The problem that follows was the September 2019 problem of the month from the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. The problem of the month is a new feature for the CEMC and consists of a problem that is meant to be quite challenging for high school students. One problem appears every month with a hint partway through the month and full solution at the end. You can check out the problem of the month and other CEMC features at https://www.cemc.uwaterloo.ca.

A point \((x, y)\) in the plane is called a lattice point if it has integer coordinates. Points \(P\), \(Q\), and \(R\) are distinct lattice points. Prove that the measure of \(\angle PQR\) cannot be \(60^\circ\).

This problem seems easy enough on the surface but you may be wondering “where do I begin?” Sometimes a problem is lacking information and we have to make an assumption in order to solve it. For example, if the problem asked: what is the next number in the sequence \(1, 2, 4, 8, 16, \ldots\) ? In this case, since all we are given are the numbers, without context, we can only assume that any pattern we find continues to hold. Thus, if we see that the numbers given satisfy \(t_{n+1} = 2t_n\), we can assume this continues to hold and the sequence continues \(32, 64, 128, \ldots\). In problems of this sort, we must make an assumption in order to get an answer because the next number could be anything. For example, searching The On-Line Encyclopedia of Integer Sequences (https://oeis.org) yields our sequence (A000079: Powers of 2), as well as

- **A027423**: Number of positive divisors of \(n!\) (starting with \(n = 1\)): 1, 2, 4, 8, 16, 30, \ldots.
- **A000127**: Maximal number of regions obtained by joining \(n\) points around a circle by straight lines. Also number of regions in 4-space formed by \(n-1\) hyperplanes.: 1, 2, 4, 8, 16, 31, \ldots.
- **A006261**: The sum of the first six terms of the \(n\)-th row in Pascal’s triangle, i.e.

\[
t_n = \sum_{i=0}^{5} \binom{n}{i}
\]

1, 2, 4, 8, 16, 32, 63, \ldots.

If you allow the sequence to start \(1, 1, 2, \ldots\) there are a few more.

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In other situations people make assumptions that lead to an incorrect solution. For example when asked to draw the smallest number of line segments so that at least one line segment has passed through each dot in the square array pictured below, many people will assume that the line segments have to stay within the square defined by the dots and conclude that five segments are needed. If we didn’t make that assumption, we might have eventually found the four segment solution on the right. We can also better the solution to three if “dots” are meant to mean “small circles” and our page is large enough.

So we have seen assumptions that we need to make in order to solve a problem as well as assumptions that people sometimes make that change the conditions of the original problem and don’t give the desired result. There are other “assumptions” that can be made that can simplify the problem, without changing it. Suppose, for example, we had $P(13,9)$, $Q(9,7)$ and $R(15,4)$ and we wanted to measure the angle $\angle PQR$. What if we moved each point four units to the left and 2 units down to get $P’(9,7)$, $Q’(5,5)$ and $R’(11,2)$ – would the angle change?

The angle stays the same because the angle is invariant under rigid transformations like translations, rotations and reflections. As a result, moving all the points by the same amount doesn’t change the angle. We can say without loss of generality, assume that $Q$ is at the origin. What this means is that we will assume that $Q$ is $(0,0)$. We do this, without loss of generality, because we could have had $Q$ anywhere, then moved it to the origin to make the numbers easier and the problem would be unchanged. We can also argue that rotating our coordinate system by a multiple of $90^\circ$ changes the location of the points, but not the size of the angle. Finally, notice that if we swap the coordinates of $P$ and $R$ the angle $\angle PQR$ remains unchanged. Thus, before we start solving the problem we will make the following assumption:

Assume, without loss of generality, that $Q$ is at the origin, $P(a,b)$ is in the first quadrant and $R(c,d)$ is in the first or fourth quadrant such that the slope of $QP$ is greater than the slope of $QR$.

