FOCUS ON...

No. 35

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The Asymptotic Behavior of Integrals

Introduction

Focus On... No 10 [2014 : 21-24] considered the integrals $I_n = \int_0^1 (\phi(x))^n \, dx$, where the quadratic function $\phi(x) = ax^2 + bx + c$ was supposed to remain positive in $[0, 1]$. The study culminated in the asymptotic behavior of such integrals, that is, the determination of a simple sequence (ω_n) such that $\lim_{n\to\infty} \frac{I_n}{\omega_n} = 1$, denoted by $I_n \sim \omega_n$. In this number, we keep the same goal but, through various sequences of integrals, present some simple ways to obtain such an asymptotic behavior. We restrict ourselves to elementary problems and methods, referring the reader to **[1]** for more complicated examples and to **[2]** for more sophisticated techniques (Laplace's method, for instance).

With monotone sequences

In our first examples, we show how the knowledge of a recursion formula for a monotone sequence of integrals can quickly lead to the sought sequence (ω_n).

Take the classical integrals

$$I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx.$$

Since $(\sin x)^{n+1} \leq (\sin x)^n$ for $x \in [0, \frac{\pi}{2}]$, the sequence (I_n) is nonincreasing. Moreover, an integration by parts yields

$$I_{n+2} = (n+1)(I_n - I_{n+2}) \quad \text{or} \quad (n+2)I_{n+2} = (n+1)I_n.$$

It follows that

$$(n+2)I_{n+2}I_{n+1} = (n+1)I_{n+1}I_n$$

for all nonnegative integers n, showing that the sequence $((n+1)I_{n+1}I_n)$ is constant. Thus

$$(n+1)I_{n+1}I_n = 1 \cdot I_1 \cdot I_0 = \frac{\pi}{2}$$

for all $n \geq 0$.

Now, $I_{n+2} \leq I_{n+1} \leq I_n$ gives

$$\frac{n+1}{n+2} I_n \leq I_{n+1} \leq I_n$$

and using the squeeze principle, we deduce $I_{n+1} \sim I_n$. Then we obtain

$$I_n^2 \sim I_n I_{n+1} = \frac{\pi}{2(n+1)} \sim \frac{\pi}{2n},$$

that is, $I_n \sim \sqrt{\frac{\pi}{2n}}$.

For another example, consider

$$J_n = \int_0^1 \frac{x^n \ln x}{x+1}\, dx.$$

Since $\lim_{x\to 0^+} x^n \ln x = 0$ when $n \geq 1$, the integrand is the restriction to $(0,1]$ of a continuous function on $[0,1]$ and the integral J_n does exist. For $x \in (0,1]$ the inequality

$$\frac{x^n \ln x}{x+1} \leq \frac{x^{n+1} \ln x}{x+1}$$

holds (since $x^{n+1} \leq x^n$ and $\ln x \leq 0$), therefore the sequence (J_n) is nondecreasing. In addition, we observe that

$$J_n + J_{n+1} = \int_0^1 x^n \ln x\, dx = -\frac{1}{(n+1)^2}$$

(readily found by an integration by parts). Now, for all $n \geq 2$, we obtain

$$-\frac{1}{n^2} = J_{n-1} + J_n \leq 2J_n \leq J_n + J_{n+1} = -\frac{1}{(n+1)^2}$$

and so $J_n \sim -\dfrac{1}{2n^2}$.

Two general results

The following theorem offers two general results of the good-to-be-known category.

Theorem 1 *Let f be a real-valued continuous function on $[0,1]$. Then*

(i) $\lim_{n\to\infty} n \cdot \int_0^1 x^n f(x^n)\, dx = \int_0^1 f(x)\, dx,$

(ii) $\lim_{n\to\infty} n \cdot \int_0^1 x^n f(x)\, dx = f(1).$

Proof of (i). The change of variables $x = u^{1/n}$ shows that

$$n \cdot \int_0^1 x^n f(x^n)\, dx = \int_0^1 u^{1/n} f(u)\, du$$

and, denoting by M the maximum value of the continuous function f on $[0,1]$, we deduce

$$\left| n \cdot \int_0^1 x^n f(x^n)\,dx - \int_0^1 f(u)\,du \right| \leq \int_0^1 |f(u)|(1-u^{1/n})\,du$$
$$\leq M \int_0^1 (1-u^{1/n})\,du$$
$$= \frac{M}{n+1}.$$

The result follows.

