SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


4441. Proposed by Mihaela Berindeanu.

Let \( \triangle ABC \) be an acute triangle, with circumcenter \( O \) and orthocenter \( H \). Let \( A', B' \) and \( C' \) be the intersection of \( AH, BH, CH \) with \( BC, AC, AB \), respectively. Let \( A_1, B_1 \) and \( C_1 \) be the intersection of \( AO, BO, CO \) with \( BC, AC, AB \), respectively. Let \( A'' \), \( B'' \) and \( C'' \) be midpoints of \( AA_1, BB_1 \) and \( CC_1 \), show that \( A'A'', B'B'' \) and \( C'C'' \) have a common intersection point.

We received 7 submissions, all of which were correct; we feature a solution by Ivko Dimitrić supplemented by a small, but critical, contribution from the solution by Marie-Nicole Gras.

The requirement that the triangle be acute is unnecessary; we shall see that for any \( \triangle ABC \) the lines \( A'A'', B'B'' \), and \( C'C'' \) all pass through the nine-point center.

When \( AB = AC \), the points \( A' \) and \( A_1 \) coincide, whence the line \( A'A'' \) is an altitude and passes through the nine-point center.

Let us therefore assume that \( AB \neq AC \), and let \( M \) be the midpoint of \( BC \), \( K \) the midpoint of \( AA_1 \), and \( N \) the intersection point of \( A'A'' \) and \( KM \). Since \( AH = 2OM \) and \( AH, OM \perp BC \), we have \( AK = OM \) and \( AK \parallel OM \), so the quadrilateral \( AKMO \) is a parallelogram and hence \( KM \parallel AA_1 \). Consequently, triangles \( NMA' \) and \( A''A_1A' \) are similar as are triangles \( A'NK \) and \( A'A''A' \), and we have

\[
\frac{NM}{A''A_1} = \frac{A'N}{A'A''} = \frac{NK}{A''A'}
\]

Because \( A''A_1 = A''A \) we have \( NM = NK \). Hence, \( N \) is the midpoint of the hypotenuse \( KM \) of \( \triangle A'MK \), and therefore the center of its circumcircle.

Since a circle is uniquely determined by any three of its points, and the nine-point circle passes through the feet of the altitudes, the midpoints of the sides and the midpoints of the segments from the orthocenter to the vertices, it follows that the circumcircle of \( \triangle A'MK \) is the nine-point circle of \( \triangle ABC \) and its center is \( N = A'A'' \cap KM \). Hence, the line \( A'A'' \) passes through the nine-point center \( N \).

In the same manner, the lines \( B'B'' \) and \( C'C'' \) also pass through the nine-point center, so that the three lines in question have the common intersection point \( N \).

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Find the following limit
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \left( \frac{1}{1 + \sqrt{2}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{2n - 1} + \sqrt{2n}} \right).
\]

We received 21 submissions, of which all but one were correct. We present several solutions.

Solution 1, by Florentin Viescu.

We show that the limit is \(1/\sqrt{2}\) as an application of Cesaro-Stolz Lemma. Let
\[
a_n = \frac{1}{1 + \sqrt{2}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{2n - 1} + \sqrt{2n}}.
\]
Then
\[
a_{n+1} - a_n = \frac{1}{\sqrt{2n+1} + \sqrt{2n+2}}.
\]
Let \(b_n = \sqrt{n}\). Then \(b_{n+1} - b_n = \sqrt{n+1} - \sqrt{n} > 0\), for all \(n \geq 1\). So \((b_n)_{n \geq 1}\) is strictly increasing and approaches \(+\infty\).

Moreover, the following limit exists
\[
\lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \to \infty} \frac{\sqrt{2n+1} + \sqrt{2n+2}}{\sqrt{n+1} - \sqrt{n}}
\]
\[
= \lim_{n \to \infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{2n+1} + \sqrt{2n+2}}
\]
\[
= \lim_{n \to \infty} \frac{\sqrt{n}\left(\sqrt{1 + \frac{1}{n}} + 1\right)}{\sqrt{n}\left(\sqrt{2 + \frac{1}{n}} + \sqrt{2 + \frac{2}{n}}\right)}
\]
\[
= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.
\]

Then according to Cesàro-Stolz Lemma \(\lim_{n \to \infty} a_n/b_n = 1/\sqrt{2}\).

Solution 2, by Nguyen Viet Hung.

