OLYMPIAD CORNER

No. 372

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

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To facilitate their consideration, solutions should be received by July 15, 2019.

OC421. Find all pairs of natural numbers $a$ and $k$ such that for every positive integer $n$ relatively prime to $a$, the number $a^{k^n+1} - 1$ is divisible by $n$.

OC422. In a scalene triangle $ABC$, $\angle C = 60^\circ$ and $\Omega$ is its circumcircle. On the angle bisectors of $\angle A$ and $\angle B$ take two points $A'$ and $B'$, respectively such that $AB' \parallel BC$ and $BA' \parallel AC$. The line $A'B'$ intersects $\Omega$ at points $D$ and $E$. Prove that triangle $CDE$ is isosceles.

OC423.

(a) Prove that there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f$, $f \circ f = g \circ g$ and $\forall x \in \mathbb{R} \ f(x) \neq g(x)$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f \circ g = g \circ f$ and $\forall x \in \mathbb{R} \ f(x) \neq g(x)$, then $\forall x \in \mathbb{R} \ (f \circ f)(x) \neq (g \circ g)(x)$.

OC424. Let $A \in \mathcal{M}_n(\mathbb{C}) \ (n \geq 2)$ with $\det \ A = 0$ and let $A^*$ be its adjoint. Prove that $(A^*)^2 = \text{tr}(A^*)A^*$, where $\text{tr}(A^*)$ is the trace of the matrix $A^*$.

OC425.

(a) Prove that for any choice of $n$ rational numbers $a_i/b_i$ from the interval $(0,1)$ with distinct pairs $(a_i, b_i)$ of positive integers, the sum of the denominators is at least $\frac{2\sqrt{2}}{3} \cdot n^2$.

(b) Prove that if we add the restriction that the rational numbers are distinct, then the sum of the denominators is at least $2 \cdot \left(\frac{2}{3}n\right)^2$. 

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 juillet 2019.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l’Université de Saint-Boniface, d’avoir traduit les problèmes.

**OC421.** Déterminer tous les couples de nombres naturels \(a\) et \(k\) tels que, pour tout entier positif \(n\) relativement premier avec \(a\), le nombre \(a^{k+1} - 1\) soit divisible par \(n\).

**OC422.** Soit \(ABC\) un triangle scalène tel que \(\angle C = 60^{\circ}\) et soit \(\Omega\) son cercle circonscrit. Les points \(A'\) et \(B'\) se trouvent sur les bissectrices de \(\angle A\) et \(\angle B\), respectivement, de façon à ce que \(AB' \parallel BC\) et \(BA' \parallel AC\). La ligne \(A'B'\) intersecte \(\Omega\) en \(D\) et \(E\). Démontrer que le triangle \(CDE\) est isocèle.

**OC423.**

(a) Démontrer qu’il existe des fonctions \(f : \mathbb{R} \rightarrow \mathbb{R}\) et \(g : \mathbb{R} \rightarrow \mathbb{R}\) telles que \(f \circ g = g \circ f\), \(f \circ f = g \circ g\) et, pour tout \(x \in \mathbb{R}\), \(f(x) \neq g(x)\).

(b) Démontrer que si \(f : \mathbb{R} \rightarrow \mathbb{R}\) et \(g : \mathbb{R} \rightarrow \mathbb{R}\) sont des fonctions continues telles que \(f \circ g = g \circ f\) et \(f(x) \neq g(x)\)\(\forall x \in \mathbb{R}\), alors, \((f \circ f)(x) \neq (g \circ g)(x)\)\(\forall x \in \mathbb{R}\).

**OC424.** Soit \(A \in \mathcal{M}_n(\mathbb{C})\) \((n \geq 2)\) telle que \(\det A = 0\) et soit \(A^*\) son adjointe. Démontrer que \((A^*)^2 = \text{tr}(A^*)A^*\), où \(\text{tr}(A^*)\) est la trace de la matrice \(A^*\).

**OC425.**

(a) Démontrer que pour tout choix de \(n\) nombres rationnels \(a_i/b_i\) dans l’intervalle \((0,1)\), avec des paires distinctes d’entiers positifs \((a_i,b_i)\), la somme des dénominateurs est au moins \(2\sqrt{2}/3 \cdot n^2\).

