SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(3), p. 115–119 and 44(4), p.160–163.

An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

4321. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Find the greatest positive real number k such that

$$(a^2+b^2+c^2+d^2+e^2)^2 \geq k(a^4+b^4+c^4+d^4+e^4)$$

for all real numbers a, b, c, d and e satisfying $a+b+c+d+e=0$.

There were 4 correct solutions and one submission using Maple. *We present the solution obtained independently by AN-anduud Problem Solving Group and Digby Smith.*

We prove that

$$(a^2+b^2+c^2+d^2+e^2)^2 \geq \frac{20}{13}(a^4+b^4+c^4+d^4+e^4)$$

with equality if and only if $\{a,b,c,d,e\} = \{4r,-r,-r,-r,-r\}$ for some real number r.

Suppose that $a^2 = \max\{a^2,b^2,c^2,d^2,e^2\}$, and let $4x^2 = b^2+c^2+d^2+e^2$. Then $x^2 \leq a^2$ and the left side of the inequality is $(a^2+4x^2)^2$. By the Cauchy-Schwarz inequality (or the quadratic-arithmetic means inequality),

$$a^2 = (b+c+d+e)^2 \leq 4(b^2+c^2+d^2+e^2) = 16x^2$$

and

$$\begin{aligned}(a^2-4x^2)^2 &= [(b+c+d+e)^2-(b^2+c^2+d^2+e^2)]^2\\ &= 4(bc+bd+be+cd+ce+de)^2\\ &\leq 24(b^2c^2+b^2d^2+b^2e^2+c^2d^2+c^2e^2+d^2e^2).\end{aligned}$$

Equality occurs if and only if $b=c=d=e=-a/4$.

Therefore

$$\begin{aligned}&12(b^4+c^4+d^4+e^4)\\ &\quad= 12(b^2+c^2+d^2+e^2)^2-24(b^2c^2+b^2d^2+b^2e^2+c^2d^2+c^2e^2+d^2e^2)\\ &\quad\leq 192x^4-(a^4-8a^2x^2+16x^4)\\ &\quad= 176x^4+8a^2x^2-a^4.\end{aligned}$$

Since $x^2 \leq a^2 \leq 16x^2$, we have

$$\begin{aligned}0 &\geq 16(16x^2 - a^2)(x^2 - a^2)\\ &= 256x^4 - 272x^2a^2 + 16a^4\\ &= 5(12a^4 + 176x^4 + 8a^2x^2 - a^4) - 39(a^4 + 8a^2x^2 + 16x^4)\\ &\geq 60(a^4 + b^4 + c^4 + d^4 + e^4) - 39(a^2 + 4x^2)^2,\end{aligned}$$

from which

$$39(a^2 + b^2 + c^2 + d^2 + e^2)^2 \geq 60(a^4 + b^4 + c^4 + d^4 + e^4).$$

The desired result follows.

4322. *Proposed by Marius Drăgan.*

Let a, b, c be the side lengths of a triangle, x, y, z be positive numbers and let $u = bz - cy$, $v = ay - bx$, $w = cx - az$. Prove that $uv + vw + wu \leq 0$.

There were 5 correct solutions. We present two of them.

Solution 1, by Michel Bataille and Digby Smith, independently.

Since $au + cv + bw = 0$, we have that

$$\begin{aligned}b(uv + vw + wu) &= buv + bw(u + v)\\ &= buv - (au + cv)(u + v)\\ &= -[au^2 + (a + c - b)uv + cv^2].\end{aligned}$$

The discriminant of the quadratic form, namely,

$$\begin{aligned}&(a + c - b)^2 - 4ac\\ &\quad = (a^2 + b^2 + c^2) - 2(ab + bc + ca)\\ &\quad = -[(a + b - c)(a + c - b) + (b + c - a)(b + a - c) + (c + a - b)(c + b - a)],\end{aligned}$$

is negative because of the inequality involving the sides of a triangle. Since a, b, c are positive, the quadratic form is never positive. Thus $uv + vw + wu \leq 0$.

Solution 2, by the proposer.

The expression $uv + vw + wu$ can be written as $-\frac{1}{2}(x, y, z)M(x, y, z)^T$, where

$$M = \begin{pmatrix} 2bc & c(c - b - a) & b(b - c - a)\\ c(c - b - a) & 2ac & a(a - b - c)\\ b(b - c - a) & a(a - b - c) & 2ab \end{pmatrix}.$$

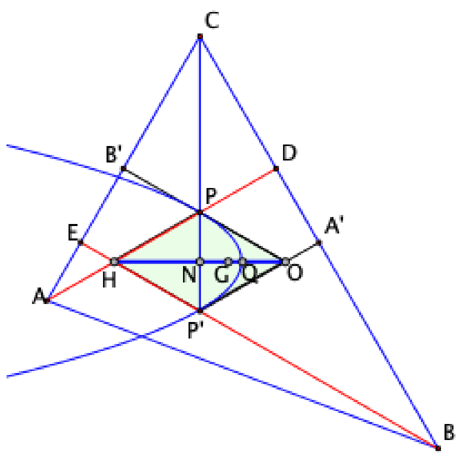
The principal minors are $2bc$,

$$\frac{c^2}{4}\left[4ab - (c - b - a)^2\right] = \frac{c^2}{4}[(c + a - b)(c + b - a) + 2c(a + b - c)]$$

and $\det M$. Because of the triangle inequality and because M annihilates the nontrivial vector $(a,b,c)^T$, these minors are nonnegative. Therefore, by Sylvester's Criterion, M is positive semidefinite and the result follows.

