OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(2), p. 52–53, and 44(3), p. 100–101.

OC366. Prove that there exist infinitely many positive integer triples (a, b, c) such that a, b, c are pairwise relatively prime, and $ab+c, bc+a, ca+b$ are pairwise relatively prime.

Originally 2016 China Western Mathematical Olympiad Day 2 Problem 5.

We received 6 *submissions. We present* 3 *solutions.*

Solution 1, by Shuborno Das.

Let $a = 2$, $b = 3$ and c be any prime number greater than 3 such that $c \equiv 4 \pmod 5$. We claim that $ab + c$, $bc + a$, $ca + b$ are pairwise co-prime for the above choice. Note that $\gcd(2, 3) = \gcd(2, c) = \gcd(3, c) = 1$. Now,

$$ab + c = c + 6, \qquad bc + a = 3c + 2, \qquad ca + b = 2c + 3.$$

Therefore,

$$\gcd(c + 6, 3c + 2) = \gcd(3c + 18, 3c + 2) = \gcd(16, 3c + 2) = 1.$$

(Note $\gcd(3, 3c + 2) = 1$).

We have

$$\gcd(c + 6, 2c + 3) = \gcd(2c + 12, 2c + 3) = \gcd(9, 2c + 3) = 1.$$

(Note $\gcd(2, 2c + 3) = 1$).

We also have

$$\gcd(3c + 2, 2c + 3) = \gcd(6c + 4, 2c + 3) = \gcd(6c + 4, 6c + 9) = \gcd(5, 6c + 4).$$

Now, $6c+4 \equiv c-1 \pmod 5$. We have assumed that $c \equiv 4 \pmod 5$, that is $c-1 \not\equiv 0 \pmod 5$, so $\gcd(5, 6c + 4) = 1$. By Dirichlet's theorem, there are infinitely many primes of the form $5k + 4$. This means there are infinitely many triples (a, b, c) satisfying the condition.

Solution 2, by Oliver Geupel.

It is enough to show that, for every positive integer k, the triple

$$(a, b, c) = (10k - 1,\ 10k,\ 10k + 1)$$

satisfies the required conditions.

Evidently, a, b, and c are pairwise relatively prime. Let

$$n = 10k, \quad u = ab + c, \quad v = bc + a, \quad w = ca + b.$$

Then,

$$(u, v, w) = (n^2 + 1,\ n^2 + 2n - 1,\ n^2 + n - 1).$$

We have

$$(n + 3)u - (n + 1)v = 4,$$

so that every common divisor of u and v is a divisor of 4. Since $u = 100k^2 + 1$ is odd, $\gcd(u, v) = 1$. From the identity

$$(n + 1)v - (n + 2)w = 1,$$

we see that v and w are coprime. Finally, we have

$$(n + 3)u - (n + 2)w = 5,$$

which implies that every common divisor of u and w is a divisor of 5. But u is not divisible by 5. Therefore, u and w are relatively prime, which completes the proof.

Solution 3, by David Manes.

For each positive integer n, let $a_n = 2^n$, $b_n = 3$ and $c_n = 5$. Then

$$(a_n, b_n, c_n) = (2^n, 3, 5)$$

define infinitely many triples such that $\gcd(2^n, 3) = \gcd(2^n, 5) = \gcd(3, 5) = 1$. Therefore, $(2^n, 3, 5)$ are pairwise relatively prime for each n. For a given positive integer n, let

$$x = a_n b_n + c_n = 3 \cdot 2^n + 5,\ y = b_n c_n + a_n = 3 \cdot 5 + 2^n,\ z = c_n a_n + b_n = 5 \cdot 2^n + 3.$$

Then x, y and z are all odd positive integers. We consider 3 cases.

1) Let $d = \gcd(x, y)$. Then d divides all linear combinations of x and y. In particular, d divides

$$3x - 1 \cdot y = 9 \cdot 2^n - 2^n = 2^{n+3}.$$

Hence, $d = 1$ since the only odd number that divides a power of 2 is 1. Therefore, x and y are relatively prime.

2) Let $d = \gcd(x, z)$. Then d divides the linear combination

$$5x - 3z = 25 - 9 = 2^4.$$

Hence, $d = 1$ so that x and z are relatively prime.

