OLYMPIAD CORNER

No. 378

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by February 15, 2020.

OC456. Solve the system of equations

\[(x^2 + 1)(x - 1)^2 = 2017yz\]
\[(y^2 + 1)(y - 1)^2 = 2017zx\]
\[(z^2 + 1)(z - 1)^2 = 2017xy,\]

where \(x \geq 1, y \geq 1, z \geq 1\).

OC457. On a blackboard are written the numbers 1!, 2!, 3!, \ldots, 2017!. What is the smallest among these numbers that should be deleted so that the product of all the remaining numbers is a perfect square?

OC458. Let \(A\) be the product of eight consecutive positive integers and let \(k\) be the largest positive integer for which \(k^4 \leq A\). Find the number \(k\) knowing that it is represented in the form \(2^p m\), where \(p\) is a prime number and \(m\) is a positive integer.

OC459. Points \(P\) and \(Q\) lie respectively on sides \(AB\) and \(AC\) of a triangle \(ABC\) such that \(BP = CQ\). Segments \(BQ\) and \(CP\) intersect at \(R\). The circumcircles of triangles \(BPR\) and \(CQR\) intersect again at point \(S\) different from \(R\). Prove that point \(S\) lies on the angle bisector \(\angle BAC\).

OC460. Prove that the set of positive integers \(\mathbb{Z}^+\) can be represented as a union of five pairwise disjoint subsets with the following property: each 5-tuple of numbers of the form \((n, 2n, 3n, 4n, 5n)\), where \(n \in \mathbb{Z}^+\), contains exactly one number from each of these five subsets.

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d’une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l’examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 février 2020.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l’Université de Saint-Boniface, d’avoir traduit les problèmes.

**OC456.** Résoudre le système d’équations

\[
\begin{align*}
(x^2 + 1)(x - 1)^2 &= 2017yz \\
(y^2 + 1)(y - 1)^2 &= 2017zx \\
(z^2 + 1)(z - 1)^2 &= 2017xy,
\end{align*}
\]

où \(x \geq 1, y \geq 1, z \geq 1\).

**OC457.** Soient les nombres 1!, 2!, 3!, …, 2017!. Lequel parmi ces nombres pourrait être supprimé, de façon à ce que le produit des nombres restants soit un carré parfait ?

**OC458.** Soit \(A\) le produit de huit entiers consécutifs et soit \(k\) le plus gros entier positif tel que \(k^4 \leq A\). Déterminer le nombre \(k\), prenant pour acquis qu’il est de la forme \(2^p m\), où \(p\) est un nombre premier et \(m\) est un entier positif.

**OC459.** Les points \(P\) et \(Q\) se situent, respectivement, sur les côtés \(AB\) et \(AC\) d’un triangle \(ABC\), de façon à ce que \(BP = CQ\). Les segments \(BQ\) et \(CP\) intersectent en \(R\). Enfin, les cercles circonscrits des triangles \(BPR\) et \(CQR\) intersectent de nouveau au point \(S\), distinct de \(R\). Démontrer que le point \(S\) se trouve sur la bissectrice de l’angle \(\angle BAC\).

**OC460.** Démontrer que l’ensemble des entiers positifs \(\mathbb{Z}^+\) peut être partitionné comme réunion de cinq ensembles disjoints avec la propriété suivante : chaque 5-tuple de nombres de la forme \((n, 2n, 3n, 4n, 5n)\), avec \(n \in \mathbb{Z}^+\), contient exactement un nombre de chacun des cinq sous ensembles.

OC431. All natural numbers greater than 1 are coloured with blue or red so that the sum of every two blue numbers (not necessarily distinct) is blue, and the product of every two red ones (not necessarily distinct) is red. It is known that the number 1024 is blue. What colour can the number 2017 be?

Originally from Moscow Math Olympiad, Problem 2, Grade 10, Final Round 2017.
We received 7 submissions. We present 2 solutions.
Solution 1, by Kathleen E. Lewis, modified by the editor.
The number 2017 can be red. We know that 1024 is blue, so 2 must be blue, because if 2 were red, every power of 2 would have to be red. Since 2 is blue, and the sum of two blues is blue, then every even number must be blue. But we cannot determine the colours of the odds. Since two colours are used, then one odd number is red, say \( a \). Assume that \( a \geq 5 \). Since \( a - (a - 2) = 2 \), then also \( a - 2 \) must be red and applying the same reasoning to \( a - 2 \) and so on, we get that all the odd numbers less than \( a \) are red. Now, all the powers \( a^k \) with \( k = 1, 2, \ldots \) are red and there is a power \( a^k > 2017 \) which is coloured with red. Using the same reasoning, we conclude that all the odd numbers less than \( a^k \) are red, therefore 2017 is red.