4323. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let ABC be a triangle with $\angle C = 60^\circ$. Let H denote the orthocenter, G the centroid, N the nine-point circle center and O the circumcenter of ABC. Let Q be the midpoint of NO. Prove that the parabola with vertex at Q and focus at G is tangent to the perpendicular bisector of both AC and BC.



We received 4 submissions, all of which were correct, and feature the solution by Ivko Dimitrić, modified by the editor.

Let A' and B' be the midpoints of sides BC and CA whose lengths are a and b, respectively, and assume that $a > b$ as in the picture (since for the points H, N, G, Q, and O to be distinct we cannot have $a = b$). Let D and E be the feet of the altitudes from A and B and let P and P' be the points of intersection of the altitudes AD, BE with the perpendicular bisectors of the sides AC and BC, respectively. Then the quadrilateral $OPHP'$ is a parallelogram.

Since $\angle C = 60^\circ$, from the right triangles ACD and BCE we get $CD = \frac{1}{2}CA = \frac{b}{2}$ and $CE = \frac{1}{2}CB = \frac{a}{2}$. Hence,

$$DA' = CA' - CD = \frac{1}{2}(a-b) \qquad \text{and} \qquad B'E = CE - CB' = \frac{1}{2}(a-b),$$

and the parallelogram $OPHP'$ is a rhombus (because the distances between its parallel sides are the same). Because the sides of $\angle POP'$ are perpendicular to the sides of $\angle ACB$, we have $\angle POP' = 60^\circ$. Since a diagonal of a rhombus bisects the angles at the pair of vertices that it connects, we conclude that

$$\angle POH = \angle HOP' = 30^\circ.$$

Because the points are arranged on the Euler line OH of any triangle so that the centroid G divides the segment joining the 9-point center N to the circumcenter O in the ratio $1:2$, we can introduce coordinates with the origin at the midpoint Q of NO and

$$N = (0,3),\ G = (0,1) \text{ and } O = (0,-3).$$

It follows that the line OP, which makes an angle of $30°$ with respect to the y-axis, has slope $\tan 60° = \sqrt{3}$ and equation $y = \sqrt{3}x - 3$, while the parabola with focus at G and vertex at Q has equation $y = \frac{1}{4}x^2$.

It is easily confirmed that OP is tangent to the parabola at $P(2\sqrt{3},\ 3)$, as desired. By symmetry the other perpendicular bisector, namely OP', is likewise tangent to the parabola (at $P'(-2\sqrt{3},\ 3)$).

Editor's comments. Properties of triangles with a 60° angle are discussed in the article "Recurring *Crux* Configurations 3" [**37**:7, November 2011, pages 449-453]. One of the results there confirms a theorem that is suggested by the figure displayed above: the line PP' (which is the perpendicular bisector of OH) is the bisector of $\angle ACB$. This is part of Problem 2855 [2003: 316; 2004: 308-309]. Note that the reflection in PP' fixes C and interchanges O with H, and the line CB with the line CA. This observation shows that the key result in the featured solution above, namely

> If exactly one of the angles of a triangle is $60°$ then the perpendicular bisectors of the sides adjacent to that vertex form an angle that is bisected by the Euler line OH.

is equivalent to the analogous theorem with the perpendicular bisectors replaced by the altitudes to those sides. The proof of that theorem was Problem M1046 from the 1987 U.S.S.R journal *Kvant* [appearing in *Crux* 1988: 165; 1990: 103].

4324. *Proposed by Michel Bataille.*

Let f be a continuous, positive function on $[0,1]$ such that

$$\mathcal{S} = \left\{ \int_0^1 (f(x))^n \, dx : n \in \mathbb{N} \right\}$$

is bounded above. Find the value of $\sup \mathcal{S}$.

We received 6 submissions of which 5 were correct and complete. We present the solution by Ivko Dimitrić.

Since a continuous function on a closed interval attains global maximum, we let $c \in [0,1]$ be a point where f attains its maximum value $M = f(c)$. Assume that $M > 1$ and let $q = \frac{1+M}{2} > 1$. By continuity, there must exist an interval $[a,b] \subset [0,1]$ containing c such that $f(x) \geq q$ on $[a,b]$. Indeed, if that is not so, we would have a sequence $\{x_n\}$ approaching c for which $f(x_n) < q$ for every n. But then

$$M = f(c) = \lim_{n\to\infty} f(x_n) \leq q = \frac{1+M}{2},$$

which is not possible. Consequently, we have

$$\int_0^1 (f(x))^n dx \geq \int_a^b (f(x))^n dx \geq \int_a^b q^n \, dx = q^n(b-a).$$

Since $q^n(b-a) \to \infty$ as $n \to \infty$, the set $\mathcal{S}$ would be unbounded, contrary to the given condition. Hence, our assumption $M > 1$ cannot hold and we have $f(x) \leq M \leq 1$ for all $x \in [0,1]$. Therefore, $(f(x))^n \leq f(x)$ and

$$\int_0^1 (f(x))^n dx \leq \int_0^1 f(x) \, dx, \qquad \text{hence} \qquad \sup_{n\in\mathbb{N}} \mathcal{S} = \int_0^1 f(x) \, dx$$

when $\mathbb{N}$ is defined as the set of integers that are greater or equal than 1.