(b) Démontrer que si on oblige les nombres à être distincts, alors la somme des dénominateurs est au moins \(2 \cdot \left(\frac{2}{3}n\right)^{1/2}\).
OLYMPIAD CORNER

SOLUTIONS


OC376. Let \( m \) be a positive integer and \( a \) and \( b \) be distinct positive integers strictly greater than \( m^2 \) and strictly less than \( m^2 + m \). Find all integers \( d \) such that \( m^2 < d < m^2 + m \) and \( d \) divides \( ab \).

Originally Problem 4 of Day 2 of the 2016 Spain Mathematical Olympiad.

We received 4 correct submissions. We present the solution by Oliver Geupel.

The numbers \( d = a \) and \( d = b \) obviously satisfy the required conditions. We show that there are no further solutions. Let \( d \) be any integer such that \( m^2 < d < m^2 + m \) and \( d \neq a \) and \( d \neq b \).

Then \( 0 < |a - d| < m \), and \( 0 < |b - d| < m \), that is,

\[
\gcd(a, d) < m \quad \text{and} \quad \gcd(b, d) < m.
\]

Hence, \( \gcd(ab, d) < m^2 < d \), which implies that \( d \) does not divide \( ab \).

OC377. Prove that \( x - \frac{1}{x} + y - \frac{1}{y} = 4 \) has no solutions over the rationals.

Originally Problem 3 of Day 1 of the 2016 Final Round Korea.

We received 4 correct submissions. We present a solution based on the submissions of Mohammed Aassila and C.R. Pranesachar.

We prove the statement by contradiction. Assume, that \( x \), \( y \) are rationals. Without loss of generality, assume that \( x > 0 \) and \( y > 0 \). Let \( x = a/b \), and \( y = c/d \) with \( a, b, c, d \in \mathbb{N} \), and \( \gcd(a, b) = 1 \), \( \gcd(c, d) = 1 \). The equation is equivalent to

\[
ab(c^2 - d^2) + cd(a^2 - b^2) = 4abcd.
\] (1)

We obtain \( ab | cd(a^2 - b^2) \), and because \( \gcd(a, b) = 1 \), we have \( ab | cd \). Similarly, \( cd | ab \). Thus \( ab = cd \). Since \( \gcd(a, b) = 1 \) and \( \gcd(c, d) = 1 \), there exists pairwise relatively prime numbers \( m, n, p, q \) such that \( a = mp, c = mq, b = np, \) and \( d = np \).

The equation (1) becomes

\[
(p^2 + q^2)(m^2 - n^2) = 4mnmq.
\] (2)

Next, we show that there are no non-zero natural numbers \( m, n, p, q \) satisfying (2).

As before, because \( \gcd(m, n) = 1 \) we have \( \gcd(m^2 - n^2, mn) = 1 \). Indeed, let \( t \) be a prime common divisor of \( m^2 - n^2 \) and \( mn \). Then either \( t | m \), or \( t | n \). If \( t | m \)
then $t \mid -(m^2 - n^2) + m^2 = n^2$, and $t \mid n$. If $t \mid n$, then $t \mid m^2$, and $t \mid m$. It follows that $t = 1$.

In order for equation (2) to hold, we must have an integer number, $k$, such that

$$p^2 + q^2 = mnk \quad \text{and} \quad 4pq = (m^2 - n^2)k.$$  \hspace{1cm} (3)

This implies

$$8(p^2 + q^2)^2 = 4(p^2 + q^2)^2 - 16p^2 q^2 + 4(p^2 + q^2)^2 + 16p^2 q^2$$

$$= 4(p^2 - q^2)^2 + 4m^2 n^2 k^2 + (m^2 - n^2)^2 k^2$$

$$= 4(p^2 - q^2)^2 + (m^2 + n^2)^2 k^2.$$  \hspace{1cm} (4)

We claim that there are no integers $m, n, p, q$ and $k$ that satisfy (4). We use the fact that the maximum power of 2 in a perfect square is always an even number. The maximum power of 2 in $8(p^2 + q^2)^2$ is odd, due to factor 8, whereas the maximum power of 2 in the sum of perfect squares $4(p^2 - q^2)^2 + (m^2 + n^2)^2 k^2$ is even. This is a contradiction and completes the proof.