Solution 2, by the Missouri State University Problem Solving Group.
Assuming that both colours must be used (otherwise, every number could be coloured blue), we will show that 2017 must be red.
We will prove, more generally, that (assuming both colours must be used) the only valid colourings are ones in which all the multiples of a given prime number are blue and the rest of the numbers are red. We need a lemma.

Lemma. If \( \gcd(a, b) = 1 \) and \( a \) is blue, then \( b \) is red.
Proof. Suppose \( b \) were blue. Then every number of the form \( ka + \ell b \) with integers \( k, \ell \geq 1 \) must be blue. It is well known that since \( \gcd(a, b) = 1 \), every integer greater than or equal to \( N = (a - 1)(b - 1) \) is of this form. Therefore every integer greater than or equal to \( N \) must be blue. Now there must be some number \( c \) that is red (since we are assuming both colours are used) so \( c^m \) is red for all positive \( m \). But for \( m \) sufficiently large, \( c^m > N \) and so must be blue. This gives a contradiction, so \( b \) must be red.
Suppose the number \( n \) is blue. Then at least one of the primes dividing it must be blue (if all of the primes dividing \( n \) were red, \( n \) would be red). Denote this prime

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by $p$. All the multiples of $p$ must be blue and any non-multiple is relatively prime to $p$ so must be red by the lemma.

In the case at hand, since 1024 is blue, 2 must be blue, so the even numbers are blue and the odd numbers are red hence 2017 is red.

Editor's Comment. The original problem included the hypothesis that it is known that when colouring the numbers both colours were used. Unfortunately, this was missed in the translation by the Editor. We apologize for that.

**OC432.** Find the smallest natural number that is a multiple of 80 such that you can rearrange two of its distinct digits and the resulting number will also be a multiple of 80.

*Originally from Moscow Math Olympiad, Problem 1, Grade 11, Final Round.*

We received 3 submissions, of which 1 was correct and complete. We present the solution by Oliver Geupel.

A number with the desired property is 1520 because it is a multiple of 80, as well as the number 5120. We prove that 1520 is the smallest number with the required property.

The multiples of 80 that are less than 1000 are 80, 160, 240, 320, 400, 480, 560, 640, 720, 800, 880, and 960. By inspection, they all do not have the property. Hence, the least number with the desired property is a four digit number 1bc0.

We cannot rearrange the leading 1 with the digit $c$, because a number that ends in 10 is not divisible by 80. If we can rearrange the leading 1 with the digit $b$, then the number $b1c0 - 1bc0$ is a multiple of 80. Hence, $80 | 900(b - 1)$, that is, $4 | b - 1$, so that $b \geq 5$. By inspection, $(b, c) = (5, 0)$ fails, and $(b, c) = (5, 2)$ yields the solution 1520.

Finally, if we can rearrange the digits $b$ and $c$, then $b$ and $c$ are even digits such that $b < c$, and the number $1bc0 - 1bc0$ is a multiple of 80, that is $80 | 90(c - b)$. Hence, $c - b$ is divisible by 8, which leads to $b = 0$ and $c = 8$. But this is impossible, because the number 1080 is not divisible by 80.

The proof is complete.

**OC433.** Consider an isosceles trapezoid $ABCD$ with bases $AD$ and $BC$. A circle $\omega$ passing through $B$ and $C$ intersects the side $AB$ and the diagonal $BD$ at points $X$ and $Y$, respectively. The tangent to $\omega$ at $C$ intersects the line $AD$ at $Z$.

Prove that the points $X$, $Y$, and $Z$ are collinear.

*Originally from Moscow Math Olympiad, Problem 2, Grade 9, Final Round 2017.*

We received 4 submissions. We present the solution by Oliver Geupel.

The line $XY$ intersects the line $AD$ at a point $Z'$ (see figure on the next page). It is enough to show that $Z = Z'$. 

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We have
\[ \angle CYZ' = 180^\circ - \angle XYC \quad \text{because } X, Y, Z' \text{ are collinear} \]
\[ = \angle CBX \quad \text{because } B, C, Y, X \text{ are concyclic} \]
\[ = 180^\circ - \angle ADC \quad \text{because } ABCD \text{ is an isosceles trapezoid} \]
\[ = \angle CDZ' \quad \text{because } A, D, Z' \text{ are collinear}. \]

Hence, the points \( C, Z', D, \) and \( Y \) are concyclic.