Editor's comments. Several comments included in the submissions are worth a mention. First, Kathleen Lewis indicated that if the set of natural numbers is assumed to include 0, then the supremum of $\mathcal{S}$ is 1. Additionally, Roy Barbara pointed out that under the same hypotheses the lower bound of the set $\mathcal{S}$ can be found. If we denote by μ the Lebesgue measure on $[0,1]$, then $f^{-1}\{1\}$ is a closed, compact set and $\mu(f^{-1}\{1\})$ exists. The infimum of $\mathcal{S}$ is $\mu(f^{-1}\{1\})$.

4325. *Proposed by Alessandro Ventullo.*

Solve in real numbers the system of equations:

$$\begin{aligned} x^4 - 2y^3 - x^2 + 2y &= -1 + 2\sqrt{5} \\ y^4 - 2x^3 - y^2 + 2x &= -1 - 2\sqrt{5}. \end{aligned}$$

We received 10 correct and complete submissions. We present the solution by the proposer. Similar solutions were submitted by Šefket Arslanagić and the group of Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith.

Adding the two equations, we get

$$\begin{aligned} &(x^4 - 2x^3 - x^2 + 2x + 1) + (y^4 - 2y^3 - y^2 + 2y + 1) = 0 \quad \iff \\ &(x^2 - x - 1)^2 + (y^2 - y - 1)^2 = 0 \quad \iff \\ &x^2 - x - 1 = 0 \quad \text{and} \quad y^2 - y - 1 = 0. \end{aligned}$$

So, $x, y \in \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$. Let $\alpha = \frac{1-\sqrt{5}}{2}$ and $\beta = \frac{1+\sqrt{5}}{2}$. Since

$$\begin{aligned} \alpha^2 &= \alpha + 1, \\ \beta^2 &= \beta + 1, \\ \alpha^3 &= \alpha(\alpha+1) = 2\alpha + 1, \\ \beta^3 &= \beta(\beta+1) = 2\beta + 1, \\ \alpha^4 &= \alpha(2\alpha+1) = 3\alpha + 2, \\ \beta^4 &= \beta(2\beta+1) = 3\beta + 2, \end{aligned}$$

then

$$\alpha^4 - 2\beta^3 - \alpha^2 + 2\beta = (3\alpha + 2) - 2(2\beta + 1) - (\alpha + 1) + 2\beta = 2\alpha - 2\beta - 1 = -1 - 2\sqrt{5}$$

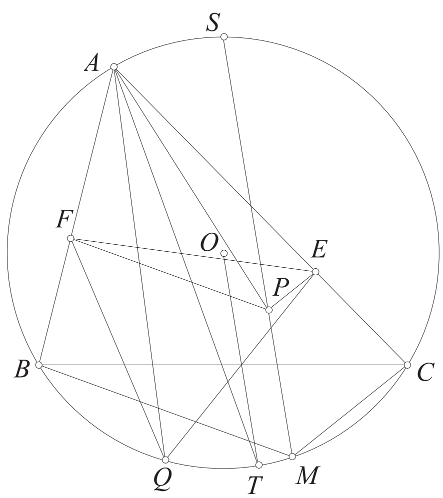
and by symmetry,

$$\beta^4 - 2\alpha^3 - \beta^2 + 2\alpha = 2\beta - 2\alpha - 1 = -1 + 2\sqrt{5}.$$

It follows that $x = \dfrac{1+\sqrt{5}}{2}$ and $y = \dfrac{1-\sqrt{5}}{2}$.

4326. *Proposed by Tran Quang Hung.*

Let ABC be a triangle inscribed in circle (O). Suppose S is the midpoint of arc BC containing A, T is a point on arc BC not containing A, M is on (O) such that $SM \parallel OT$, P is a point on SM. Let points E and F lie on CA and AB, respectively, such that $PE \parallel MC$ and $PF \parallel MB$. Finally, let Q be on (O) such AT is bisector of $\angle PAQ$. Prove that $QE = QF$.



We received 4 submissions, of which 3 were correct and one was incomplete. We feature the proposer's solution modified by the editor, who introduced directed angles. The proposer's original argument depended upon his diagram.

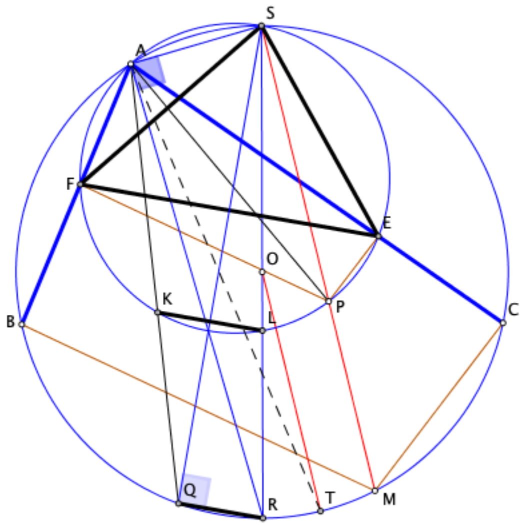
All angles are directed (between 0° and 180°). Observe that $\angle AFP = \angle ABM = \angle ASM = \angle ASP$, whence F lies on circle (ASP). Similarly, E also lies on (ASP). Because AS is the external bisector of $\angle FAE$ $(= \angle BAC)$, we have

$$SE = SF. \tag{1}$$

Define K and L to be the points where AQ and SO again intersect (ASP), while we let R be the other end of the diameter of (O) through S. We see $\angle AKL = \angle ASL = \angle ASR = \angle AQR$. From this

$$QR \parallel KL \quad \text{and (because } SR \text{ is a diameter of } (O)) \ QR \perp QS.$$

We will prove that $KL \parallel EF$, in which case we would have $QS \perp EF$ so that from (1) we will be able to conclude, finally, that $QE = QF$.