*Editor’s Comment.* C.R. Pranesachar pointed out that relations (3) cannot hold because $u^2 + v^2$ and $u^2 - v^2$ are discordant forms, in other words, for any non-zero integers $u$ and $v$, $u^2 + v^2$ and $u^2 - v^2$ cannot be both squares. In our case we can select $u = 2(p^2 + q^2)$ and $v = 4pq$. One reference on concordant-discordant forms is [History of the Theory of Numbers, L.E. Dickson, Carnegie Institution of Washington, Washington, Vol II, chapter XVI, page 473, 1920](#).

**OC378.** Define a sequence $\{a_n\}$ by

$$S_1 = 1, \quad S_{n+1} = \frac{(2 + S_n)^2}{4 + S_n} (n = 1, 2, 3, \ldots),$$

where $S_n$ the sum of the first $n$ terms of sequence $\{a_n\}$. For any positive integer $n$, prove that

$$a_n \geq \frac{4}{\sqrt{9n + 7}}.$$  

*Originally Problem 5 of Day 2 of the 2016 China Girls Mathematical Olympiad.*

*We received 6 correct submissions. We present two solutions.*

**Solution 1, by Mohammed Aassila**

First $a_1 = S_1 = 1$ and $a_{n+1} = S_{n+1} - S_n = \frac{4}{4 + S_n}$ for all integers $n \geq 1$. Thus

$$\frac{4}{a_{n+1}} - \frac{4}{a_n} = 4 + S_n - 4 - S_{n-1} = S_n - S_{n-1} = a_n \quad \text{and}$$

$$\frac{4}{a_{n+1}} = a_n + \frac{4}{a_n}.$$  \hspace{1cm} (1)

*Crux Mathematicorum, Vol. 45(4), April 2019*
Apply AM-GM inequality to (1) to find that $\frac{4}{a_{n+1}} \geq 4$, and $1 \geq a_{n+1}$ for all integers $n \geq 1$. Square both sides of (1) to get that

$$\frac{16}{a_{n+1}^2} = \frac{16}{a_n^2} + a_n^2 + 8 \quad \text{for all} \quad n \geq 1. \quad (2)$$

Apply successively (2) to find

$$\frac{16}{a_{n+1}^2} = \frac{16}{a_1^2} + a_1^2 + \cdots + a_n^2 + 8n \quad \text{for all} \quad n \geq 1. \quad (3)$$

Use $1 \geq a_n$ and $a_1 = 1$ in (3) to find

$$\frac{16}{a_{n+1}^2} \leq 16 + n + 8n = 9(n + 1) + 7.$$ 

Hence $a_n \geq \frac{4}{\sqrt{9n+7}}$ for all $n \geq 2$. Since $a_1 = 1$, this is true for $n = 1$.

**Solution 2, by IISER Mohali Problem Solving Group.**

Clearly, $a_1 = 1 \geq \frac{4}{\sqrt{16}} = 1$. We define a new sequence $\{T_n\}$ by $T_n = 4 + S_n$. Make note that $T_{n+1} - T_n = 4/T_n = a_{n+1}$. Thus, proving $a_n \geq \frac{4}{\sqrt{9n+7}}$ for all $n \geq 1$ reduces to showing

$$\frac{4}{\sqrt{9(n+1)+7}} \leq \frac{4}{T_n}, \quad \text{or} \quad T_n^2 \leq 9n + 16$$

for all $n \geq 1$.

We prove this by induction. The base case holds trivially: $9 \cdot 1 + 16 = 25 \geq 25 = T_1^2$. Assume that for some integer $k > 1$, the following holds:

$$T_k^2 \leq 9k + 16. \quad (1)$$

We know that $T_{k+1} - T_k = 4/T_k$, and hence,

$$T_{k+1}^2 = T_k^2 + 8 + 16/T_k^2.$$

We add $8 + 16/T_k^2$ to both sides of inequality (1), and we get

$$T_k^2 + 8 + \frac{16}{T_k^2} \leq 9k + 8 + \frac{16}{T_k^2} + 16.$$

Moreover, $T_k = 4 + S_k \geq 4$ hence, we have $16/T_k^2 \leq 1$ and

$$T_{k+1}^2 = T_k^2 + 8 + \frac{16}{T_k^2} \leq 9k + 8 + \frac{16}{T_k^2} + 16 \leq 9(k + 1) + 16.$$