Proof of (ii). To keep the same level of simplicity, we suppose that f is continuously differentiable on the interval $[0,1]$ (the reader will find a proof in the general case, together with an example of application, in my solution to problem **4010** [2015 : 28,30 ; 2016 : 43-4]).

Let μ be the maximum of f' on $[0,1]$. Then, for $x \in [0,1]$ we have

$$|f(1)-f(x)| = \left| \int_x^1 f'(t)\,dt \right| \leq \int_x^1 |f'(t)|\,dt \leq \mu(1-x).$$

Since

$$\left| f(1) - (n+1) \cdot \int_0^1 x^n f(x)\,dx \right| = (n+1) \left| \int_0^1 x^n (f(1)-f(x))\,dx \right|$$
$$\leq (n+1) \int_0^1 x^n |f(1)-f(x)|\,dx,$$

we obtain

$$\left| f(1) - (n+1) \cdot \int_0^1 x^n f(x)\,dx \right| \leq (n+1)\mu \int_0^1 x^n (1-x)\,dx = \frac{\mu}{n+2}$$

and the conclusion readily follows. □

An application of (i)

Interestingly, part (i) can be applied twice in a problem of the 2005 Romanian Olympiad proposed in Mathproblems, Vol. 5, Issue 2 (here slightly adapted):

Show that $\lim\limits_{n\to\infty} n \displaystyle\int_0^1 \frac{x^n}{1+x^{2n}}\,dx = \frac{\pi}{4}$ and then find $\lim\limits_{n\to\infty} n \left(\dfrac{\pi}{4} - n \displaystyle\int_0^1 \frac{x^n}{1+x^{2n}}\,dx \right)$.

Part (i) directly yields

$$\lim_{n\to\infty} n \int_0^1 \frac{x^n}{1+x^{2n}}\,dx = \int_0^1 \frac{dx}{1+x^2} = [\arctan x]_0^1 = \frac{\pi}{4}.$$

Then, integrating by parts, we obtain

$$\begin{aligned} n\int_0^1 \frac{x^n}{1+x^{2n}}\,dx &= \int_0^1 x\cdot\frac{nx^{n-1}}{1+(x^n)^2}\,dx \\ &= [x\arctan(x^n)]_0^1 - \int_0^1 \arctan(x^n)\,dx \\ &= \frac{\pi}{4} - \int_0^1 \arctan(x^n)\,dx \end{aligned}$$

and the problem now reduces to finding $\lim\limits_{n\to\infty} n\displaystyle\int_0^1 \arctan(x^n)\,dx$.

Let f be the continuous function defined on $[0,1]$ by $f(x) = \frac{\arctan x}{x}$ if $x \neq 0$ and $f(0) = 1$. We observe that $\arctan(x^n) = x^n f(x^n)$ and therefore

$$\lim_{n\to\infty} n\int_0^1 \arctan(x^n)\,dx = \lim_{n\to\infty} n\int_0^1 x^n f(x^n)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \frac{\arctan x}{x}\,dx.$$

The latter integral is equal to the constant of Catalan $G = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$. Note that we have proved that $\displaystyle\int_0^1 \frac{x^n}{1+x^{2n}}\,dx = \frac{\pi}{4n} - \frac{G}{n^2} + o(1/n^2)$.