Indeed, we have by angle chasing,

$$\begin{aligned}
\angle RAL &= \angle RAT + \angle TAL \\
&= \angle RAT + (\angle TAP + \angle PAL) \\
&= \angle RAT + \angle QAT + \angle PSL \\
&= \angle RAT + \angle QAT + \angle MSR \\
&= \angle RAT + \angle QAT + \angle TOR \\
&= \angle QAT + \angle RAT + 2\angle TAR \\
&= \angle QAT + \angle TAR \;=\; \angle QAR.
\end{aligned}$$

Therefore AR bisects $\angle QAL = \angle KAL$; but AR also bisects $\angle BAC = \angle FAE$, so that $KL \parallel EF$ as claimed. Now we are done.

4327. *Proposed by Daniel Sitaru.*

Prove the following inequality for all $x > 0$:

$$\arctan(x)\arctan\left(\frac{1}{x}\right) < \frac{\pi}{2(x^2+1)}.$$

We received 8 submissions besides the original proposal. All submitted solutions pointed out that the conclusion in the given problem was incorrect. This error was caused by a small typo when the problem was printed. The right hand side of the given inequality should be $\frac{\pi x}{2(x^2+1)}$ *instead of* $\frac{\pi}{2(x^2+1)}$. *The printed inequality* *was clearly incorrect since if $x = \sqrt{3}$, then*

$$\arctan(x) \cdot \arctan(\tfrac{1}{x}) = \left(\frac{\pi}{3}\right)\left(\frac{\pi}{6}\right) = \frac{\pi^2}{18} > \frac{\pi}{8} = \frac{\pi}{2((\sqrt{3})^2+1)}.$$

This was given by a few solvers. In addition, several solvers also provided, with proof, the correct inequality. We present the proof by Michel Bataille, enhanced by the editor.

Let

$$f(x) = \frac{\pi x}{2(x^2+1)} - \arctan(x) \cdot \arctan(\frac{1}{x})$$

for $x > 0$. Since $\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}$, we have

$$f(x) = \frac{\pi x}{2(x^2+1)} - \frac{\pi}{2}\arctan(x) + (\arctan(x))^2.$$

We need to show that $f(x) > 0$ for all $x > 0$.

Since $f(\frac{1}{x}) = f(x)$, it suffices to show that $f(x) > 0$ for $x \in (0, 1]$.

By straightforward computations, we have

$$\begin{aligned} f'(x) &= \frac{\pi}{2}\left(\frac{1-x^2}{(x^2+1)^2} - \frac{1}{x^2+1}\right) + \frac{2\arctan(x)}{x^2+1} \\ &= \frac{\pi}{2} \cdot \frac{-2x^2}{(x^2+1)^2} + \frac{2\arctan(x)}{x^2+1} \\ &= \frac{2g(x)}{x^2+1}, \end{aligned}$$

where $g(x) = \arctan(x) - \dfrac{\pi}{2} \cdot \dfrac{x^2}{x^2+1}$.

Now,

$$g'(x) = \frac{1}{x^2+1} - \frac{\pi x}{(x^2+1)^2} = \frac{x^2 - \pi x + 1}{(x^2+1)^2},$$

so setting $g'(x) = 0$ we have

$$x = \frac{1}{2}\left(\pi \pm \sqrt{\pi^2 - 4}\right).$$

Since $\frac{1}{2}\left(\pi + \sqrt{\pi^2-4}\right) > 1$, we have $x = \frac{1}{2}\left(\pi - \sqrt{\pi^2-4}\right)$.

Let $\alpha = \frac{1}{2}\left(\pi - \sqrt{\pi^2-4}\right)$. Then $\alpha \in (0,1), g'(\alpha) = 0, g'(x) > 0$ for $x \in (0, \alpha)$, and $g'(x) < 0$ for $x \in (\alpha, 1]$.

Since $g(1) = \arctan(1) - \frac{\pi}{4} = 0 = g(0)$, it follows that $g(x) > 0$ for all $x \in (0,1)$ so f is increasing on $(0, 1]$. Finally, since $\lim\limits_{x\to 0^+} f(x) = 0$, we conclude that $f(x) > 0$ for all $x \in (0, 1]$ as claimed.

4328. *Proposed by Van Khea and Leonard Giugiuc.*

A circle I is inscribed in a triangle ABC and the points of tangency on the sides BC, CA and AB are D, E and F, respectively. The rays AD, BE and CF cut the circle I in points X, Y, Z, respectively. Prove that

$$\frac{1}{\frac{AX}{XD}+\frac{1}{4}}+\frac{1}{\frac{BY}{YE}+\frac{1}{4}}+\frac{1}{\frac{CZ}{ZF}+\frac{1}{4}}=4.$$

We received 6 submissions, all of which were correct, and present the solution by C.R. Pranesachar.