This solves the problem.
OC379. Let \( n \geq 3 \) and \( a_1, a_2, \ldots, a_n \in \mathbb{R}^+ \), such that
\[
\frac{1}{1 + a_1^4} + \frac{1}{1 + a_2^4} + \cdots + \frac{1}{1 + a_n^4} = 1.
\]
Prove that:
\[
a_1a_2 \cdots a_n \geq (n - 1)^\frac{2}{n}.
\]

*Originally Problem 5 of the 2016 Macedonia National Olympiad.*

We received 4 correct submissions. We present a solution that was submitted independently by Paolo Perfetti and Sundara Narasimhan.

Let \( x_k = \frac{1}{1 + a_k^4} \). Then \( x_1 + \cdots + x_n = 1 \) and
\[
a_k = \left( \frac{1 - x_k}{x_k} \right)^\frac{1}{4}.
\]

The inequality becomes
\[
\frac{1 - x_1}{x_1} \times \frac{1 - x_2}{x_2} \times \cdots \times \frac{1 - x_n}{x_n} \geq (n - 1)^n,
\]
which is equivalent to
\[
\frac{x_2 + \cdots + x_n}{x_1} \times \frac{x_3 + \cdots + x_n + x_1}{x_2} \times \cdots \times \frac{x_1 + \cdots + x_{n-1}}{x_n} \geq (n - 1)^n. \quad (1)
\]

Apply AM-GM inequality to obtain
\[
x_2 + \cdots + x_n \geq (n - 1) \times (x_2 \cdots x_n)^\frac{1}{n-1}.
\]

Consequently, the left hand side of (1) is greater than or equal to
\[
\frac{(n - 1)(x_2 \cdots x_n)^\frac{1}{n-1}}{x_1} \times \frac{(n - 1)(x_3 \cdots x_n x_1)^\frac{1}{n-1}}{x_2} \times \cdots \times \frac{(n - 1)(x_1 \cdots x_{n-1} x_n)^\frac{1}{n-1}}{x_n},
\]
which equals \((n - 1)^n\) establishing the inequality.

OC380. Let \( \triangle ABC \) be an acute-angled triangle with altitudes \( AD \) and \( BE \) meeting at \( H \). Let \( M \) be the midpoint of segment \( AB \), and suppose that the circumcircles of \( \triangle DEM \) and \( \triangle ABH \) meet at points \( P \) and \( Q \) with \( P \) on the same side of \( CH \) as \( A \). Prove that the lines \( ED, PH, \) and \( MQ \) all pass through a single point on the circumcircle of \( \triangle ABC \).

*Originally Problem 5 of the 2016 Canadian Mathematical Olympiad.*

We received 2 submissions. We present the solution by Mohammed Aassila.

Let \( f \) be the inversion with center \( M \) and radius \( r = MA \). Let \( \odot ABC \) denote the circumcircle of triangle \( ABC \).
Let \( \{U, V\} = MH \cap (\odot ABC) \) with \( U \) in the arc \( \widehat{ACB} \) and denote \( X' = f(X) \). Then,
\[
MU \cdot MV = MA \cdot MB = MH \cdot MU \implies f(H) = U,
\]
thus we get \( f(\odot AHB) = \odot ABC \). Furthermore \( P', E, D, Q' \) are collinear with \( P', Q' \in \odot ABC \). Hence, \( \{Q'\} = MQ \cap DE \).

Now, consider \( g \) as the composition of an inversion with center \( H \) and radius \( \sqrt{HA \cdot HD} \), and reflection with respect to \( H \), then \( g(\odot ABC) = \odot MED \) and \( g(DE) = \odot ABH \). Therefore,
\[
\{g(Q')\} = \odot MED \cap \odot ABH = \{P\}
\]
(the latter follows since \( g(Q') \) is on the same side of \( A \) with respect to \( MH \)), hence we proved that the lines \( ED, PH, \) and \( MQ \) all pass through a single point on the circumcircle of \( \triangle ABC \).