Some applications of (ii)

As a first example, we examine again the integral $J_n = \displaystyle\int_0^1 \frac{x^n \ln x}{x+1}\,dx$. The result (ii) gives $\lim\limits_{n\to\infty} n\cdot J_n = 0$, which does not lead to the sought ω_n. To get round this difficulty, we first integrate by parts:

$$\int_\varepsilon^1 \frac{x^n \ln x}{x+1}\,dx = \int_\varepsilon^1 \frac{x\ln x}{1+x}\,d\left(\frac{x^n}{n}\right) = \left[\frac{x^{n+1}\ln x}{n(1+x)}\right]_\varepsilon^1 - \frac{1}{n}\int_\varepsilon^1 x^n g(x)\,dx$$

where $\varepsilon \in (0,1)$ and $g(x) = \dfrac{1+x+\ln x}{(1+x)^2}$. Letting $\varepsilon \to 0^+$, we obtain

$$J_n = -\frac{1}{n}\int_0^1 x^n g(x)\,dx = -\frac{1}{n^2}\left(n\int_0^1 x^n g(x)\,dx\right)$$

and, since $\lim\limits_{n\to\infty} n\displaystyle\int_0^1 x^n g(x)\,dx = g(1) = \frac{1}{2}$, this confirms the result $J_n \sim -\frac{1}{2n^2}$ found above.

Integration by parts will also play a role in our second example, the integral $K_n = \displaystyle\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx$. Incidentally, this integral is equal to $\left|\frac{\pi}{4} - \sum\limits_{k=0}^{n} \frac{(-1)^k}{2k+1}\right|$, hence estimates the error when π is approximated *via* the partial sums of Gregory's series. We first prove $K_n \sim \frac{1}{4n}$ and then improve this result by showing that $K_n = \frac{1}{4n} - \frac{1}{4n^2} + o(1/n^2)$. The substitution $x = \sqrt{u}$ leads to

$$K_n = \int_0^1 u^n \frac{\sqrt{u}}{2(1+u)}\, du$$

and an application of (ii) gives $\lim_{n\to\infty} nK_n = \dfrac{1}{4}$, that is, $K_n \sim \frac{1}{4n}$.

Now, let $f(u) = \frac{\sqrt{u}}{2(1+u)}$. First integrating $u^n f(u) = (uf(u)) \cdot u^{n-1}$ by parts on $[\varepsilon, 1]$ and then letting $\varepsilon \to 0^+$ (as above), an easy calculation leads to

$$K_n = \frac{f(1)}{n} - \frac{1}{n}\int_0^1 u^n \cdot \frac{(3+u)\sqrt{u}}{4(1+u)^2}\, du = \frac{1}{4n} - \frac{1}{n^2}\left(n\int_0^1 u^n \cdot \frac{(3+u)\sqrt{u}}{4(1+u)^2}\, du\right)$$

and so $K_n = \frac{1}{4n} - \frac{1}{n^2}\left(\frac{1}{4} + o(1)\right)$, the desired result.

Part (ii) of the theorem can sometimes be used to find the asymptotic behavior of integrals of the form $\displaystyle\int_a^b (h(x))^n\, dx$. Typically, suppose that h is a positive, continuously differentiable function on $[a, b]$ such that $h'(x) > 0$ for $x \in [a, b]$ and $h(a) = 0, h(b) = 1$. Under these hypotheses, the following holds:

$$\int_a^b (h(x))^n\, dx \sim \frac{1}{nh'(b)}.$$

Indeed, the function h, being strictly increasing and continuous on $[a, b]$, is a bijection from $[a, b]$ onto $[h(a), h(b)] = [0, 1]$. The change of variables $x = h^{-1}(y)$ yields $dx = \dfrac{dy}{h'(h^{-1}(y))}$ and so

$$\int_a^b (h(x))^n\, dx = \int_{h(a)}^{h(b)} y^n \frac{dy}{h'(h^{-1}(y))} = \int_0^1 y^n \frac{dy}{h'(h^{-1}(y))}.$$

Since $\dfrac{1}{h'(h^{-1}(1))} = \dfrac{1}{h'(b)}$, the result follows.

