One of the challenges in teaching mathematics is allowing time for processing while ensuring that individuals finishing quickly do not diminish opportunities for satisfaction and accomplishment of others. For instance, blurting out an answer diminishes both motivation and satisfaction for others attempting to determine a solution. One avenue to address this concern is to provide problems with multiple solutions. An elementary example of this is posed here with the challenge being to create a magic triangle. Magic triangles have equal sums of numbers along each of the three sides.

**Problem**

Using the digits 1, 2, 3, 4, 5, and 6, fill in the circles so that the numbers along each side of the triangle add up to the same amount.

```
  
  o   o

  o   o   o
```

**The Solving Process**

Typically people play with the numbers moving them around until a magic triangle appears, or some frustration perhaps emerges. By that time someone in the class will have found a magic triangle. The instinct upon finding a solution is to stop, until either another person gets a triangular arrangement with a different magic sum, or a teacher prompts the class suggesting multiple solutions are possible.

Suppose that the first person mentions getting a magic triangle with sums of 11 along each side, and then the next person gets a magic sum of 9. As a teacher, my inclination would be to tell all of the students that it is possible to find a magic triangle with a sum of 9 along each side while mentioning that there are more sums that work. Those of you looking for a first solution can work with the total being 9, whereas, others can try to find more solutions.

The pedagogical value of the problem is enhanced as students engage at a level suited to their own experience with the problem to that point. Further, the class...
has a common goal of trying to identify the various solutions. We will reach a place where magic triangles with sums of 9, 10, 11 and 12 have been found. This represents all of the possible solutions.

_The Underlying Mathematics_

How do we know that there are no other solutions? Would it be practical for us to find these solutions without being so random in our approach? How might the mathematics figure into this problem?

Let us begin to consider the possible sums for the magic triangle using the digits from 1 through 6. First, we note that the sum of these numbers is 21. Secondly, note that each of the numbers will appear in one sum, and those in the vertices will appear a second time. Hence, the largest possible total of the three sums would be 21 + (4 + 5 + 6) and likewise, the smallest would be 21 + (1 + 2 + 3). The case with the larger total of 36 corresponds to a magic triangle with a sum of 12 along each side, whereas, the smaller total would require sums of 9 along the sides. Hence, the only possible magic sums are 9, 10, 11, and 12.

Consider one of the extreme cases as an example. Suppose we want the magic sum to be 9. Recall that necessitates the placement of 1, 2, and 3 in the vertices. The middle numbers are placed accordingly to produce sums of 9, as shown.

\[
\begin{array}{c}
1 \\
6 & 5 \\
2 & 4 & 3
\end{array}
\]

Suppose that we had wanted to know whether it would be possible to have a magic sum of 12. (Of course, we now know that it is.) How could we proceed?

Observe that 3 \times 12 = 36 and 1 + 2 + 3 + 4 + 5 + 6 = 21. Hence, the difference of 15 must be the sum of the numbers in the vertices. This can only be if we place the digits 4, 5, and 6 in the corners. (Note that with a triangle the ordering of these does not matter.)

\[
\begin{array}{c}
5 \\
1 & 3 \\
6 & 2 & 4
\end{array}
\]

It turns out that the two extreme cases are easier to analyze as they only give us one choice for the set of numbers that must appear in the vertices. Suppose that we wanted to consider whether it would be possible to make the magic sum 11. Proceeding as above, we require 3 \times 11 - 21 = 12 as the sum of the numbers in the vertices. This gives three possibilities: (1, 5, 6), (2, 4, 6), and (3, 4, 5). Mixed results will emerge with efforts to complete the triangles. For example, 5 + 6 = 11
already without adding a third number. The \((3, 4, 5)\) arrangement is unworkable since the side with 3 and 4 will require a second 4 to make 11. Hence, only one magic triangle results with a sum of 11. It is shown below.

\[
\begin{array}{ccc}
2 \\
3 & 5 \\
6 & 1 & 4 \\
\end{array}
\]

Analysis of the case with the magic sum of 10 is included in the concluding set of questions to consider.

**Concluding comments**

In summary, a problem that is accessible to students in elementary school can serve as a starting point for deeper investigation. The simplicity of the concept allows for valuable problem solving qualities such as playfulness, conjecturing and pattern seeking at the forefront. The relative absence of any mathematical complication allows for elaboration and development of the balance between finding a solution and analyzing a problem. The simplicity of the task brings attention to the process and the mathematics.

For those who want to delve deeper into the idea here, it is worth noting that any discussion becomes considerably more advanced by having four numbers along each side and using the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9. Suggestions are offered for consideration beginning with #3 in the closing set of ideas for consideration.

**Ideas to Consider**

1. Complete the analysis of the elementary magic triangle problem by considering all possible cases for the vertices in a magic triangle with a sum of 10 along each side.

2. The smallest and largest digits from our example are 1 and 6. Their sum is 7. Revisit the solutions for each of the magic sums (9, 10, 11, and 12) that worked. For each solution, create a complementary solution by replacing each number in the solution with its positive difference from 7. That is, for each of \(x = 1, 2, 3, 4, 5,\) and 6, replace \(x\) with \((7 - x)\). What do you notice?

3. Extend the ideas from the elementary example discussed to the next level using the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9. Each magic triangle will have four digits on each side in this case. Begin by determining the range of possible magic sums. What are the smallest and largest possible magic sums?

4. Select either the largest or smallest possible magic sums from Question 3, and create all possible magic triangles for that sum. Note that simply interchanging the middle two numbers (those not in the vertices) along a side will not be considered to be creating a new magic triangle.
5. Choose any other possible magic sum between the extreme values. Complete the analysis for that case and determine how many (if any) magic triangles with that sum are possible.

6. Observe that the extreme digits of 1 and 9 add to 10. Apply the ideas of Question 2 above to any of the magic triangles you found in questions 4 and/or 5. Comment on your findings.

The author welcomes feedback via email [johnm@unb.ca](mailto:johnm@unb.ca). Comments from teachers or students would be particularly helpful as we move into the second year of this feature, *Teaching Problems*. 