Let the tangents from the vertices of triangle ABC to its incircle be denoted by

$$x = AF = AE, \quad y = BD = BF, \quad \text{and} \quad z = CD = CE.$$

Then by Stewart's theorem,

$$(y+z)AD^2 = y(z+x)^2 + z(x+y)^2 - \frac{yz}{y+z}(y+z)^2.$$

Simplification gives

$$AD^2 = x^2 + \frac{4xyz}{y+z}.$$

By the tangent-secant theorem, we have $AX \cdot AD = AF^2 = x^2$. Hence $AX = \dfrac{x^2}{AD}$. Also

$$XD = AD - AX = AD - \frac{x^2}{AD} = \frac{AD^2 - x^2}{AD}.$$

Therefore

$$\frac{AX}{XD} = \frac{x^2}{AD^2 - x^2} = \frac{x^2}{\frac{4xyz}{(y+z)}} = \frac{x(y+z)}{4yz},$$

with analogous expressions for $\dfrac{BY}{YE}$ and $\dfrac{CZ}{ZF}$. Hence

$$\begin{aligned}\frac{1}{\frac{AX}{XD}+\frac{1}{4}}+\frac{1}{\frac{BY}{YE}+\frac{1}{4}}+\frac{1}{\frac{CZ}{ZF}+\frac{1}{4}} &= \frac{1}{\frac{x(y+z)}{4yz}+\frac{1}{4}}+\frac{1}{\frac{y(z+x)}{4zx}+\frac{1}{4}}+\frac{1}{\frac{z(x+y)}{4xy}+\frac{1}{4}}\\ &= \frac{4yz}{yz+zx+xy}+\frac{4zx}{yz+zx+xy}+\frac{4xy}{yz+zx+xy}\\ &= 4.\end{aligned}$$

This completes the proof.

Editor's Comments. Both Pranesachar and Volkhard Schindler reported that a similar problem was proposed by Abdul Hanjnan of Chennai, India:

> Prove that
> $$\frac{AX}{XD}+\frac{BY}{YE}+\frac{CZ}{ZF}=\frac{R}{r}-\frac{1}{2}.$$

It appeared as Problem 12027 in the *American Mathematical Monthly* **125**:3, page 276. Curiously, both problems appeared in the March 2018 issue of their respective journals and had the same deadline.

4329. *Proposed by Mihaela Berindeanu.*

For $x, y, z \geq 1$, show that

$$\frac{\log_2 xy}{(\log_2 2z)^2} + \frac{\log_2 yz}{(\log_2 2x)^2} + \frac{\log_2 xz}{(\log_2 2y)^2} \geq \frac{\log_2 xyz}{1 + \left(\log_2 \sqrt[3]{xyz}\right)^2}.$$

We received 9 submissions, all of which were correct. We present the solution by the AN-anduud Problem Solving Group.

Let $a = \log_2 x$, $b = \log_2 y$, and $c = \log_2 z$. Then $a, b, c > 0$ and the given inequality is equivalent to

$$\frac{a+b}{(1+c)^2} + \frac{b+c}{(1+a)^2} + \frac{c+a}{(1+b)^2} \geq \frac{a+b+c}{1 + \left(\frac{a+b+c}{3}\right)^2}$$

or

$$\sum_{cyc} \frac{a+b}{2(a+b+c)} \cdot \frac{1}{(1+c)^2} \geq \frac{a+b+c}{2\left(1 + \left(\frac{a+b+c}{3}\right)^2\right)} \qquad (1)$$

Let $f(x) = \dfrac{1}{(1+x)^2}$, $x \geq 0$. Then $f''(x) = \dfrac{6}{(1+x)^4} > 0$ so f is convex on $[0, \infty)$. By Jensen's Inequality we have

$$\sum_{cyc} \frac{a+b}{2(a+b+c)} \cdot f(c) \geq f\left(\frac{(a+b)c + (b+c)a + (c+a)b}{2(a+b+c)}\right) = f\left(\frac{ab+bc+ca}{a+b+c}\right)$$

or

$$\sum_{cyc} \frac{a+b}{2(a+b+c)} \cdot \frac{1}{(1+c)^2} \geq \frac{1}{\left(1 + \frac{ab+bc+ca}{a+b+c}\right)^2} \qquad (2)$$

By AM-GM Inequality, we have

$$\begin{aligned} 1 + \left(\frac{a+b+c}{3}\right)^2 &\geq 2\left(\frac{a+b+c}{3}\right) \\ &= \frac{2}{3} \cdot \frac{(a+b+c)^2}{a+b+c} \\ &\geq \frac{2}{3} \cdot \frac{3(ab+bc+ca)}{a+b+c} \\ &= \frac{2(ab+bc+ca)}{a+b+c}. \end{aligned} \qquad (3)$$

Also,

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

implies that

$$\frac{a+b+c}{3} \geq \frac{ab+bc+ca}{a+b+c}. \tag{4}$$

From (3) and (4) we have

$$1+2\left(\frac{a+b+c}{3}\right)^2 \geq \frac{2(ab+bc+ca)}{a+b+c}+\left(\frac{ab+bc+ca}{a+b+c}\right)^2$$

or

$$2\left(1+\left(\frac{a+b+c}{3}\right)^2\right) \geq \left(1+\frac{ab+bc+ca}{a+b+c}\right)^2 \tag{5}$$

From (2) and (5), we then obtain (1), completing the proof.

4330★. *Proposed by Mohammed Aassila.*

Let a and b be integers such that $a^2 - 20b + 24 = 0$. Find the complete set of solutions of the following equation over integers:

$$5x^2 + axy + by^2 = 11.$$

There were 8 correct solutions and 1 incorrect submission.

Since $a^2 + 4$ is a multiple of 10, $a = 10c \pm 4$ for some integer c, whereupon $b = 5c^2 \pm 4c + 2$.

Multiplying the equation by 20 and completing the square yields

$$(10x + ay)^2 = 4(55 - 6y^2).$$

Since $55 - 6y^2$ has to be square, $y = \pm 1$ or $y = \pm 3$. Since (x, y) satisfies the equation if and only if $(-x, -y)$ does, we consider the cases $y = -1$ and $y = -3$.