We show that the limit is \(1/\sqrt{2}\) as an application of the Squeeze Theorem. We have clearly
\[
S_n = \frac{1}{\sqrt{1 + \sqrt{2}}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{2n - 1} + \sqrt{2n}}
\]
\[
< \frac{1}{\sqrt{1} + \sqrt{1}} + \frac{1}{\sqrt{2} + \sqrt{2}} + \cdots + \frac{1}{\sqrt{2n-2} + \sqrt{2n-1}}.
\]
Therefore,

\[ 2S_n < 1 + \frac{1}{\sqrt{1 + \sqrt{2}}} + \frac{1}{\sqrt{2 + \sqrt{3}}} + \cdots + \frac{1}{\sqrt{2n - 1 + \sqrt{2n}}} \]

\[ = 1 + \frac{\sqrt{1} - \sqrt{2}}{-1} + \frac{\sqrt{2} - \sqrt{3}}{-1} + \cdots + \frac{\sqrt{2n - 1} - \sqrt{2n}}{-1} \]

\[ = \sqrt{2n}. \quad (1) \]

On the other hand,

\[ S_n = \frac{1}{\sqrt{1 + \sqrt{2}}} + \frac{1}{\sqrt{2 + \sqrt{3}}} + \cdots + \frac{1}{\sqrt{2n - 1 + \sqrt{2n}}} \]

\[ > \frac{1}{\sqrt{2 + \sqrt{3}}} + \frac{1}{\sqrt{4 + \sqrt{5}}} + \cdots + \frac{1}{\sqrt{2n + \sqrt{2n + 1}}} \]

Therefore,

\[ 2S_n > \frac{1}{\sqrt{1 + \sqrt{2}}} + \frac{1}{\sqrt{2 + \sqrt{3}}} + \cdots + \frac{1}{\sqrt{2n - 1 + \sqrt{2n}}} + \frac{1}{\sqrt{2n + \sqrt{2n + 1}}} \]

\[ = \frac{\sqrt{1} - \sqrt{2}}{-1} + \frac{\sqrt{2} - \sqrt{3}}{-1} + \cdots + \frac{\sqrt{2n} - \sqrt{2n + 1}}{-1} \]

\[ = \sqrt{2n + 1} - 1. \quad (2) \]

From (1) and (2), we find that

\[ \frac{\sqrt{2n + 1} - 1}{2} < S_n < \frac{\sqrt{2n}}{2}. \]

We apply Squeeze Theorem to find

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{n}} = \frac{1}{\sqrt{2}}. \]

**Solution 3, by Rob Downes.**

We show that the limit is 1/\(\sqrt{2}\) as an application of the Squeeze Theorem. Let

\[ S_n = \frac{1}{\sqrt{1 + \sqrt{2}}} + \frac{1}{\sqrt{2 + \sqrt{3}}} + \cdots + \frac{1}{\sqrt{2n - 1 + \sqrt{2n}}} \]

\[ = (\sqrt{2} - 1) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{2n} - \sqrt{2n - 1}). \]

Consider \( f(x) = \sqrt{2x} - \sqrt{2x - 1} \). Note that \( f(x) \) is a strictly decreasing function for \( x \geq 1 \). Therefore, as illustrated in the left figure below, we have:

\[ S_n \geq \int_1^{n+1} (\sqrt{2x} - \sqrt{2x - 1}) \, dx \]

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Additionally, as illustrated in the right figure we have:

\[ S_n \leq \sqrt{2} - 1 + \int_1^n (\sqrt{2x} - \sqrt{2x - 1}) \, dx. \]

Evaluating the integrals, dividing by \( \sqrt{n} \), and simplifying yields the compound inequality:

\[ \frac{(2n + 2) \frac{2}{3} - (2n + 1) \frac{2}{3} - (2^{\frac{3}{2}} - 1)}{3\sqrt{n}} \leq \frac{S_n}{\sqrt{n}} \leq \frac{(2n) \frac{2}{3} - (2n - 1) \frac{2}{3} - (2^{\frac{3}{2}} - 1) + 3(\sqrt{2} - 1)}{3\sqrt{n}}. \]

Next, we use the Squeeze Theorem to evaluate the given limit. For the expression on the left in (1), we have:

\[
\lim_{n \to \infty} \frac{(2n + 2)^{\frac{2}{3}} - (2n + 1)^{\frac{2}{3}} - (2^{\frac{3}{2}} - 1)}{3\sqrt{n}} = \lim_{n \to \infty} \frac{12n^2 + 18n + 7}{3(\sqrt{n(2n + 2)^{\frac{3}{2}} + \sqrt{n(2n + 1)^{\frac{3}{2}}}})} = \frac{1}{\sqrt{2}}.
\]