3) Let $d = \gcd(y, z)$. Then d divides

$$5z - y = 24 \cdot 2^n = 3 \cdot 2^{n+3}.$$

The only odd positive divisors of $3 \cdot 2^{n+3}$ are 1 and 3. If $d = 3$, then d divides y and d divides 15 implies d divides $y - 15 = 2^n$, a contradiction. Hence, $d = 1$ so that y and z are relatively prime.

Therefore, x, y and z are pairwise relatively prime for each positive integer n. Summarizing, if for each positive integer n, $(2^n, 3, 5)$ define infinitely many positive integer triples that are pairwise relatively prime, then the triples

$$(3 \cdot 2^n + 5, 5 \cdot 3 + 2^n, 5 \cdot 2^n + 3)$$

are also pairwise relatively prime.

Editor's Comments. These are not the only triples that satisfy the conditions of the problem. Mohammed Aassila found the triples $(a, b, c) = (2, 3, 30k + 5)$, where $k \in \mathbb{N}$, Richard Hess found the triples $(a, b, c) = (2, 3, 30k + 7)$, where $k \in \mathbb{N}$ and the Missouri State University Problem Solving Group found the triples $(a, b, c) = (10k + 5, 10k + 3, 2)$, where $k \in \mathbb{N}$. We leave to the reader the pleasure to check that these triples will do the trick.

OC367. A mathematical contest had 3 problems, each of which was given a score between 0 and 7, inclusive. It is known that, for any two contestants, there exists at most one problem in which they have obtained the same score (for example, there are no two contestants whose ordered scores are $7, 1, 2$ and $7, 1, 5$, but there might be two contestants whose ordered scores are $7, 1, 2$ and $7, 2, 1$). Find the maximum number of contestants.

Originally 2016 Italian Mathematical Olympiad Problem 2.

We received 2 submissions. We present the solution by Mohammed Aassila.

Consider the lattice points (x, y, z) such that $x, y, z \in [0, 7]$ and $x, y, z \in \mathbb{Z}$. Colour a lattice point (x, y, z) red if there exists a student who got x, y, and z as their scores for the first, second, and third problems, respectively.

The problem condition is equivalent to the condition that if (x, y, z) is red, then (x, y, k), (x, k, z), (k, y, z) are all not red for $k \in [0, 7]$ (with the exception of when the point is equal to (x, y, z)), which correspond to the up-down, left-right, and front-back rows that contain (x, y, z).

Looking at all lattice points (x, y, k) such that $x, y \in [0, 7]$ and k is a constant, by Pigeonhole principle there exists at most 8 red lattice points, else there exists two red lattice points in the same row, contradiction.

Thus, the maximum possible is $8 \times 8 = 64$ and it remains to show this is attainable. Indeed, just take all points of the form $(x, y, x - y)$ where values are calculated mod 8 and $x, y \in [0, 7]$.

OC368. Let n be a positive integer. Find the number of solutions of

$$x^2 + 2016y^2 = 2017^n$$

as a function of n.

Originally 2016 Korea National Olympiad Day 1 Problem 1.

Problem in not complete as stated. The statement should specify that the solutions sought are positive integer solutions.

We received one correct submission. We present the solution by Shengda Hu.

For $n = 1$, the only integral solutions are $(\pm 1, \pm 1)$.

We see that the following generate integral solutions of the equation for integers $l, m \geq 0$:

$$a_{m,l} + 2b_{m,l}\sqrt{-504} := 2017^m(1 + 2\sqrt{-504})^l \Longrightarrow a_{m,l}^2 + 2016b_{m,l}^2 = 2017^{l+2m}$$

We prove that for each n the set of integral solutions is

$$\{(\pm a_{m,l}, \pm b_{m,l}),\ \text{where } l + 2m = n\}.$$

For $k > 1$, assume that solutions for $n < k$ are given above. Let (u, v) be a solution of the equation for $n = k > 1$, we compute

$$u^2 + 2016v^2 = 2017^k \Longrightarrow u^2 \equiv v^2 \pmod{2017} \Longrightarrow u \equiv \pm v \pmod{2017}.$$