**OC381.** The integers 1, 2, 3, \ldots, 2016 are written on a board. You can choose any two numbers on the board and replace them with one copy of their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

a) Prove that there is a sequence of replacements that will make the final number equal to 2.

b) Prove that there is a sequence of replacements that will make the final number equal to 1000.
Originally Problem 1 of the 2016 Canadian Mathematical Olympiad.
We received 3 submissions. We present the solution by Ramanujan Srihari.

For brevity, we use an arrow to show a replacement as described in the problem. As an example,

\[(a, b, c, d) \rightarrow (a, (b + c)/2, d) \rightarrow ((a + d)/2, (b + c)/2).\]

Further, we write \(\text{S} \rightarrow\) to denote a sequence \(S\) of such replacements.

For \(n \geq 3\), we claim that there is a sequence \(L_n\) of such replacements which replaces the numbers 1, 2, \ldots, \(n\) with \(n - 1\), or \((1, 2, \ldots, n) \overset{L_n}{\rightarrow} (n - 1)\). To prove this statement, we will use induction on \(n\). The following replacements are obvious:

\[(1, 2, 3) \rightarrow (2, 2) \rightarrow (2)\]

so that \((1, 2, 3) \overset{L_3}{\rightarrow} (2)\).

Now for \(k > 2\), let us assume (induction hypothesis) that there exists \(L_k\) such that \((1, 2, \ldots, k) \overset{L_k}{\rightarrow} (k - 1)\). Then we have

\[(1, 2, \ldots, k, k + 1) \overset{L_k}{\rightarrow} (k - 1, k + 1) \rightarrow (k)\]

where we replaced \(k - 1\) and \(k + 1\) with their average, \(k\).

Similarly, for \(n \geq 3\), we claim that there is a sequence \(R_n\) of such replacements so that \((2016 - n + 1, 2016 - n + 2, \ldots, 2016) \overset{R_n}{\rightarrow} (2016 - n + 2)\). One can easily prove this claim using induction, and the argument is very similar to the one given for \(L_n\). Then we have:

a) \((1, 2, \ldots, 2016) \overset{R_{2016}}{\rightarrow} (2)\)

b) \((1, 2, \ldots, 2016) \overset{L_{1009}}{\rightarrow} (998, 1000, 1001, \ldots, 2016) \overset{R_{1016}}{\rightarrow} (998, 1000, 1002)\)

\[\rightarrow (1000, 1000) \rightarrow (1000)\]

and we are done.

**Generalization.** It is not hard to see that if \(1, 2, \ldots, m\) are written on a board, then the following numbers can be left on the board after \(m - 1\) replacements:

a) \(2; (1, 2, \ldots, m) \overset{R_m}{\rightarrow} (2)\)

b) \(m - 1; (1, 2, \ldots, m) \overset{L_m}{\rightarrow} (m - 1)\)

c) \(n\), where \(3 < n < m - 2; (1, 2, \ldots, m) \overset{L_{m-1}}{\rightarrow} (n - 2, n, n + 1, \ldots, m) \overset{R_{m-n}}{\rightarrow} (n - 2, n, n + 2) \rightarrow (n, n) \rightarrow (n)\)

where \(L_k\) replaces 1, 2, \ldots, \(k\) by \(k - 1\) and \(R_k\) replaces \(m - k + 1, m - k + 2, \ldots, m\) by \(m - k + 2\). In other words,

\[(1, 2, \ldots, k) \overset{L_k}{\rightarrow} (k - 1)\] and \((m - k + 1, m - k + 2, \ldots, m) \overset{R_k}{\rightarrow} (m - k + 2)\).

On the other hand, the number \(m\) cannot be "obtained" since the average of any two numbers among 1, 2, \ldots, \(m\) is always less than \(m\). Similarly 1 cannot be obtained through such replacements.

*Crux Mathematicorum*, Vol. 45(4), April 2019
OC382. There are \( n > 1 \) cities in a country and some pairs of cities are connected by two-way non-stop flights. Moreover, every two cities are connected by a unique route (possibly with stopovers). A mayor of every city \( X \) counted the number of labelings of the cities from 1 to \( n \) so that every route beginning with \( X \) has the rest of the cities on that route occurring in ascending order. Every mayor, except one, noticed that the resulting number of their labelings is a multiple of 2016. Prove that the last mayor’s number of labelings is also a multiple of 2016.