Problem 2007 of *Mathematics Magazine* proposed in December 2016 can be solved using the ideas just developed. Here is the statement of this problem and a variant of solution:

> Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Evaluate
>
> $$\lim_{n\to\infty}\left(n \cdot \int_0^1 \left(\frac{2\left(x - \frac{1}{2}\right)^2}{x^2 - x + \frac{1}{2}}\right)^n f(x)\, dx\right).$$

Let $\phi(x) = \dfrac{2\left(x-\frac{1}{2}\right)^2}{x^2-x+\frac{1}{2}}$ and $L_n = \displaystyle\int_0^1 (\phi(x))^n f(x)\,dx$. We prove that

$$\lim_{n\to\infty} n\cdot L_n = \frac{f(0)+f(1)}{2}.$$

A quick study of $\phi(x) = 2 - \frac{1}{2x^2-2x+1}$ (left to the reader) shows that the restriction ϕ_0 of ϕ to $[0, \frac{1}{2}]$ is a bijection onto $[0,1]$ with

$$\phi_0^{-1}(u) = \frac{1}{2}\left(1-\sqrt{\frac{u}{2-u}}\right),$$

and that the restriction ϕ_1 of ϕ to $[\frac{1}{2}, 1]$ is a bijection onto $[0,1]$ whose inverse is given by

$$\phi_1^{-1}(u) = \frac{1}{2}\left(1+\sqrt{\frac{u}{2-u}}\right).$$

Now, let

$$U_n = \int_0^{1/2} (\phi_0(x))^n f(x)\,dx \qquad \text{and} \qquad V_n = \int_{1/2}^1 (\phi_1(x))^n f(x)\,dx.$$

The changes of variables $x = \phi_0^{-1}(u)$ in U_n and $x = \phi_1^{-1}(u)$ in V_n and a short calculation give

$$U_n = \frac{1}{2}\int_0^1 u^n g_0(u)\,du, \qquad V_n = \frac{1}{2}\int_0^1 u^n g_1(u)\,du$$

where $g_k(u) = u^{-1/2}(2-u)^{-3/2} f(\phi_k^{-1}(u))$ $(k = 0, 1)$. Thus

$$\lim_{n\to\infty} n\cdot U_n = \frac{1}{2}g_0(1) = \frac{1}{2}f(0) \qquad \text{and} \qquad \lim_{n\to\infty} n\cdot V_n = \frac{1}{2}g_1(1) = \frac{1}{2}f(1)$$

and so

$$\lim_{n\to\infty} n\cdot L_n = \lim_{n\to\infty} n(U_n+V_n) = \lim_{n\to\infty} n\cdot U_n + \lim_{n\to\infty} n\cdot V_n = \frac{f(0)+f(1)}{2}.$$

As usual we conclude with a bunch of exercises.

Exercises

1. Let $X_n = \displaystyle\int_0^{\frac{\pi}{4}} (\tan x)^n\,dx$. Compute $X_n + X_{n+2}$ and deduce that $X_n \sim \dfrac{1}{2n}$.

2. Let $Y_n = \displaystyle\int_0^1 \frac{x^n \ln x}{x^n - 1}\,dx$. Show that $Y_n \sim \dfrac{\alpha}{n^2}$ for some positive α.

3. Let $f_1(t) = \dfrac{1}{1+t}$ and for $k \geq 2$, let $f_k(t) = \dfrac{d}{dt}(tf_{k-1}(t))$. Prove that for any integer $m \geq 1$,

$$\int_0^1 \frac{t^n}{1+t}\,dt = \frac{f_1(1)}{n} - \frac{f_2(1)}{n^2} + \cdots + (-1)^{m-1}\frac{f_m(1)}{n^m} + o(1/n^m)$$

as $n \to \infty$.

4. Let $Z_n = \displaystyle\int_0^1 (ax^2 + bx + c)^n\,dx$ where a, b are negative numbers and $a + b + c$ is positive. Use this number of Focus On... to obtain $Z_n \sim \dfrac{c^{n+1}}{-nb}$ already found in No. 10.

References

[**1**] O. Furdui, *Limits, Series, and Fractional Integrals*, Springer, 2013, pp. 9-16.

[**2**] H. S. Wilf, *Mathematics for the Physical Sciences*, Wiley, 1962, chapter 4.