Suppose that $u \equiv v \mod 2017$ (the other case reduces to this one by changing the sign of v), then

$$\frac{u + 2v\sqrt{-504}}{1 + 2\sqrt{-504}} = \frac{(u + 2v\sqrt{-504})(1 - 2\sqrt{-504})}{2017} = \frac{u + 2016v}{2017} + 2\frac{v - u}{2017}\sqrt{-504},$$

which implies that

$$(a, b) := \left(\frac{u + 2016v}{2017}, \frac{v - u}{2017}\right)$$

is an integral solution to the equation for $n = k - 1$. By the inductive assumption, we see that

$$(a, b) \in \{2017^m(\pm a_l, \pm b_l) : l + 2m = k - 1\}$$

and so

$$a + 2b\sqrt{-504} \in \{\pm 2017^m(1 \pm 2\sqrt{-504})^l : l + 2m = k - 1\}$$

and

$$u + 2v\sqrt{-504} \in \{\pm 2017^m(1 \pm 2\sqrt{-504})^l(1 + 2\sqrt{-504}) : l + 2m = k - 1\}.$$

We compute that

$$\pm(1 + 2\sqrt{-504})^l(1 + 2\sqrt{-504}) = \pm(1 + 2\sqrt{-504})^{l+1} = \pm(a_{0,l+1} + 2b_{0,l+1}\sqrt{-504}),$$

$$\begin{aligned}\pm(1 - 2\sqrt{-504})^l(1 + 2\sqrt{-504}) &= \pm 2017(1 - 2\sqrt{-504})^{l-1} \\ &= \pm 2017(a_{0,l-1} - 2b_{0,l-1}\sqrt{-504}).\end{aligned}$$

Thus (u, v) is of the claimed form.

For each (m, l), among the four solutions $(\pm a_{m,l}, \pm b_{m,l})$, there is only one positive solution. It follows that the number of positive integral solutions to the equation in the problem for $n > 0$ coincides with the number of non-negative integral solutions to the equation below

$$l + 2m = n, \text{ where } l, m \geq 0$$

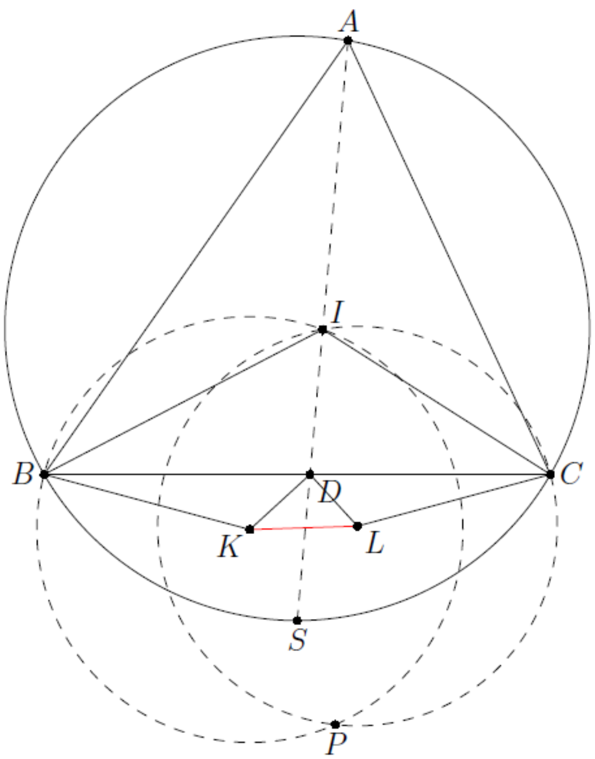
Thus the original equation has $\left\lfloor \frac{n}{2} \right\rfloor + 1$ positive solutions.

We note that the explicit description above also shows that the number of integer solutions to the equation in the problem is $2(n + 1)$.

OC369. Let I be the incenter of $\triangle ABC$. Let D be the point of intersection of AI with BC and let S be the point of intersection of AI with the circumcircle of ABC ($S \neq A$). Let K and L be incenters of $\triangle DSB$ and $\triangle DCS$. Let P be a reflection of I with respect to KL. Prove that $BP \perp CP$.

Originally 2016 Polish Mathematical Olympiad Finals Day 2 Problem 6.