The circle \( \omega \) intersects the side \( CD \) at a point \( X' \). It holds
\[ \angle Z'CD = \angle Z'YD \quad \text{because } C, Z', D, Y \text{ are concyclic} \]
\[ = \angle XYB \quad \text{because } Y \text{ is intersection of } BD \text{ and } XZ' \]
\[ = \angle CYX' \quad \text{because arcs } XB \text{ and } CX' \text{ have equal lengths} \]
\[ = \angle ZCD \quad \text{angle between tangent and chord}. \]

Therefore, \( Z = Z' \).

**OC434.** The acute isosceles triangle \( ABC \) \((AB = AC)\) is inscribed in a circle with center \( O \). The rays \( BO \) and \( CO \) intersect the sides \( AC \) and \( AB \) at the points \( B' \) and \( C' \), respectively. A line \( l \) parallel to the line \( AC \) passes through point \( C' \). Prove that the line \( l \) is tangent to the circumcircle \( \omega \) of the triangle \( B'OC \).

*Originally from Moscow Math Olympiad, Problem 2, Grade 10, Final Round 2017.*

We received 4 submissions. We present 2 solutions.

**Solution 1,** by Oliver Geupel.

Let \( A' \) denote the point of intersection of the lines \( l \) and \( AO \). Let \( \varphi \) denote the size of the angle \( \angle OAC \).
Since the triangle $AOC$ is isosceles, we have $\angle ACO = \varphi$. Also, $\angle OA'C' = \angle A'C'O = \varphi$, because the triangle $A'O'C'$ is homothetic to the triangle $AOC$. Moreover, $\angle B'A'O = \angle OB'A' = \varphi$, since the triangle $A'B'O$ is the reflection of the triangle $A'C'O$ in the line $AO$.

Since $\angle B'A'O$ has the same size as the inscribed angle $\angle B'CO$ in $\omega$, it follows that $A'$ lies on $\omega$.

The angle between the chord $A'O$ and the line $l$ has the size $\varphi$, which is identical to the size of the inscribed angle $\angle OB'A'$ and, thus, to the size of the angle between the chord $A'O$ and the tangent to $\omega$ in $A'$.

We conclude that $l$ is the tangent to $\omega$ in $A'$.

Solution 2, by Ivko Dimitrić.

Let $A = \angle BAC$. The line $q = \overrightarrow{AO}$ is the axis of symmetry of $\triangle ABC$. Let $D$ be the intersection point of $l$ and $q$. Because of the axial symmetry of the triangle $ABC$ about $q$ in which the lines $AB, AC$ and $BB', CC'$ each correspond to the other in the pair, it follows that $B'D \parallel AB$ just as $C'D \parallel AC$. That implies

$$\angle B'DO = \angle B'DA = \angle DAB = \frac{A}{2}$$

and $\angle DB'C = \angle BAC = A$. Since $O$ is the circumcenter of the triangle $ABC$ then

$$\angle DOC = \frac{1}{2} \angle BOC = \angle BAC = A.$$ 

Because $\angle DB'C = \angle DOC = A$, we conclude that the quadrilateral $DCB'O$ is cyclic and its circumcircle is $\omega$, implying

$$\angle B'CO = \angle B'DO = \angle OAB = \frac{A}{2}.$$
By symmetry, \( \angle OBC' = A/2 \) and since \( B'D \parallel AB \) it follows

\[
\angle OCD = \angle O'B'D = \angle OBC' = \frac{A}{2}.
\]

Thus,

\[
\angle B'CD = \angle B'CO + \angle OCD = \frac{A}{2} + \frac{A}{2} = A.
\]

Hence, since \( \angle B'CD = \angle DB'C \), the triangle \( DB'C \) is isosceles with base \( B'C \).

The common circumcenter \( S \) of \( \triangle B'O'C \) and \( \triangle B'DC \) belongs to the perpendicular bisector \( s \) of \( B'C \) that is also perpendicular to \( l \parallel B'C \) and passes through the vertex \( D \). Since the circumcircle \( \omega \) of \( \triangle B'O'C \) passes through \( D \) and the line \( l \) is perpendicular to the radius \( SD \) of \( \omega \) at \( D \), it follows that \( l \) is tangent to the circumcircle of \( \triangle B'O'C \) at point \( D \).