Suppose $y = -1$. Then

$$10x - a = 14 \quad \text{or} \quad 10x - a = -14.$$

In the first case, a must have the form $10c - 4$, whence $x = c + 1$. In the second case, $a = 10c + 4$ and $x = c - 1$.

Suppose $y = -3$. Then

$$10x - 3a = 2 \quad \text{or} \quad 10x - 3a = -2.$$

If $10x-3a = 2$, then $a = 10c-4$ and $x = 3c-1$. If $10x-3a = -2$, then $a = 10c+4$ and $x = 3c + 1$.

Thus, for the equation to be solvable, we need $(a, b) = (10c \pm 4, 5c^2 \pm 4c + 2)$. The solutions of

$$5x^2 + (10c + 4)xy + (5c^2 + 4c + 2)y^2 = 11$$

are

$$(x, y) = (c - 1, -1), (-c + 1, 1), (3c + 1, -3), (-3c - 1, 3)$$

and the solutions of

$$5x^2 + (10c - 4)xy + (5c^2 - 4c + 2)y^2 = 11$$

are

$$(x, y) = (c + 1, -1), (-c - 1, 1), (3c - 1, -3), (-3c + 1, 3).$$

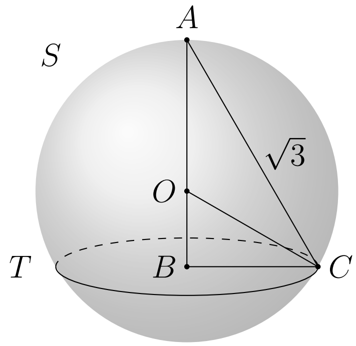
These work.

4331. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let S be a unit sphere. Suppose that the surface of S is coloured with 4 distinct colours. Prove that there exist two points $X, Y \in S$ of the same colour with $|XY| \in \{\sqrt{3}, \sqrt{3/2}\}$.

We received 8 solutions. We present the solution by Srihari Ramanujan, slightly edited.

Consider the unit sphere S, centred at O. Let A be a point on the sphere. The points on S which are at a distance of $\sqrt{3}$ away from A form a circle T centred at B with radius BC (for any point C on T), as shown in the figure below.



Since $\angle OBC$ is a right angle, we have

$$AC^2 - AB^2 = OC^2 - OB^2$$

which yields $OB = 1/2$ (using $OC = 1$ and $AB = 1 + OB$) and

$$BC^2 = OC^2 - OB^2 = 3/4.$$

Therefore the diameter of T is $\sqrt{3}$.

Now consider a square $CDEF$ inscribed in T. The diagonal has length $\sqrt{3}$ and thus the sidelength is $\sqrt{3/2}$. So the distances between the four points C, D, E, and F are either $\sqrt{3/2}$ or $\sqrt{3}$, whereas the distance between A and any of the four points is $\sqrt{3}$. By the pigeonhole principle two of the points A, C, D, E, and F must receive the same colour, yielding two points on S of the same colour of distance $\sqrt{3}$ or $\sqrt{3/2}$.

4332. *Proposed by S. Muralidharan.*

Draw the family of circles of radius $\frac{1}{2}$ with centers at (i, j) where i, j are integers. Prove that a line joining centers of any two of these circles cannot be tangent to any circle in the family.

Of the 13 submissions, 12 were correct; we feature a shortened version of the solution by the Missouri State University Problem Solving Group.

Assume that P, Q, and R are points with integer coordinates such that the line PQ is tangent to the circle centered at R with radius $\frac{1}{2}$. After a translation, we may assume $P = (0, 0)$, $Q = (a, b)$ and $R = (h, k)$. We may also assume $\gcd(a, b) = 1$ since otherwise we may replace Q with

$$\left(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)}\right).$$

If we consider the vectors $v = \overrightarrow{PQ}$ and $u = \overrightarrow{PR}$, then the distance from R to the line PQ is

$$\frac{|u \times v|}{|v|} = \frac{1}{2}.$$

Since

$$|u \times v| = |ak - bh| \quad \text{and} \quad |v| = \sqrt{a^2 + b^2},$$

this gives

$$4(ak - bh)^2 = a^2 + b^2.$$

But the only way for $a^2 + b^2$ to be a multiple of 4 is if both a and b are even, which would contradict the assumption that $gcd(a, b) = 1$.

More generally, if the radius of the circles is a rational number $r = p/q$ with $\gcd(p, q) = 1$, then there is a line through the centers of two circles that is tangent to a third circle if and only if $q = 1$ or all the prime factors of q are of the form $4i + 1$.

Editor's comments. The authors provided a proof of their generalization, but it is simply the featured solution together with an exercise in elementary number theory that relies on two related facts: the congruence

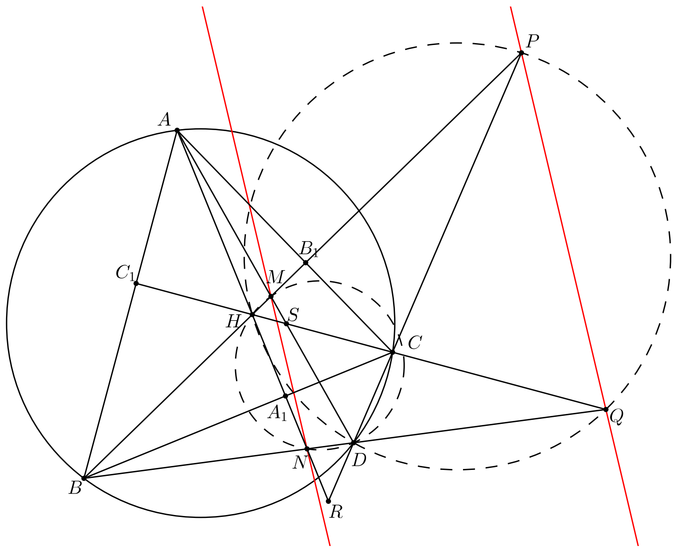
$$x^2 \equiv -1 \bmod (4i + 3)$$

has no solution, and a prime of the form $4i + 1$ can be written as the sum of two squares.