We received 3 submissions. We present the solution by Shuborno Das.



We first prove that $BKDI$ and $CLDI$ are cyclic.

As K and I are incenters of $\triangle DSB$ and $\triangle ABC$, respectively, we have

$$\begin{aligned}\angle BKD + \angle BID &= (180^\circ - \angle DBK - \angle BDK) + \left(\frac{\angle A + \angle B}{2}\right)\\ &= \left(180^\circ - \frac{\angle DBS}{2} - \frac{\angle A}{4} - \frac{\angle B}{2}\right) + \left(\frac{\angle A + \angle B}{2}\right)\\ &= \left(180^\circ - \frac{\angle A}{2} - \frac{\angle B}{2}\right) + \left(\frac{\angle A + \angle B}{2}\right)\\ &= 180^\circ,\end{aligned}$$

where the penultimate equality follows from

$$\angle DBS = \angle CBS = \angle A/2.$$

So, $BKDI$ is cyclic and similarly $CLDI$ is cyclic.

Next, we prove that the circumcenter of $\triangle BPI$ is K and the circumcenter of $\triangle CPI$ is L. We have

$$\angle SBI = \angle SBC + \angle CBI = \frac{\angle A}{2} + \frac{\angle B}{2} = \angle BAI + \angle ABI = \angle SIB,$$

so $SB = SI$. As K is the incenter of $\triangle DSB$, then KS bisects $\angle ISB$, which gives $KI = KB$.

As point P is the reflection of point I over the line KL, then $IK = PK$, so $KB = IK = PK$. Therefore, the circumcenter of $\triangle BPI$ is point K and similarly the circumcenter of $\triangle CPI$ is point L.

Now, we have

$$\angle BPC = \angle BPI + \angle CPI = \frac{\angle BKI + \angle CLI}{2} = \frac{\angle BDI + \angle CDI}{2} = \frac{\angle BDC}{2} = 90^\circ,$$

which gives the desired conclusion.

OC370. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is winnable if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

Originally 2016 USAMO Day 2 Problem 6.

We received 2 submissions. We present the solution by Oliver Geupel.

We assert that the game is winnable if and only if $k < n$.

First, suppose $k < n$. We give a way to win in not more than $n + 3$ moves.

Denote the positions of cards by the integers $1, \ldots, 2n$. In the first move, we point to the cards at positions $1, \ldots, k$. If the game is not over, we point to the cards at positions $2, \ldots, k + 1$ in the second move. By inspecting the card at position $k + 1$ and the distinction of the labels shown in both moves, we can determine the label of the card at position 1. We proceed, if necessary, pointing to cards positions $3, \ldots, k + 2$ in the third move and so on. We point to cards at positions $n + 2, \ldots, n + k + 1$ in the $(n + 2)$nd move which gives us enough information to determine the label of the card at position $n + 1$.

At that time, we have learned the labels of $n+1$ cards. By the Pigeonhole principle, we know two matching cards, and we can point to them and $k-2$ redundant cards in the $(n + 3)$rd move, which wins the game.

It remains to show that there is no winning strategy when $k = n$.

Let P_1 be the set of positions of cards we point to in the first move. Assume that there is no match in the first move. Then, after the first move, we have the situation that we know a collection P_1 of n positions with no matching pair of cards, but we do not know the specific position of any label in P_1. The latter is because the wizard has arbitrarily permuted the n chosen cards.

In the second move, it will not help to point to the set P_1 or to its complement $\overline{P_1}$, which has also no matching pair. So we will point to a combination P_2 of some positions in P_1 and some positions in $\overline{P_1}$. It may happen that the positions chosen from P_1 give exactly the labels that are missing at the positions chosen from $\overline{P_1}$. Then, after the second move, we have the situation that we know a collection P_2 of n positions with no matching pair of cards, but we do not know the specific position of any label in P_2.

Proceeding this way, it may happen that, after every move, we know a set of n positions with no matching pair of cards, but we do not know the specific position of any label in this set. This prevents us from winning certainly in the next move. So there is no fixed number m of moves, in which we can certainly win the game.