Since we are only interested in the angle $\angle PQR$, then if $\gcd(a,b) = d > 1$, then $P’(\frac{a}{d}, \frac{b}{d})$ is on the segment $QP$ and hence $\angle PQR = \angle P’QR$. Similarly with the coordinates of $R$. So we can further assume, without loss of generality, that $\gcd(a,b) = \gcd(c,d) = 1$ and we are ready to start our proof.
**Solution 1:** Construct the triangle \( PQR \) and inscribe it in a rectangle whose sides are parallel to the coordinate axes.

![Diagram of triangle PQR inscribed in a rectangle](image)

Thus the side lengths of the rectangle are integer, which means so is its area. The rectangle is partitioned into four triangles. Three of the triangles are right angled with integer legs, which means the area of these triangles are either an integer or half of an integer. This means that twice the area of triangle \( PQR \) must be an integer.

But,

\[
[PQR] = \frac{1}{2}|PQ||QR|\sin(\angle PQR) = \frac{\sin(\angle PQR)\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}}{2},
\]

hence, if \( \angle PQR = 60^\circ \)

\[
2[PQR] = \frac{\sqrt{3}}{2}\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}.
\]

Clearly the left side of the equation is an integer, so the right side must be as well. Since we have a factor of \( \sqrt{3} \) on the right, we need either \( 3 \mid (a^2 + b^2) \) or \( 3 \mid (c^2 + d^2) \) in order to rationalize the \( \sqrt{3} \). Since \( 1^2 \equiv 2^2 \equiv 1 \pmod{3} \), the only way to get \( 3 \mid (a^2 + b^2) \) is to have \( 3 \mid a \) and \( 3 \mid b \), which, since we assumed \( \gcd(a, b) = 1 \) is impossible. Similarly, we cannot have \( 3 \mid (c^2 + d^2) \), so we cannot make the angle 60°.

**Solution 2:** As in solution 1, we assume that \( Q \) is at the origin, \( P(a, b) \) is in the first quadrant and the slope of \( QR \) is smaller than the slope of \( PQ \). Then let \( \theta \) and \( \phi \) represent the directed angles between the positive \( x \)-axis and \(QP \) and \( QR \), respectively. That is, if \( R \) is in the first quadrant \( \phi > 0 \), and if \( R \) is in the fourth quadrant \( \phi < 0 \).

Thus, \( \angle PQR = \theta - \phi \), but \( \tan \theta = \frac{b}{a} \) and \( \tan \phi = \frac{d}{c} \). Hence

\[
\tan(\angle PQR) = \tan(\theta - \phi) = \frac{\tan(\theta) - \tan(\phi)}{1 + \tan(\theta)(\tan(\phi))} = \frac{\frac{b}{a} - \frac{d}{c}}{1 + \frac{bd}{ac}}.
\]
which is defined and rational as long as \(1 + \frac{bd}{ac} \neq 0\). Given that \(\tan 60^\circ = \sqrt{3}\) is not rational, and if \(1 + \frac{bd}{ac} = 0\), we would get \(\angle Q = 90^\circ\), then we conclude that \(\angle Q \neq 60^\circ\).

Judicially use assumptions in your proofs where they simplify the process without altering the problem itself. You may want to check out the official solutions to this problem which are slight variations on mine (or mine is a variation of theirs). For your enjoyment, the current problem of the month, for December, is reproduced below. You may want to check the others out.

Let \(a, b, c, \) and \(d\) be rational numbers and \(f(x) = ax^3 + bx^2 + cx + d\).

Suppose \(f(n)\) is an integer whenever \(n\) is an integer and that

\[
\frac{1}{3}n^3 - n - \frac{2}{3} \leq f(n) \leq \frac{1}{3}n^3 + n^2 + 2n + \frac{4}{3}
\]

for every integer \(n\) with the possible exception of \(n = -2\).

(a) Show that \(a = \frac{1}{3}\).

(b) Find \(f(10^{2019}) - f(10^{2019} - 1)\).