Dividing the numerator and denominator by \( n^2 \) and taking the limit yields the result \( 1/\sqrt{2} \). Similarly, for the expression on the right in (1), we have:

\[
\lim_{n \to \infty} \frac{(2n)^{\frac{2}{3}} - (2n - 1)^{\frac{2}{3}} - (2^{\frac{3}{2}} - 1) + 3(\sqrt{2} - 1)}{3\sqrt{n}} = \lim_{n \to \infty} \frac{12n^2 - 6n + 1}{3(\sqrt{8n^3} + \sqrt{n(2n - 1)^{\frac{3}{2}}})} = \frac{1}{\sqrt{2}}.
\]

Since the limits of the two outer expressions in (1) are equal, we have by the Squeeze Theorem

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{n}} = \frac{1}{\sqrt{2}}. \]

_Editor’s comments._ The solutions received used either Cesàro-Stolz Lemma or Squeeze Theorem to calculate the limit. The authors used various inequalities or
approximations to be able to apply these theorems. Many solutions were similar and we chose to feature the representative ones.

4443. Proposed by Andrew Wu.

Acute scalene \( \triangle ABC \) has circumcircle \( \Omega \) and altitudes \( \overline{BE} \) and \( \overline{CF} \). Point \( N \) is the midpoint of \( \overline{EF} \) and line \( \overline{AN} \) meets \( \Omega \) again at \( Z \). Let lines \( \overline{ZF} \) and \( \overline{ZE} \) meet \( \Omega \) again at \( V \) and \( U \), respectively, and let lines \( \overline{CV} \) and \( \overline{BU} \) meet at \( P \). Prove that \( \overline{UV} \) and \( \overline{BC} \) meet on the tangent from \( P \) to the circumcircle of \( \triangle APN \).

We received 3 solutions. We present the solution by Andrea Fanchini.

Use barycentric coordinates with reference to \( \triangle ABC \). We have \( A(1 : 0 : 0) \), \( B(0 : 1 : 0) \) and \( C(0 : 0 : 1) \). Denote the side lengths of the triangle by \( a, b \) and \( c \). We use Conway’s notation, in particular \( S \) for twice the area of \( \triangle ABC \), and the shorthand \( S_A, S_B \) and \( S_C \), where \( S_\alpha = S \cot(\alpha) \) for an angle \( \alpha \).

The equation of the circumcircle \( \Omega \) is \( a^2yz + b^2zx + c^2xy = 0 \). As the feet of the altitudes from \( B \) and \( C \) respectively, \( E \) and \( F \) have coordinates \( E(S_C : 0 : S_A) \) and \( F(S_B : S_A : 0) \).

Point \( N(c^2S_C + b^2S_B : b^2S_A : c^2S_A) \) is the midpoint of \( EF \). The line through \( A \) and \( N \) has equation \( c^2y - b^2z = 0 \), and meets \( \Omega \) again at \( Z(-a^2 : 2b^2 : 2c^2) \).

The line through \( Z \) and \( F \) has equation
\[
2b^2S_Ax - 2c^2S_By + (a^2S_A + 2b^2S_B)z = 0
\]
and meets \( \Omega \) again at \( V(2S_B(a^2S_A + 2b^2S_B) : S_A(a^2S_A + 2b^2S_B) : -2c^2S_AS_B) \).

Similarly, the line through \( Z \) and \( E \) has equation
\[
2b^2S_Ax + (a^2S_A + 2c^2S_C)y - 2b^2S_Cz = 0
\]

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and meets $\Omega$ again at $U(2S_C(a^2S_A + 2c^2S_C)) : -2b^2S_AS_C : S_A(a^2S_A + 2c^2S_C)$.
The lines $CV : S_Ax - 2SBy = 0$ and $BU : S_Ax - 2SCz = 0$ meet at the point $P(2SBSC : S_AS_C : S_A$ $S_B$).
The circumcircle of $\triangle APN$ thus has equation
$$a^2yz + b^2zx + c^2xy - \frac{4S^2 - a^2S_A}{4S^2 - 2a^2S_A}(SBy + SCz)(x + y + z) = 0.$$
The tangent to this circumcircle at $P$ has equation
$$t : -2S_AS^2x + S_B(a^2S_A + 2c^2S_C)y + SC(a^2S_A + 2b^2S_B)z = 0.$$
The lines $UV : -a^2S_A^2x + 2SB(a^2S_A + 2c^2S_C)y + SC(a^2S_A + 2b^2S_B)z = 0$ and $BC : x = 0$ meet at the point $X(0 : SC(a^2S_A + 2b^2S_B) : -S_B(a^2S_A + 2c^2S_C))$, which we can easily verify is on the tangent $t$, concluding the proof.