OC371. Let a, b and $c \in \mathbb{R}^+$ such that $abc = 1$. Prove that

$$\frac{a+b}{(a+b+1)^2} + \frac{b+c}{(b+c+1)^2} + \frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}.$$

Originally 2016 Iranian Mathematical Olympiad 3rd Round Algebra Problem 2.

We received 5 correct submissions. We present the solution submitted independently by Šefket Arslanagić and Shuborno Das.

The inequality in the statement will be obtained via AM-GM inequality and Cauchy-Schwarz inequality.

First, we establish that for any positive real numbers x, y, and, z, we have

$$\frac{xyz}{x^3+y^3+xyz}+\frac{xyz}{y^3+z^3+xyz}+\frac{xyz}{z^3+x^3+xyz}\le 1 \tag{1}$$

Indeed AM-GM inequality implies that

$$\frac{x^3+x^3+y^3}{3}\ge x^2y,\quad \frac{x^3+y^3+y^3}{3}\ge xy^2$$

and so $x^3+y^3\ge x^2y+xy^2$. Similarly

$$y^3+z^3\ge y^2z+yz^2 \quad\text{and}\quad z^3+x^3\ge z^2x+zx^2.$$

These consequences of AM-GM inequality imply and prove (1)

$$\begin{aligned}
&\frac{xyz}{x^3+y^3+xyz}+\frac{xyz}{y^3+z^3+xyz}+\frac{xyz}{z^3+x^3+xyz}\\
&\le \frac{xyz}{x^2y+xy^2+xyz}+\frac{xyz}{y^2z+yz^2+xyz}+\frac{xyz}{z^2x+zx^2+xyz}\\
&=\frac{xyz}{xy(x+y+z)}+\frac{xyz}{yz(y+z+x)}+\frac{xyz}{zx(z+x+y)}\\
&=\frac{z+x+y}{x+y+z}\\
&=1.
\end{aligned}$$

Next we take $a=x^3$, $b=y^3$, and $c=z^3$ in (1). Taking into consideration that $abc=1$, we obtain

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\le 1,$$

$$2=3-1\le 1-\frac{1}{a+b+1}+1-\frac{1}{b+c+1}+1-\frac{1}{c+a+1}\quad\text{and,}$$

$$4\le\left(\frac{a+b}{a+b+1}+\frac{b+c}{b+c+1}+\frac{c+a}{c+a+1}\right)^2 \tag{2}$$

Lastly, Cauchy-Schwarz inequality provides an upper bound for the right side of inequality (2), and allows us to write

$$4\le[(a+b)+(b+c)+(c+a)]\left[\frac{a+b}{(a+b+1)^2}+\frac{b+c}{(b+c+1)^2}+\frac{c+a}{(c+a+1)^2}\right] \tag{3}$$

After dividing both sides of (3) by $2(a+b+c)$, we obtain the statement inequality. Equality holds if and only if $a=b=c=1$.

OC372. In the circumcircle of a triangle ABC, let A_1 be the point diametrically opposite to the vertex A. Let A' be the intersection point of A_1' and BC. The perpendicular to the line AA_1 from A' meets the sides AB and AC at M and N, respectively. Prove that the points A, M, A_1 and N lie on a circle whose center lies on the altitude from A of the triangle ABC.

Originally 2016 Spain Mathematical Olympiad Day 1 Problem 3.

We received 4 correct submissions. We present the solution by Ivko Dimitrić.

We solve the question in several cases.

Case 1. If $\angle B = \angle C$, then $\triangle ABC$ is isosceles, $M = B$, and $N = C$. In this case, it is obvious that the statement holds.

Case 2. Assume $\angle B < \angle C$. Let D be the foot of the altitude from A.

Case 2a. Assume $\angle B < \angle C < 90°$. If vertices A, B, and C are arranged in counterclockwise order, then rays AB, AA_1, AD, and AB are also in counterclockwise order. Since AA_1 is a diameter of the circumcircle, we have

$$\angle MBA_1 = \angle ABA_1 = 90^\circ$$

and

$$\angle NCA_1 = 180^\circ - \angle ACA_1 = 90^\circ.$$

In addition, as $AA' \perp MN$, $\angle MA'A_1 = 90^\circ$ and $\angle NA'A_1 = 90^\circ$. Therefore the quadrilaterals MBA_1A' and $NCA'A_1$ are cyclic, inscribed in circles whose centers are midpoints of MA_1 and NA_1, respectively.