*Originally Problem 6 (Grade 11) of Day 2 of the 2016 AllRussian Olympiad.*

We received 1 incomplete submission.

OC383. Let \( ABC \) be a triangle. Let \( r \) and \( s \) be the angle bisectors of \( \angle ABC \) and \( \angle BCA \), respectively. The points \( E \) in \( r \) and \( D \) in \( s \) are such that \( AD \parallel BE \) and \( AE \parallel CD \). The lines \( BD \) and \( CE \) cut each other at \( F \). The point \( I \) is the incenter of \( ABC \). Show that if \( A, F \) and \( I \) are collinear, then \( AB = AC \).

*Originally Problem 1 of Day 1 of the 2016 Brazil National Olympiad.*

We received 6 submissions. We present 2 solutions.

*Solution 1, by IISER Mohali Problem solving group.*

Let \( AI \) meet \( BC \) at \( K \). Apply Ceva’s Theorem to \( \triangle FBC \), where the cevians \( FK, BE, CD \) meet at \( I \). Then, we get

\[
\frac{BK}{KC} \cdot \frac{CE}{EF} \cdot \frac{FD}{DB} = 1.
\]

Since \( \frac{FD}{DB} = \frac{FA}{AI} \) and \( \frac{CE}{EF} = \frac{AI}{FA} \), then we see immediately that 

\[
BK = KC,
\]

which implies that the angle bisector of \( \triangle ABC \) is its median, which shows that \( AB = AC \).
Solution 2, by Ivko Dimitrić.

We use barycentric coordinates in reference to triangle $ABC$. Let $a$, $b$ and $c$ be the side lengths of this triangle and $q = AI$, $r = BI$ and $s = CI$ be the internal angle bisectors at vertices $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$, respectively. Each of them is a cevian that passes through one of the vertices and the incenter $I(a:b:c)$. Since an equation of a general line is $ux + vy + wz = 0$, the equations of these bisectors are respectively given by

$q: cy - bz = 0, \quad r: cx - az = 0, \quad s: bx - ay = 0.$

Since the line at infinity has an equation $x + y + z = 0$, the points at infinity on bisectors $r$ and $s$ are $r_\infty = (-a : a + c : -c)$ and $s_\infty = (a : b : -a - b)$, respectively. Hence, the line through $A$ parallel to $r$ has an equation

$$\begin{vmatrix} 1 & 0 & 0 \\ -a & a + c & -c \\ x & y & z \end{vmatrix} = 0 \iff cy + (a + c)z = 0.$$ 

This line intersects bisector $s$ at point $D\left(a:b: \frac{-bc}{a+c}\right)$. Likewise, the line through $A$ parallel to $s$ has an equation

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & -a - b \\ x & y & z \end{vmatrix} = 0 \iff (a + b)y + bz = 0$$

and intersects bisector $r$ at point $E\left(a: \frac{-bc}{a+b}:c\right)$. Consequently, the line $DB$ has an equation

$$\begin{vmatrix} a & b & \frac{-bc}{a+c} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = 0 \iff \frac{bc}{a+c} x + az = 0$$

and the line $EC$ an equation

$$\begin{vmatrix} a & \frac{-bc}{a+b} & c \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = 0 \iff \frac{bc}{a+b} x + ay = 0.$$ 

These two lines intersect at point $F\left(-a: \frac{bc}{a+b}: \frac{bc}{a+c}\right)$. The points $A$, $F$ and $I$ are then collinear if and only if $F$ belongs to bisector $q$ iff

$$\frac{bc^2}{a+b} = \frac{b^2c}{a+c}.$$ 

This in turn reduces to

$$c(a + c) = b(a + b) \iff a(b - c) + b^2 - c^2 = (a + b + c)(b - c) = 0.$$ 

Since $a + b + c \neq 0$, it follows that $c = AB$ is equal to $b = AC$. 

*Crux Mathematicorum*, Vol. 45(4), April 2019
OC384. Solve the equation \(xyz + yzt + xzt + ytx = xyzt + 3\) over the set of natural numbers.

*Originally Problem 3 of the 2016 Macedonia National Olympiad. We received 8 submissions, of which 6 were correct and complete.*

We present the solution by Ieko Dimitrić, slightly modified.