4444. Proposed by Michel Bataille.
Let $n$ be a positive integer. Evaluate in closed form
$$\sum_{k=0}^{n-1} \left( \tan^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{4} \right) + \cot^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{4} \right) \right).$$
We received 9 correct solutions. We present the solution by Angel Plaza.

Since
$$\tan^2 x + \cot^2 x = 4 \cot^2(2x) + 2 = 4 \tan^2(\pi/2 - 2x) + 2,$$
the sum can be written as
$$4 \sum_{k=0}^{n-1} \cot^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{2} \right) + 2n = 4 \sum_{k=1}^{n} \tan^2 \left( \frac{2k}{2n+1} \cdot \frac{\pi}{2} \right) + 2n.$$
We will prove that
$$\sum_{k=0}^{n} \tan^2(2k\pi/2(2n+1)) = n(2n+1).$$
Observe that, for $1 \leq k \leq n$,
$$1 = (-1)^{2k} = \left( \cos \left( \frac{2k}{2n+1} \cdot \frac{\pi}{2} \right) + i \sin \left( \frac{2k}{2n+1} \cdot \frac{\pi}{2} \right) \right)^{2n+1}$$
$$= \sum_{j=0}^{2n+1} \binom{2n+1}{j} \left( \cos \left( \frac{2k}{2n+1} \cdot \frac{\pi}{2} \right) \right)^j \left( i \sin \left( \frac{2k}{2n+1} \cdot \frac{\pi}{2} \right) \right)^{2n+1-j}.$$
Taking the imaginary parts of both sides and dividing by
\[ \cos^{2n+1}\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right) \cdot \tan\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right) \]
yields
\[ 0 = \sum_{j=0}^{n} \left(\frac{2n+1}{2j}\right) \left(i\tan\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right)\right)^{2n-2j}. \]

Thus, \( \tan^2\left(\frac{2k}{(2n+1)}\right) \) with \( 1 \leq k \leq n \) are the zeros of the polynomial
\[ \sum_{j=0}^{n} \left(\frac{2n+1}{2j}\right) (-z)^{n-j}. \]
The sum of these zeros is
\[ \frac{\binom{2n+1}{2}}{\binom{2n+1}{0}} = n(2n+1). \]

Therefore,
\[ \sum_{k=1}^{n} \tan^2\left(\frac{2k}{2n+1}\cdot\frac{\pi}{2}\right) = n(2n+1) \]
and the proposed sum is
\[ 4n(2n+1) + 2n = 2n(4n+3). \]

Editor’s comments. Brian Bradie and Oliver Geupel, independently, rendered the sum as
\[ \sum_{k=0}^{2n} \tan^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{4}\right) - 1 = \sum_{k=1}^{4n+1} \tan^2\left(\frac{k\pi}{8n+4}\right) - \sum_{k=1}^{2n} \tan^2\left(\frac{k\pi}{4n+2}\right) - 1 = \frac{(8n+3)(4n+1)}{3} - \frac{2n(4n+1)}{3} - 1 = (2n+1)(4n+1) - 1 = 8n^2 + 6n. \]

Eight of the solutions relied on the value of a trigonometric sum of the type in the foregoing solutions. Three of the solvers justified it by a polynomial argument like Plaza’s, and one used a partial fractions representation. Two people gave a reference to the literature, while the remaining two regarded it as “well-known” and “classical”. The ninth solution appealed to the series development
\[ \frac{\pi^2}{\sin^2(\pi x)} = \sum_{m=-\infty}^{+\infty} \frac{1}{(m+x)^2}, \]
and through some delicate bookkeeping got the result from the sum of the odd square reciprocals.

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4445. Proposed by Leonard Giugiuc and Dan Stefan Marinescu.

Let $ABC$ be a triangle with $AC > BC > AB$, incenter $I$ and centroid $G$.

1. Prove that point $A$ lies in one half-plane of the line $GI$, while points $B$ and $C$ lie in the other half-plane.

2. The line $GI$ intersects the sides $AB$ and $AC$ at $M$ and $N$, respectively. Prove that $BM = CN$ if and only if $\angle BAC = 60^\circ$.

We received 5 submissions, all correct. Four of the solutions relied on coordinates; we have chosen one example of such a solution to feature together with the solution that avoided coordinates.

Solution 1, by Marie-Nicole Gras.