Furthermore, $\angle AMN = \angle BA_1A = \angle C$ as an exterior angle of the cyclic quadrilateral MBA_1A', and $\angle AA_1N = \angle C$, as $\angle C$ is an exterior angle of cyclic quadrilateral $NCA'A_1$. As $\angle AMN = \angle AA_1N$ it follows that AMA_1N is a cyclic quadrilateral.

Let P be the center of the circumcircle of AMA_1N. We have

$$\angle NAP = \frac{180^\circ - \angle APN}{2} = \frac{180^\circ - 2\angle AMN}{2} = 90^\circ - \angle C.$$

On the other hand, since $AD \perp BC$, $\angle CAD = 90^\circ - \angle C$. Hence $\angle NAP = \angle CAD$ and P belongs to AD . Its location is found as the intersection point of the perpendicular bisector of the diameter AA_1 with altitude AD.

Case 2b. Assume $\angle B < \angle C = 90^\circ$. Then $A' = A_1 = M = B$ and the quadrilateral AMA_1N degenerates into a right triangle, $\triangle ABN$, with hypotenuse, AN, along AC. However AC is the altitude from A of $\triangle ABC$. Hence the circumcenter of quadrilateral AMA_1N, now $\triangle ABN$, belongs to that altitude from A of $\triangle ABC$.

Case 2c. Assume $\angle B < 90^\circ < \angle C$. The proof for this case follows closely the proof of *Case 2a.* If vertices A, B, and C are arranged in counterclockwise order, then rays AA_1, AB, AC, and AD are also in counterclockwise order and P and B are on different sides of AC. Moreover, $\angle CAD = \angle C - 90°$ and

$$\angle APN = 2\,(180° - \angle AMN) = 2\,\angle BMA' = 2\,\angle BA_1A = 2\,(180° - \angle C),$$

leading to

$$\angle NAP = \frac{180° - \angle APN}{2} = \angle C - 90° = \angle CAD,$$

so, again, P belongs to the altitude AD.

OC373. Let a and b be positive integers. Denote by $f(a,b)$ the number of sequences $s_1, s_2, \dots, s_a \in \mathbb{Z}$ such that $|s_1| + |s_2| + \cdots + |s_a| \leq b$. Show that $f(a,b) = f(b,a)$.

Originally 2016 Polish Mathematical Olympiad Finals Day 1 Problem 3.

We received 2 correct submissions. We present both solutions modified for editorial presentation.

Solution 1, by Mohammed Aassila.

Let $f(a,b,0) = 1$. Let k be an integer such that $1 \leq k \leq \min(a,b)$. Let $f(a,b,k)$ be the number of a-tuples, $(s_1, s_2, \dots, s_a)$, of integers such that $|s_1|+|s_2|+\cdots+|s_a| \leq b$ and exactly k of $s_1, \dots, s_a$ are nonzero. Such a tuple is uniquely defined by the k positions of the nonzero s_i's, the signs, $+$ or $-$, of the nonzero s_i's, and an ordered k-tuple of positive integers with sum b or less. There are $\binom{a}{k}$ ways to choose the unordered subset of k positions from the fixed set of positions $1, \dots, a$. There are two ways to choose the sign of a specific nonzero s_i, and there are 2^k ways to choose the signs of the nonzero s_i's. There are $\binom{b}{k}$ ways to choose the ordered k-tuple of positive integers that sum up to b or less. This counting result follows from the next observation. Choosing the ordered k-tuple of nonzero positive integers $u_1, \dots, u_k$ with $u_1 + \cdots + u_k \leq b$ is equivalent to choosing k numbers $u_1, u_1 + u_2, \dots, u_1 + \cdots + u_k$, disregarding the order, from b numbers $1, \dots, b$.

By the multiplicative priciple of counting

$$f(a,b,k) = 2^k \binom{a}{k}\binom{b}{k},$$

for any $a \geq 1$, $b \geq 1$, and $k \geq 1$. Moreover, the above formula holds for $k = 0$, as $f(a,b,0) = 1$.