4333. *Proposed by Mihai Miculiţa and Titu Zvonaru.*

Let $ABCD$ be a cyclic quadrilateral and let A_1, B_1 and C_1 be orthogonal projections of the points A, B and C onto the lines BC, CA and AB, respectively. We denote $M = DA \cap BB_1$, $N = DB \cap AA_1$, $P = DC \cap BB_1$, $Q = DB \cap CC_1$, $R = DC \cap AA_1$ and $S = DA \cap CC_1$. Prove that $MN||PQ||RS$.

We received 4 submissions, all of which were correct, and feature the solution by Michel Bataille.



Let H be the orthocenter of the triangle ABC. Since A, B, C, D are concyclic, we have

$$\angle(DB, DA) = \angle(CB, CA) \neq 0 \pmod{\pi}$$

and so

$$\angle(DN, DM) = \angle(DB, DA) = \angle(CB, CA) = \angle(HA_1, HB_1) = \angle(HN, HM) \neq 0 \pmod{\pi}$$

(using $HA_1 \perp CB$ and $HB_1 \perp CA$).

It follows that H, D, M, N lie on a circle. In the same way, H, D, P, Q lie on a circle.

Now, the result $MN \parallel PQ$ is deduced from

$$\begin{aligned}
\angle(MN, PQ) &= \angle MN, ND) + \angle(DQ, QP) \quad (N, D, Q \text{ collinear}) \\
&= \angle(HM, HD) + \angle(HD, HP) \quad (\text{concyclicity}) \\
&= \angle(HM, HP) \\
&= 0 \pmod{\pi} \qquad (H, M, P \text{ collinear}).
\end{aligned}$$

Similarly, we prove that $RS \parallel PQ$ and the required result follows.

4334. *Proposed by George Stoica.*

Let $(a_n)_{n\geq 1}$ and $(x_n)_{n\geq 1}$ be sequences of real numbers such that $\frac{a_n}{x_n} \searrow 0$. Put $b_1 = 0$ and

$$b_n = a_1 + \cdots + a_{n-1} - \frac{x_1 + \cdots + x_{n-1}}{x_n} a_n$$

for $n \geq 2$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $(b_n)_{n\geq 1}$ converges and that, in this case, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} b_n$.

There was one correct solution, one partial and one incorrect solution. We present both the correct and the partial solutions.

Full solution, by Madhav Modak.

We will suppose that all the elements of the sequences $\{a_n\}$ and $\{x_n\}$ are positive.

Let $u_n = a_1 + a_2 + \cdots + a_n$ and $v_n = x_1 + x_2 + \cdots + x_n$ for each positive integer n. Then

$$b_n = a_1 + a_2 + \cdots + a_{n-1} - \frac{x_1 + \cdots + x_{n-1}}{x_n} a_n = u_n - v_n \frac{a_n}{x_n}.$$

Using the fact that when $p/q \leq r/s$ for two positive fractions, then

$$\frac{p}{q} \leq \frac{p+r}{q+s} \leq \frac{r}{s},$$

and applying an induction argument, we have that, for $n > m \geq 1$,

$$\frac{a_n}{x_n} \leq \frac{a_{m+1} + \cdots + a_n}{x_{m+1} + \cdots + x_n} \leq \frac{a_{m+1}}{x_{m+1}}$$

whence

$$(v_n - v_m)\frac{a_n}{x_n} \leq u_n - u_m. \tag{1}$$

Suppose that $\sum a_n$ converges. Let $\epsilon > 0$ be given. Fix p so that, for all $n \geq p$, $0 < u_n - u_p < \epsilon/2$. Now fix $q \geq p$, so that for $n \geq q$, $v_p(a_n/x_n) < \epsilon/2$. Then

$$0 < v_n \frac{a_n}{x_n} = v_p \frac{a_n}{x_n} + (v_n - v_p)\frac{a_n}{x_n} \leq v_p \frac{a_n}{x_n} + (u_n - u_p) < \epsilon.$$

Hence $\lim_{n\to\infty} v_n(a_n/x_n) = 0$, and therefore $\lim_{n\to\infty} b_n$ exists and is equal to

$$\lim_{n\to\infty} u_n = \sum_{n=1}^{\infty} a_n.$$

On the other hand, suppose $\sum a_n$ diverges. Let $M > 0$ be given, and choose p so that $u_p > M + 1$. Select $q \geq p$ so that $0 < v_p(a_q/x_q) < 1$. Then, by (1),

$$b_q = u_q - v_q \frac{a_q}{x_q} > u_p - v_p \frac{a_q}{x_q} > M + 1 - 1 = M.$$

Hence the sequence $\{b_n\}$ is unbounded and therefore diverges.

Partial solution, by the proposer.

We prove that the convergence of $\sum a_n$ implies the convergence of $\{b_n\}$, and also that $\sum a_n = \lim b_n$. This proof is valid when a_n and x_n are allowed to take both positive and negative values in tandem.

Since
$$b_n = a_1 + \cdots + a_n - \frac{x_1 + \cdots + x_n}{x_n} a_n$$
for $n \geq 2$, it suffices to prove that $a_n(x_1 + \cdots + x_n)/x_n$ converges to 0. For $n > m$, define $s_n = a_{m+1} + \cdots + a_n$.