For triangle $ABC$ we denote by $a$, $b$, $c$ the lengths of the sides $BC$, $CA$ and $AB$; by assumption, $b > a > c$. We put $\theta = \frac{\angle BAC}{2}$.

The bisectors at vertex $A$ define an orthonormal system with origin $A$; the cartesian coordinates of $A$, $B$, $C$ are

$$A(0,0), \ B(c \cos \theta, c \sin \theta), \ C(b \cos \theta, -b \sin \theta);$$

using well-known formulas, we get those of $G$ and $I$:

$$G\left(\frac{b+c}{3} \cos \theta, \frac{c-b}{3} \sin \theta\right), \ I\left(\frac{2bc}{a+b+c} \cos \theta, 0\right).$$

The point $M$ is the intersection of the lines $GI$ and the line $AB$; the equation of the line $GI$ is

$$\frac{c-b}{3} x \sin \theta - \left(\frac{b+c}{3} - \frac{2bc}{a+b+c}\right) y \cos \theta = \frac{2bc}{a+b+c} \left(\frac{c-b}{3} \sin \theta \cos \theta;\right)$$
since the equation of \( AB \) is \( y = x \tan \theta \), we obtain

\[
\left( \frac{c-b}{3} - \frac{b+c}{3} + \frac{2bc}{a+b+c} \right) x \sin \theta = \frac{2bc}{a+b+c} c - \frac{b}{3} \sin \theta \cos \theta;
\]

and we arrive at the coordinates of \( M \):

\[
M \left( \frac{c(b-c)}{a+b-2c} \cos \theta, \frac{c(b-c)}{a+b-2c} \sin \theta \right).
\]

Exchanging \( b \) and \( c \), and the sign of \( \theta \), we obtain

\[
N \left( \frac{b(c-b)}{a+c-2b} \cos \theta, \frac{b(c-b)}{a+c-2b} \sin(-\theta) \right).
\]

Since \( b > a > c \) we have

\[
\frac{c(b-c)}{a+b-2c} = \frac{b(c-b)}{(a-c) + (b-c)} > 0, \quad \frac{b(c-b)}{a+c-2b} = \frac{b(b-c)}{(b-a) + (b-c)} > 0,
\]

and then

\[
AM = \frac{c(b-c)}{a+b-2c}, \quad AN = \frac{b(b-c)}{2b-a-c}.
\]

1. Because

\[
MB = AB - AM = c - \frac{c(b-c)}{a+b-2c} = \frac{c(a-c)}{a+b-2c} > 0,
\]

we deduce that \( M \) is between \( A \) and \( B \), and

\[
BM = \frac{c(a-c)}{a+b-2c}.
\]

Exchanging \( b \) and \( c \), we get that \( N \) is between \( A \) and \( C \), and

\[
CN = \frac{b(a-b)}{a+c-2b} = \frac{b(b-a)}{2b-a-c}.
\]

As points \( G \) and \( I \) are inside \( ABC \), they belong to the line segment \( MN \), and then point \( A \) lies in one half-plane of the line \( GI \), while points \( B \) and \( C \) lie in the other half-plane. We remark that \( A \) is the vertex of \( ABC \) such that \( \angle BAC \) is between \( \angle ABC \) and \( \angle BCA \).

2. Finally, we have

\[
BM = CN \iff \frac{c(a-c)}{a+b-2c} = \frac{b(b-a)}{2b-a-c} \iff \frac{c(a-c)(b-a+b-c)}{2b-a-c} = b(b-a)(a-c+b-c) \iff (b-c)(ac-c^2-b^2+ab) = (b-c)(ab-bc-a^2+ac) \iff (b-c)(b^2+c^2-a^2-bc) = 0.
\]

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By assumption, \( b \neq c \); then \( BM = CN \) is equivalent to
\[
a^2 = b^2 + c^2 - bc \iff \cos(\angle BAC) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2} \iff \angle BAC = 60^\circ.
\]

Remark. If \( \angle BAC = 60^\circ \) and \( b \neq c \), then \( a^2 - c^2 = b(b - c) \neq 0 \) and
\[
CN = BM = \frac{c(a - c)(a + c)}{((a - c) + (b - c))(a + c)} = \frac{cb(b - c)}{b(b - c) + (b - c)(a + c)} = \frac{bc}{a + b + c}.
\]

Let \( r \) be the inradius of \( \triangle ABC \); then the area of \( \triangle ABC \) is
\[
\frac{a + b + c}{2} r = \frac{1}{2} bc \sin(\angle BAC) = \frac{\sqrt{3}}{4} bc;
\]
we find, finally, that
\[
BM = CN = \frac{2r\sqrt{3}}{3} = \frac{2}{3} r \tan 60^\circ.
\]

**Solution 2, by the proposers.**

The argument is based on a two-part lemma.