The only solutions are the following quadruples for \((x, y, z, t)\):

\[(1, 1, 1, 1), (2, 3, 9, 17), (2, 3, 7, 39), (3, 3, 4, 11), (3, 3, 5, 7)\]

and other quadruples obtained from these by arbitrary permutations of \(x, y, z\) and \(t\), given that the equation is completely symmetric in the four variables. It is easily checked that these quadruples satisfy the equation. Conversely, we prove that any solution must be one of the listed or obtained from it by a permutation.

Let \(s = \text{xyzt}\), \(u = yzt = \frac{s}{x}\) and assume without loss of generality that \(x \leq y \leq z \leq t\).

If \(x \geq 4\), then \(y, z, t \geq 4\) and we would have

\[s + 3 = \frac{s}{x} + \frac{s}{y} + \frac{s}{z} + \frac{s}{t} \leq 4 \cdot \frac{s}{4} = s,\]

which is not possible. Hence, \(x \leq 3\).

**Case 1.** If \(x = 1\), from the given condition we would have \(yz + zt + yt = 3\), and since the numbers involved are positive integers, it follows that \(yz = zt = yt = 1\), implying \(y = z = t = 1\), so we get one solution \((1, 1, 1, 1)\).

**Case 2.** If \(x = 2\), the equation reduces to

\[2(yz + zt + yt) = yzt + 3\]  \hspace{1cm} (1)

or

\[u + 3 = 2 \left( \frac{u}{y} + \frac{u}{z} + \frac{u}{t} \right).\]  \hspace{1cm} (2)

Clearly, \(u = yzt\) must be odd, so all three of \(y, z, t\) are odd. If \(7 \leq y \leq z \leq t\) from (2) we would have

\[u + 3 \leq 2 \cdot 3 \cdot \frac{u}{7} = \frac{6}{7} u,\]

which is not possible. If \(y = 5\) and \(7 \leq z \leq t\), from (2) we get

\[u + 3 \leq 2 \left( \frac{u}{5} + \frac{2u}{7} \right) = \frac{34}{35} u,\]

which is not possible. Moreover, if \(y = 5\) and \(z = 5\), for any choice of \(t \geq 5\) the left hand side of (1) is divisible by 5, whereas the right hand side is not, contradiction! Thus, \(y = 3\) and the equation reduces to

\[6(z + t) = zt + 3 \iff (z - 6)(t - 6) = 33,\]

Since \(3 \leq z \leq t\), then \(-3 \leq z - 6 \leq t - 6\) and we get \((z - 6, t - 6) \in \{(1, 33), (3, 11)\}\), i.e. \((z, t) \in \{(7, 39), (9, 17)\}\). So, we get the solutions \((2, 3, 7, 39)\) and \((2, 3, 9, 17)\).
Case 2. If $x = 3$, the equation becomes

$$3(yz + zt + yt) = 2yzt + 3,$$

or

$$2u + 3 = 3 \left( \frac{u}{y} + \frac{u}{z} + \frac{u}{t} \right).$$

Thus, $3 \mid yzt$ and one of the numbers $y, z$ or $t$ must be divisible by 3. If $y \geq 4$, then one of $z$ or $t$ is $\geq 6$ and the other one is $\geq 4$. Then,

$$2u + 3 \leq 3 \left( \frac{u}{4} + \frac{u}{4} + \frac{u}{6} \right) = 2u,$$

which is not possible. Hence, $y = 3$ and the equation is further reduced to

$$3(z + t) = zt + 1 \iff (z - 3)(t - 3) = 8.$$

Since $3 \leq z \leq t$, then $0 \leq z - 3 \leq t - 3$ and we get $(z - 3, t - 3) \in \{(1, 8), (2, 4)\}$, i.e. $(z, t) \in \{(4, 11), (5, 7)\}$. So, we get the solutions $(3, 3, 4, 11)$ and $(3, 3, 5, 7)$.

Therefore, all the solutions are those listed and their permutations.

**OC385.** A subset $S \subset \{0, 1, 2, \cdots, 2000\}$ satisfies $|S| = 401$. Prove that there exists a positive integer $n$ such that there are at least 70 positive integers $x$ such that $x, x + n \in S$.

*Originally Problem 8 of Day 2 of the 2016 Korea National Olympiad.*

*We received no submissions.*