Let $\epsilon > 0$ and choose $m > 0$ such that $|s_n| < \epsilon/4$ for $n > m$. Then

$$\begin{aligned}
&\left|\frac{a_n}{x_n}(x_{m+1} + \cdots + x_n)\right| \\
&= \frac{a_n}{x_n}\left|\frac{x_{m+1}}{a_{m+1}} \cdot a_{m+1} + \cdots + \frac{x_n}{a_n} \cdot a_n\right| \\
&= \frac{a_n}{x_n}\left|\frac{x_{m+1}}{a_{m+1}} \cdot s_{m+1} + \frac{x_{m+2}}{a_{m+2}} \cdot (s_{m+2} - s_{m+1}) + \cdots + \frac{x_n}{a_n} \cdot (s_n - s_{n-1})\right| \\
&= \frac{a_n}{x_n}\left|\left(\frac{x_{m+1}}{a_{m+1}} - \frac{x_{m+2}}{a_{m+2}}\right) \cdot s_{m+1} + \cdots + \left(\frac{x_{n-1}}{a_{n-1}} - \frac{x_n}{a_n}\right) \cdot s_{n-1} + \frac{x_n}{a_n} \cdot s_n\right| \\
&\leq \frac{a_n}{x_n}\left[\left(\frac{x_{m+2}}{a_{m+2}} - \frac{x_{m+1}}{a_{m+1}}\right)|s_{m+1}| + \cdots + \left(\frac{x_n}{a_n} - \frac{x_{n-1}}{a_{n-1}}\right)|s_{n-1}| + \frac{x_n}{a_n}|s_n|\right] \\
&\leq \left(\frac{\epsilon}{4}\right)\frac{a_n}{x_n}\left[\left(\frac{x_{m+2}}{a_{m+2}} - \frac{x_{m+1}}{a_{m+1}}\right) + \cdots + \left(\frac{x_n}{a_n} - \frac{x_{n-1}}{a_{n-1}}\right) + \frac{x_n}{a_n}\right] \\
&\leq \left(\frac{\epsilon}{4}\right)\frac{a_n}{x_n}\left(2 \cdot \frac{x_n}{a_n} - \frac{x_{m+1}}{a_{m+1}}\right) < \frac{\epsilon}{2}.
\end{aligned}$$

Since $a_n/x_n \to 0$, there is an index $p > m$ such that
$$\left|\frac{a_n}{x_n}(x_1 + \cdots + x_m)\right| < \frac{\epsilon}{2}$$
for all $n > p$. Thus
$$\left|\frac{a_n}{x_n}(x_1 + \cdots + x_n)\right| \leq \left|\frac{a_n}{x_n}(x_1 + \cdots + x_m)\right| + \left|\frac{a_n}{x_n}(x_{m+1} + \cdots + x_n)\right| < \epsilon$$
for all $n > p$. The desired result follows.

Comment by the editor. In the problem, there seems to be a tacit assumption that the sequences $\{a_n\}$ and $\{x_n\}$ both be positive. As we have seen, the result holds when $\sum a_n$ is convergent but not necessarily positive (with the x_n changing sign in tandem with a_n). However, it is possible for $\sum a_n$ to diverge while $\{b_n\}$ converges.

Let

$$a_n = (-1)^{n-1} \quad \text{and} \quad x_n = (-1)^{n-1} n$$

for each positive integer n, so that $a_n/x_n = 1/n$. Then, with the notation of the first solution,

$$u_n = 1 \quad \text{and} \quad v_n = \frac{1}{2}(n+1) \quad \text{when } n \text{ is odd,}$$
$$u_n = 0 \quad \text{and} \quad v_n = -\frac{1}{2}n \quad \text{when } n \text{ is even.}$$

Thus $b_n = 1 - \dfrac{n+1}{2n}$ when n is odd and $b_n = 0 + \dfrac{n}{2n}$ when n is even. Hence $\{b_n\}$ converges to the limit $\frac{1}{2}$.

However, with the same sequence $\{a_n\}$ and $x_n = (-1)^{n-1}2^{n-1}$, the sequence $\{b_n\}$ will diverge.

4335. *Proposed by Leonard Giugiuc.*

Let a and b be fixed positive real numbers and let $n \geq 2$ be an integer. Prove that for any nonnegative real numbers $x_i, i = 1, \ldots, n$ such that $x_1 + \cdots + x_n = 1$, we have

$$\sqrt[3]{ax_1 + b} + \sqrt[3]{ax_2 + b} + \cdots + \sqrt[3]{ax_n + b} \geq \sqrt[3]{a+b} + (n-1)\sqrt[3]{b}.$$

We received 11 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Consider the function

$$f(x) = \sqrt[3]{ax+b}, \; x \geq 0.$$

Note that $f(x)$ function is concave on $[0, +\infty)$. Using $x_k = (1 - x_k) \cdot 0 + x_k \cdot 1$, we have

$$f(x_k) \geq (1 - x_k) f(0) + x_k f(1) \quad \iff \quad \sqrt[3]{ax_k + b} \geq (1 - x_k) \cdot \sqrt[3]{b} + x_k \cdot \sqrt[3]{a+b}.$$

Adding the above inequalities and using $x_1 + x_2 + \cdots + x_n = 1$, gives our inequality. Equality holds only for $\{x_1, x_2, \ldots, x_n\} = \{1, 0, \ldots, 0\}$ and permutations.