) = \angle(CD,BD),
\]
we finally obtain
\[
\angle(CD,CQ) = \angle(CD,BD),
\]
which proves that $CQ \parallel BD$.
Conversely, suppose that $CQ$ is parallel to $BD$. Then, we have
\[
\angle(CS,CQ) = \angle(DS,DP) = \angle(AS,AP) = \angle(AS,AQ),
\]
so that $Q$ is on the circumcircle $\gamma'$ of $\triangle CSA$. Since $R$ is also on this circle, we see that $\gamma'$ is the circumcircle of $\triangle QRS$. Since $A$ is on $\gamma'$ and on the altitude $QP$, the point $A$ is the reflection of the orthocentre $E$ of $\triangle QRS$ and so $P$ is the midpoint of $AE$.

Editor’s comments. In terms of the notation of the foregoing problem, Problem OC266 (referred to by the proposer) states that

For any point $P$ on the side $BD$ of an arbitrary triangle $ABD$, let $O_1,O_2$ denote the circumcentres of triangles $ABP$ and $APD$, respectively. Prove that the line joining the circumcentre of triangle $ABD$ to the orthocentre $H'$ of $\triangle O_1O_2P$ is parallel to $BC$.

To see that this is a special case of the foregoing problem, define $Q$ to be the intersection of $AP$ with the line through $C$ parallel to $BD$. Then, according to Problem 4347, the dilatation with center $A$ and ratio 2 will take $\triangle O_1O_2P$ to $\triangle RSE$, the orthocentre $H'$ (of $\triangle O_1O_2P$) to the orthocentre $Q$ (of $\triangle RSE$), and the circumcentre of $\triangle ABD$ to $C$. Consequently the line joining the circumcentre of $\triangle ABD$ to the orthocentre $H'$ is parallel to $CQ$, which is parallel to $BC$.

4348. Proposed by Marius Drăgan.

Let $p \in [0,1]$. Then for each $n > 1$, prove that
\[
(1-p)^n + p^n \geq (2p^2 - 2p + 1)^n + (2p - 2p^2)^n.
\]

We received 11 correct solutions. We present the solution by Digby Smith.
Note that for $p = 0$ and $p = 1$, there is equality. Let $f(t) = t^n$ for $t \in [0, 1]$. Since $n > 1$, it follows that $f$ is convex. Suppose that $0 < t < 1$. Applying convexity, it follows that

$$f(p) + f(1-p) = [pf(p) + (1-p)f(1-p)] + [(1-p)f(p) + pf(1-p)] \
\geq f[p(p) + (1-p)(1-p)] + f[(1-p)p + p(1-p)] \
= f[p^2 + (1-p)^2] + f[p(1-p) + p(1-p)] \
= f(2p^2 - 2p + 1) + f(2p - 2p^2),$$

with equality if and only if $p = 1 - p$. That is, there is equality if and only if $p = 1/2$.

It follows that

$$(1-p)^n + p^n \geq (2p^2 - 2p + 1)^n + (2p - 2p^2)^n,$$

with equality if and only if $p \in \{0, 1/2, 1\}$.

4349. Proposed by Hoang Le Nhat Tung.

Let $x, y$ and $z$ be positive real numbers such that $x + y + z = 3$. Find the minimum value of

$$\frac{x^3}{y\sqrt{x^3 + 8}} + \frac{y^3}{z\sqrt{y^3 + 8}} + \frac{z^3}{x\sqrt{z^3 + 8}}.$$

We received 7 correct solutions. We present the solution by Leonard Giugiuc.

Due to Cauchy’s inequality

$$\left(\frac{x^3}{y\cdot\sqrt{x^3 + 8}} + \frac{y^3}{z\cdot\sqrt{y^3 + 8}} + \frac{z^3}{x\cdot\sqrt{z^3 + 8}}\right)(xy + yz + zx) \geq \left(\frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}}\right)^2,$$

implying that

$$\frac{x^3}{y\cdot\sqrt{x^3 + 8}} + \frac{y^3}{z\cdot\sqrt{y^3 + 8}} + \frac{z^3}{x\cdot\sqrt{z^3 + 8}} \geq \left(\frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}}\right)^2.$$

But

$$3(xy + yz + ax) \leq (x + y + z)^2 = 9,$$

so

$$\left(\frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}}\right)^2 \geq \left(\frac{x^2}{3\sqrt{x^3 + 8}} + \frac{y^2}{3\sqrt{y^3 + 8}} + \frac{z^2}{3\sqrt{z^3 + 8}}\right)^2.$$

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Thus
\[
\frac{x^3}{y \cdot \sqrt{x^3 + 8}} + \frac{y^3}{z \cdot \sqrt{y^3 + 8}} + \frac{z^3}{x \cdot \sqrt{z^3 + 8}} \geq \left( \frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}} \right)^2.
\]

We will show that
\[
\left( \frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}} \right)^2 \geq 1,
\]
or equivalently
\[
\frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}} \geq \sqrt{3}.
\]
Consider the function \( f : (0, 3) \to \mathbb{R} \) defined by
\[
f(t) = \frac{t^2}{\sqrt{t^3 + 8}}.
\]
For all \( t \in (0, 3) \), we have
\[
f''(t) = k \left( 5t^6 - 64t^3 + 2048 \right) (t^3 + 8)^{-9/4},
\]
where \( k \) is a positive constant, so that \( f''(t) > 0 \). Hence \( f \) is convex.

By Jensen’s inequality,
\[
f(x) + f(y) + f(z) \geq 3f(1) = \sqrt{3};
\]
that is,
\[
\frac{x^2}{\sqrt{x^3 + 8}} + \frac{y^2}{\sqrt{y^3 + 8}} + \frac{z^2}{\sqrt{z^3 + 8}} \geq \sqrt{3}.
\]
Thus
\[
\frac{x^3}{y \cdot \sqrt{x^3 + 8}} + \frac{y^3}{z \cdot \sqrt{y^3 + 8}} + \frac{z^3}{x \cdot \sqrt{z^3 + 8}} \geq 1.
\]
Note that if \( x = y = z = 1 \), then the left-hand side of the last inequality equals 1. In conclusion,
\[
\min \left( \frac{x^3}{y \cdot \sqrt{x^3 + 8}} + \frac{y^3}{z \cdot \sqrt{y^3 + 8}} + \frac{z^3}{x \cdot \sqrt{z^3 + 8}} \right) = 1.
\]
4350. Proposed by Leonard Giugiuc.

Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a decreasing, differentiable and concave function. Prove that

\[
f(a) + f(b) + f(c) + f(d) \leq 3f(0) + f(d - c + b - a),
\]

for any real numbers \( a, b, c, d \) such that \( 0 \leq a \leq b \leq c \leq d \leq 1 \).

We received 5 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Since \( f(x) \) is decreasing, we have

\[
f(a) \leq f(0), \ f(b) \leq f(0), \ f(c) \leq f(0), \tag{1}
\]

and

\[
0 \leq (d - c) + (b - a) = d - (c - b) - a \leq d \leq 1,
\]

so that

\[
f(d) \leq f(d - c + b - a). \tag{2}
\]

From (1) and (2), we get

\[
f(a) + f(b) + f(c) + f(d) \leq 3f(0) + f(d - c + b - a).
\]

Editor's comments. Note that all of the solvers, other than the proposer, explicitly or implicitly noted that neither concavity nor differentiability were needed.