**OC435.** There are \( n \) positive numbers \( a_1, a_2, \ldots, a_n \) written on a blackboard. Under each number \( a_i \), Vasya wants to write a number \( b_i \geq a_i \) so that for every pair of numbers chosen from \( b_1, b_2, \ldots, b_n \), the ratio of one of them to the other is an integer. Prove that Vasya can write out the required numbers so that

\[
b_1b_2 \cdot \ldots \cdot b_n \leq 2^{(n-1)/2}a_1a_2 \cdot \ldots \cdot a_n.
\]

*Originally from Moscow Math Olympiad, 4th Problem, Grade 10, Final Round 2017.*

*We received only 1 submission. We present the solution by Oliver Geupel.*

There are \( n \) positive numbers \( a_1, a_2, \ldots, a_n \) written on a blackboard. Under each number \( a_i \), Vasya wants to write a number \( b_i \geq a_i \) so that for every pair of numbers chosen from \( b_1, b_2, \ldots, b_n \), the ratio of one of them to the other is an integer. Prove that Vasya can write out the required numbers so that

\[
b_1b_2 \cdot \ldots \cdot b_n \leq 2^{(n-1)/2}a_1a_2 \cdot \ldots \cdot a_n.
\]

For \( i, j \in \{1, 2, \ldots, n\} \), let \( c_{i,j} = \lfloor \log_2 a_i - \log_2 a_j \rfloor \). We prove that one of the following \( n \) sequences satisfies the conditions of the problem:

\[
\begin{align*}
2^{c_{1,1}} \cdot a_1, & \quad 2^{c_{2,1}} \cdot a_1, & \quad 2^{c_{3,1}} \cdot a_1, & \quad \ldots, & \quad 2^{c_{n-1,1}} \cdot a_1, & \quad 2^{c_{n,n}} \cdot a_1, \\
2^{c_{1,2}} \cdot a_2, & \quad 2^{c_{2,2}} \cdot a_2, & \quad 2^{c_{3,2}} \cdot a_2, & \quad \ldots, & \quad 2^{c_{n-1,2}} \cdot a_2, & \quad 2^{c_{n,n}} \cdot a_2, \\
2^{c_{1,3}} \cdot a_3, & \quad 2^{c_{2,3}} \cdot a_3, & \quad 2^{c_{3,3}} \cdot a_3, & \quad \ldots, & \quad 2^{c_{n-1,3}} \cdot a_3, & \quad 2^{c_{n,n}} \cdot a_3, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2^{c_{1,n}} \cdot a_n, & \quad 2^{c_{2,n}} \cdot a_n, & \quad 2^{c_{3,n}} \cdot a_n, & \quad \ldots, & \quad 2^{c_{n-1,n}} \cdot a_n, & \quad 2^{c_{n,n}} \cdot a_n.
\end{align*}
\]

In the \( j \)-th sequence, we have

\[
b_1 = 2^{c_{i,j}} \cdot a_j = 2^{\lfloor \log_2 a_i - \log_2 a_j \rfloor} \cdot a_j \geq 2^{\log_2 a_i - \log_2 a_j} \cdot a_j = a_i.
\]

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Moreover, the ratio of every two members of the sequence is a power of 2 with an integer exponent. Hence, the ratio of one of them to the other is an integer.

Note that for real numbers \(x\) and \(y\) it holds \(\lceil x \rceil + \lceil y \rceil \leq \lceil x + y \rceil + 1\). Thus, for \(i, j \in \{1, 2, \ldots, n\}\), it holds

\[
c_{i,j} + c_{j,i} = \lceil \log_2 a_i - \log_2 a_j \rceil + \lceil \log_2 a_j - \log_2 a_i \rceil \leq \lceil 0 \rceil + 1 = 1.
\]

As a consequence, the product of all of the \(n^2\) members of the \(n\) sequences is

\[
2^{\left(\sum_{i,j=1}^{n} c_{i,j}\right)} \cdot a_1^n a_2^n \cdots a_n^n = 2^{\left(\sum_{i<j}^{n} c_{i,j} + c_{j,i}\right)} \cdot a_1^n a_2^n \cdots a_n^n \\
\leq 2^{(n-1)n/2} \cdot a_1^n a_2^n \cdots a_n^n.
\]

We conclude that the product of the members of at least one of the \(n\) sequences is not greater than \(2^{(n-1)/2} \cdot a_1 a_2 \cdots a_n\). This completes the proof.