**Lemma.** Let \( M \) be a point on side \( AB \) of \( \triangle ABC \) and \( N \) a point on side \( AC \); then

a) the line \( MN \) passes through the centroid \( G \) if and only if \( \frac{BM}{MA} + \frac{CN}{NA} = 1 \), and

b) the line \( MN \) passes through the incenter \( I \) if and only if \( b \cdot \frac{BM}{MA} + c \cdot \frac{CN}{NA} = a \).

**Editor’s comment:** Both parts follow immediately from a theorem of Nicolae Mihaioreanu that deserves to be better known outside Romania. Because its proof would make a nice challenge for *Crux* readers, instead of including it here, we have turned it into one of this month’s problem proposals, problem number 4495.

Back to the problem. We assume that \( M,G,I, \) and \( N \) are collinear and set \( x = \frac{BM}{MA}, y = \frac{CN}{NA} \). Our lemma tells us that \( x \) and \( y \) satisfy the simultaneous equations,
\[
\begin{align*}
x + y &= 1 \\
bx + cy &= a,
\end{align*}
\]
which, because \( b > a > c \), gives us
\[
x = \frac{a - c}{b - c} > 0 \quad \text{and} \quad y = \frac{b - a}{b - c} > 0.
\]

Because there is a unique point, namely \( M \), that divides segment \( BA \) in the ratio \( a - c : b - c \), and a unique point, namely \( N \), that divides segment \( CA \) in the ratio...
$b-a : b-c$, our lemma tells us that with this choice of the points $M$ and $N$ we have both $\frac{BM}{MA} + \frac{CN}{NA} = 1$, so that $MN$ passes through $G$, and $b \cdot \frac{BM}{MA} + c \cdot \frac{CN}{NA} = a$, so that $MN$ passes through $I$. We conclude that the line $GI$ separates $A$ from both $B$ and $C$. Furthermore, because

\[
BM = \frac{a-c}{b-c},
\]

we have

\[
BM = \frac{a-c}{(a-c)+(b-c)},
\]

whence

\[
BM = \frac{c(a-c)}{a+b-2c}.
\]

Similarly,

\[
CN = \frac{b(b-a)}{2b-a-c}.
\]

The argument concludes as in part 2 of the first solution.

**Editor's comments.** For a compilation of many other properties of triangles with an angle of 60°, see Chris Fisher's article “Recurring Crux Configurations 3: Triangles Whose Angles Satisfy $2B = C + A$” [37:7 (November 2011) pages 449-453]. Problem 4445 is evidently the first *Crux* problem involving the Nagel line $GI$ of such a triangle.

**4446. Proposed by Florin Stanescu.**

Let $n$ be a prime number greater than 4 and let $A \in M_{n-1}(Q)$ be such that $A^n = I_{n-1}$. Evaluate $\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})$ in terms of $n$.

We received 6 submissions, 5 of which were correct and complete solutions. We present the solution by Brian Bradie, slightly edited.

The condition $A^n = I_{n-1}$ implies

\[
(A - I_{n-1})(A^{n-1} + A^{n-2} + A^{n-3} + \cdots + A + I_{n-1}) = 0.
\]

Because $n$ is prime, the polynomial

\[
x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1
\]

is irreducible over $Q$. Thus, either $A = I_{n-1}$ or $A$ is similar to the $(n-1) \times (n-1)$ matrix

\[
C_{n-1} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & -1 \\
1 & 0 & 0 & \cdots & 0 & -1 \\
0 & 1 & 0 & \cdots & 0 & -1 \\
0 & 0 & 1 & \cdots & 0 & -1 \\
\vdots & & & & & \\
0 & 0 & 0 & \cdots & 1 & -1
\end{bmatrix},
\]

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which is the companion matrix associated with the characteristic polynomial

\[ x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1. \]

We now consider two cases.

**Case 1:** If \( A = I_{n-1} \), then

\[
\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1}) = \det \left( \frac{(n-1)I_{n-1}}{2} \right) = \left( \frac{n-1}{2} \right)^{n-1}.
\]

**Case 2:** If \( A \) is similar to \( C_{n-1} \), then

\[
\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1}) = \det(C_{n-1}^{n-2} + 2C_{n-1}^{n-3} + 3C_{n-1}^{n-4} + \cdots + (n-2)C_{n-1} + (n-1)I_{n-1}).
\]

Now note that the companion matrix has the property that

\[ C_{n-1}^k = (v_{k+1}|v_{k+2}| \cdots |v_{k+n-1}), \]

where \( v_i = v_{n+i} = e_i, (i = 1, \ldots, n-1) \) is the \( i \)th canonical base vector and \( v_n = (-1, -1, \ldots, -1)^T \).