Subsequently,

$$f(a,b) = \sum_{k=0}^{\min(a,b)} f(a,b,k) = \sum_{k=0}^{\min(a,b)} 2^k \binom{a}{k}\binom{b}{k}.$$

As the above is symmetric in a and b, it follows that $f(b,a) = f(a,b)$ for any $a \geq 1$, $b \geq 1$.

Solution 2, by Missouri State University Problem Solving Group.

First, $f(a, b) = f(b, a)$ is true if either $a = 1$ or $b = 1$. This is because the number of 1-tuples, (s_1), with $|s_1| \leq b$ is $2b + 1$, hence $f(1, b) = 2b + 1$. Also, the number of a-tuples such that $|s_1| + \cdots + |s_a| \leq 1$ is $2a + 1$, hence $f(a, 1) = 2a + 1$.

For $a \geq 2$ and $b \geq 2$, $f(a, b) = f(b, a)$ is established as a consequence of the more general result

$$f(a, b) = f(a, b - 1) + f(a - 1, b) + f(a - 1, b - 1). \tag{1}$$

Relation (1) follows from writing the set T of a-tuples $(s_1, \ldots, s_a)$ of integers such that $|s_1| + \cdots + |s_a| \leq b$ as the union of three mutually exclusive sets. Namely,

$$T_1 = \{(s_1, \ldots, s_a) \mid |s_1| + \cdots + |s_a| \leq b - 1\},$$

$$T_2 = \{(s_1, \ldots, s_a) \mid |s_1| + \cdots + |s_a| = b, s_a > 0\},$$

$$T_3 = \{(s_1, \ldots, s_a) \mid |s_1| + \cdots + |s_a| = b, s_a \leq 0\}.$$

The cardinal of set T_1 is $f(a, b - 1)$. The set T_2 is in one-to-one correspondence with

$$\{(s_1, \ldots, s_{a-1}) \mid |s_1| + \cdots + |s_{a-1}| \leq b\}$$

and has cardinal $f(a - 1, b)$. Lastly, T_3 is in one-to-one correspondence with

$$\{(s_1, \ldots, s_{a-1}) \mid |s_1| + \cdots + |s_{a-1}| \leq b - 1\}$$

and has cardinal $f(a - 1, b - 1)$. Relation (1) follows.

Editor's Comments. Solution 2 was submitted as a proof by induction. However, the editor felt that the induction step can be avoided.

OC374. Let p be an odd prime and let $a_1, a_2, \ldots, a_p$ be integers. Prove that the following two conditions are equivalent:

1) There exists a polynomial $P(x)$ of degree less than or equal to $\frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$.

2) For any natural number $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

where indices are taken modulo p.

Originally 2016 China National Olympiad Day 1 Problem 3.

We received one correct submission. We present the solution by Mohammed Aassila with editorial changes.

First, we establish the following two results.

(A) Let p be a prime number, then for natural numbers $k \geq 0$

$$S = \sum_{i=1}^{p} i^k \equiv \begin{cases} -1 \pmod p & \text{if} \quad p-1 \mid k \\ 0 \pmod p & \text{if} \quad p-1 \nmid k. \end{cases}$$

Proof of A. If $p-1 \mid k$, the result follows from Fermat's little theorem. For any $1 \leq i \leq p-1$, i is not divisible by p, hence we have $i^{p-1} \equiv 1(\text{mod } p)$, and $i^k \equiv 1(\text{mod } p)$. Therefore

$$\sum_{i=1}^{p} i^k \equiv p-1+p^k(\text{mod } p) \equiv -1(\text{mod } p).$$

If $p-1 \nmid k$, consider g a primitive root of p. Any odd prime number has a primitive root. A primitive root is an integer number, g, that is coprime with p, such that the powers $g, g^2, \ldots, g^{p-1}$ are congruent to $1, 2, \ldots, p-1$ modulo p, not necessarily in the specified order. In addition, by Fermat's little theorem we know that $g^{p-1} \equiv 1(\text{mod } p)$. Therefore,

$$g^k S = g^k \sum_{i=1}^{p} i^k \equiv g^k \sum_{i=1}^{p-1} (g^i)^k (\text{mod } p) \equiv \sum_{i=2}^{p} (g^i)^k (\text{mod } p) \equiv S(\text{mod } p).$$

Since $p-1 \nmid k$, we have $g^k \not\equiv 1(\text{mod } p)$. This together with $g^k S \equiv S(\text{mod } p)$ implies that $S \equiv 0(\text{mod } p)$, which establishes the second part of (A).