We obtain

\[
C_{n-1}^{n-2} + 2C_{n-1}^{n-3} + 3C_{n-1}^{n-4} + \cdots + (n-2)C_{n-1} + (n-1)I_{n-1} =
\begin{bmatrix}
  n-1 & -1 & -1 & \cdots & -1 & -1 \\
  n-2 & n-2 & -2 & \cdots & -2 & -2 \\
  n-3 & n-3 & n-3 & \cdots & -3 & -3 \\
  \vdots & & & & & \\
  2 & 2 & 2 & \cdots & 2 & 2-n \\
  1 & 1 & 1 & \cdots & 1 & 1
\end{bmatrix}.
\]

Adding \( j \) times row \( n-1 \) to row \( j \) for \( j = 1, 2, 3, \ldots, n-2 \) yields

\[
\begin{bmatrix}
  n & 0 & \cdots & 0 & 0 \\
  n & n & \cdots & 0 & 0 \\
  n & n & \cdots & 0 & 0 \\
  \vdots & & & & \\
  n & n & \cdots & n & 0 \\
  1 & 1 & 1 & \cdots & 1 & 1
\end{bmatrix}.
\]

Thus,

\[
\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1}) = n^{n-2}.
\]
4447. Proposed by Lorian Saceanu.

Let $ABC$ be a scalene triangle. Prove that
\[
2 + \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} \geq \sin^2 A + \sin^2 B + \sin^2 C.
\]

We received 19 submissions, all of which are correct. Most of the solutions used several known identities and are very similar to one another. We present the proof by Ioan Viorel Codreanu.

Let $r, R,$ and $s$ denote the inradius, circumradius, and semiperimeter of $\Delta ABC$, respectively. The following identities are all well known:
\[
\prod_{\text{cyc}} \sin A = \frac{sr}{2R^2},
\]
\[
\sum_{\text{cyc}} \sin A = \frac{s}{R},
\]
and
\[
\sum_{\text{cyc}} \sin^2 A = \frac{s^2 - r(4R + r)}{2R^2}.
\]

Using (1), the proposed inequality is equivalent to
\[
2 + \frac{r}{2R} \geq \frac{s^2 - r(4R + r)}{2R^2}, \quad \text{or}
\]
\[
s^2 \leq 4R^2 + 5Rr + r^2, \quad \text{or}
\]
\[
s^2 \leq (4R^2 + 4Rr + 3r^2) + r(R - 2r),
\]
which is true by Gerretsen’s Inequality [Editor’s comment: See for reference, for example, Item 5.8 on p. 50 of Geometric Inequalities by O. Bottema et al], and $R \geq 2r$ by Euler’s Inequality.


Let $a, b, c$ and $d$ be non-zero complex numbers such that $|a| = |b| = |c| = |d|$ and Arg$(a) < \text{Arg}(b) < \text{Arg}(c) < \text{Arg}(d)$. Prove that
\[
|(a - b)(c - d)| = |(a - d)(b - c)| \iff (a - b)(c - d) = (a - d)(b - c).
\]

There were 6 correct solutions. Three took the approach of Solution 1 and the other three of Solution 2.

Solution 1, by Oliver Geupel.

The reverse implication is clear. Suppose that $|(a - b)(c - d)| = |(a - d)(b - c)|$.
Then for some $r$,
\[
\frac{|a-b|}{|c-b|} = r = \frac{|a-d|}{|c-d|},
\]
so that
\[
\frac{a-b}{c-b} = re^{i\phi} \quad \text{and} \quad \frac{a-d}{c-d} = re^{i\psi}
\]
for some angles $\phi$ and $\psi$; note that these angles have opposite directions. Since the points $a, b, c, d$ in the complex plane form a concyclic quadrilateral $\phi + (-\psi) = \pi$. Therefore
\[
\frac{a-d}{c-d} = re^{i\psi} = re^{i(\phi-\pi)} = -re^{i\phi} = -\frac{a-b}{c-b},
\]
whence $(a-b)(c-d) = (a-d)(b-c)$.