(B) Let $Q(x,d) = (P(x+d)-P(x))^2$. Let k be the degree of P as a polynomial in x. Then $Q(x,d) = \sum_m C_m(d)x^m$ can be viewed as a polynomial in x with coefficients $C_m(d)$ that are polynomials in d. The following can easily be established:

(B1) The degree of Q in x is less than or equal to $2(k-1)$.

(B2) The degree of Q in d is less than or equal to $2k$.

(B3) The degree of $C_m(d)$ is less than or equal to $2k-m$.

We now will prove the main results.

(1) $\Rightarrow$ (2)

Fix d. Note that the sum in question is $\sum_{i=1}^{p} Q(i)$, where $Q(x)$ is the polynomial defined at (B). The degree of Q in x is less than or equal to

$$2(\deg(P)-1) = 2((p-1)/2-1) = p-3.$$

Therefore $\sum_{i=1}^{p} Q(i)$ is a linear combination of $\sum_{i=1}^{p} i^k$, where $0 \leq k \leq p-3$. By result (A),

$$\sum_{i=1}^{p} Q(i) \equiv 0(\text{mod } p).$$

(2) $\Rightarrow$ (1)

Lagrange polynomials and modular multiplicative inverse under p can be used to construct a polynomial P of degree at most $p-1$, such that $P(i) \equiv a_i (\text{mod } p)$ for any $1 \leq i \leq p$. The polynomial P can be thought of as interpolating the data set of points $\{(i, a_i),\, 1 \leq i \leq p\}$ mod p.

Assuming (2), we will prove by contradiction that the degree of P is at most $(p-1)/2$, mod p.

Assume $\deg(P) \geq (p-1)/2 + 1$, mod p. Let k be the degree of P mod p, and let R be the coefficient of x^k in P, $R \not\equiv 0 (\text{mod } p)$. From (B) we know that the degree of Q in x is at most $2(k-1) \leq 2(p-2)$. Note that since (2) is valid for any $1 \leq d \leq (p-1)/2$, in fact, it is valid for $0 \leq d \leq p-1$. Due to (A) and (2) we have that the coefficient of x^{p-1} in $Q(x)$, $C_{p-1}(d) \equiv 0 (\text{mod } p)$ for any $0 \leq d \leq p-1$. However, $C_{p-1}(d)$ is a polynomial in d of degree at most $2k - (p-1) \leq p-1$. Having more roots modulo p than its degree, it follows that $C_{p-1}(d)$ is the zero polynomial modulo p and that all its coefficients are 0 modulo p.

The coefficient of d^{2k-p+1}, the monomial with largest exponent in $C_{p-1}(d)$, originates from the expansion of $R^2(x+d)^{2k} - 2R^2(x+d)^k x^k$ and is a combination of binomial coefficients

$$R^2 \left(\binom{2k}{p-1} - 2\binom{k}{2k-p+1} \right).$$

However, notice that p divides $\binom{2k}{p-1}$ and does not divide $\binom{k}{2k-p+1}$, since $p+1 \leq 2k$ and $k \leq p-1$. It follows that $R \equiv 0 (\text{mod } p)$ which is a contradiction. It must be that $\deg(P) \leq (p-1)/2$.

Editor's Comments. The question and its solution can be better understood when viewed in the finite field, $\mathbb{Z}_p$, of integers mod p.

OC375. Let $ABCD$ be a non-cyclic convex quadrilateral with no parallel sides. Suppose the lines AB and CD meet in E. Let $M \neq E$ be the intersection of circumcircles of ADE and BCE. Further, suppose that the internal angle bisectors of $ABCD$ form a convex cyclic quadrilateral with circumcenter I while the external angle bisectors of $ABCD$ form a convex cyclic quadrilateral with circumcenter J. Show that I, J, M are collinear.

Originally 2016 Brazil National Olympiad Day 2 Problem 6.

We received no solutions to this problem.