**Solution 2, by Florentin Visescu.**

Let $a = r(cos m + i sin m)$, $b = r(cos n + i sin n)$, $c = r(cos p + i sin p)$, $d = r(cos q + i sin q)$. Then, after standard trigonometric manipulations, we find that
\[
(a-b)(c-d) = -4r^2 \sin \frac{n-m}{2} \sin \frac{q-p}{2} (cos s + i sin s)
\]
and
\[
(a-d)(b-c) = -4r^2 \sin \frac{q-m}{2} \sin \frac{p-n}{2} (cos s + i sin s),
\]
where $2s = m + n + p + q$. From this, the desired result follows.

4449. Proposed by Arsalan Wares.

The figure shows two congruent overlapping squares inside a larger square. The vertices of the overlapping smaller squares divide each of the four sides of the largest square into three equal parts. If the area of the shaded region is 50, find the area of the largest square.

We received 18 submissions of which 17 were correct and complete. We present the solution by Brian Beastley.
We model the diagram by placing the vertices of the largest square at \((0,0), (0,3), (3,3),\) and \((3,0)\). Then the shaded region (which we denote by \(R\)) is inside the smaller square with vertices at \((0,1), (1,3), (3,2),\) and \((2,0)\), yielding an area of 5 for the smaller square. Since the smaller square may be partitioned into \(R\) and four remaining congruent triangles, we calculate the area of \(R\) by subtracting the areas of the four congruent triangles from 5. By solving for the intersection of lines, it is easy to obtain the coordinates of \(A\) and \(B\) as \((2/3, 2/3)\) and \((1/4, 3/2)\) respectively. Thus the right-triangle formed by \((0,1), A,\) and \(B\) has area
\[
\frac{1}{2} \cdot \frac{\sqrt{5}}{4} \cdot \frac{\sqrt{5}}{3} = \frac{5}{24}.
\]
Hence in our model, the area of \(R\) is \(5 - 5/6 = 25/6\), so in general the ratio of the areas of \(R\) and the largest square is \(25/6\). Thus we conclude that an area of 50 for \(R\) must correspond to an area of 108 for the largest square.


Let \(n \geq 3\) be an integer and consider positive real numbers \(a_1, a_1, \ldots, a_n\) such that \(a_n \geq a_1 + a_2 + \cdots + a_{n-1}\). Prove that
\[
(a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq 2((n-1)^2 + 1).
\]

We received 18 solutions, all correct. We present the proof by Angel Plaza, modified slightly by the editor.

Let \(S_{n-1} = \sum_{k=1}^{n-1} a_k\) and \(T_{n-1} = \sum_{k=1}^{n-1} \frac{1}{a_k}\). Then the proposed inequality reads as
\[
(S_{n-1} + a_n) \left( T_{n-1} + \frac{1}{a_n} \right) \geq 2((n-1)^2 + 1). \tag{1}
\]
Now, by the AM-HM inequality, we have

\[ S_{n-1}T_{n-1} = \left( \sum_{k=1}^{n-1} a_k \right) \left( \sum_{k=1}^{n-1} \frac{1}{a_k} \right) \geq (n-1)^2, \]

so

\[ (S_{n-1} + a_n) \left( T_{n-1} + \frac{1}{a_n} \right) \geq (n-1)^2 + 1 + a_nT_{n-1} + \frac{S_{n-1}}{a_n}. \]  \hspace{1cm} (2)

Since \( a_n \geq S_{n-1} \), we have \( a_n = S_{n-1} + d \) for some \( d \geq 0 \). Then

\[
a_nT_{n-1} + \frac{S_{n-1}}{a_n} = (S_{n-1} + d)T_{n-1} + \frac{S_{n-1}}{S_{n-1} + d}
\]

\[
\geq (n-1)^2 + 1 + dT_{n-1} + \left( \frac{S_{n-1}}{S_{n-1} + d} - 1 \right)
\]

\[
= (n-1)^2 + 1 + \frac{dT_{n-1}S_{n-1} + d^2T_{n-1} - d}{S_{n-1} + d}
\]

\[
\geq (n-1)^2 + 1 + \frac{d(n-1)^2 + d^2T_{n-1} - d}{S_{n-1} + d} \geq (n-1)^2 + 1. \hspace{1cm} (3)
\]

(Therefore \( d(n-1)^2 - d = d(n^2 - 2n) \geq 0 \).)

Finally, (1) follows by substituting (3) into (2).

Editor’s comment. Note that equality holds if and only if \( a_1 = a_2 = \cdots = a_{n-1} = c \) and \( a_n = (n-1)c \) for some \( c > 0 \).