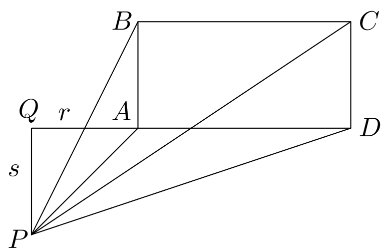
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(1), p. 28–32.

4301. *Proposed by Bill Sands.*

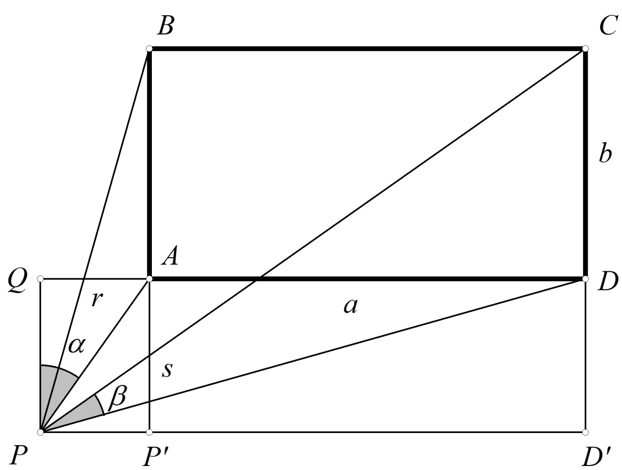
Four trees are situated at the corners of a rectangle $ABCD$. You are standing at a point P outside the rectangle, the nearest point of the rectangle to you being its corner A. To you in this position, the four trees, in the order B, A, C, D as in the diagram, appear to be equally spaced apart. Let Q be the foot of the perpendicular from P to line AD, and set $r = QA$, $s = PQ$.



a) Find the lengths of the sides of the rectangle in terms of r and s.

b) Find the range of $\angle APQ$.

We received 3 correct solutions and one incorrect submission. We present the solution by Cristobal Sanchez-Rubio.



(a) Calling $\beta = \angle BPA = \angle APC = \angle CPD$ and $\alpha = \angle QPA$, we have $2\beta = \angle BPC = \angle APD$. Also,

$$\tan\angle BPC = \tan\left(\angle BPP' - \angle CPD'\right) = \frac{\frac{b+s}{r} - \frac{b+s}{a+r}}{1 + \frac{(b+s)^2}{r(a+r)}} = \frac{a\,(b+s)}{r\,(a+r) + (b+s)^2}$$

and

$$\tan\angle APD = \tan\left(\angle APP' - \angle DPD'\right) = \frac{\frac{s}{r} - \frac{s}{a+r}}{1 + \frac{s^2}{r(a+r)}} = \frac{sa}{r\,(a+r) + s^2}.$$

Equating these expressions gives successively

$$\begin{aligned} \frac{b+s}{r\,(a+r) + (b+s)^2} &= \frac{s}{r\,(a+r) + s^2} \\ r(b+s)(a+r) + s^2(b+s) &= rs(a+r) + s(b+s)^2 \\ rb(a+r) &= sb(b+s) \\ \frac{r}{s} &= \frac{b+s}{a+r}. \end{aligned}$$

Thus $\alpha = \angle CPD'$, implying that $\beta = 90^\circ - 2\alpha$, so that $\tan 2\alpha = \frac{1}{\tan\beta}$. From

$$\tan 2\alpha = \frac{2\frac{r}{s}}{1 - \frac{r^2}{s^2}} = \frac{2rs}{s^2 - r^2}$$

and

$$\begin{aligned} \tan\beta &= \tan\left(\angle BPA\right) = \tan(\angle BPP' - \angle APP') \\ &= \frac{\frac{b+s}{r} - \frac{s}{r}}{1 + \frac{bs+s^2}{r^2}} = \frac{br}{s^2 + r^2 + bs}, \end{aligned}$$

we have

$$\frac{2rs}{s^2 - r^2} = \frac{s^2 + r^2 + bs}{br},$$

implying that

$$b = \frac{s^4 - r^4}{3r^2 s - s^3}.$$

Substituting this expression for b in

$$\frac{r}{s} = \frac{b+s}{a+r},$$

we now obtain

$$a = \frac{4\left(rs^2 - r^3\right)}{3r^2 - s^2}.$$

(b) From $3\beta < 90^\circ$, $2\alpha < 90^\circ$, and $\beta = 90^\circ - 2\alpha$, we have $30^\circ < \alpha < 45^\circ$.

4302. *Proposed by Martin Lukarevski.*

Let A be a $m \times n$ matrix with $m \geq n$ and X be any $n \times m$ matrix such that XA is invertible. Find the eigenvalues of the matrix $A(XA)^{-1}X$.

We received 3 solutions and will feature just one of them here. Solution by Missouri State University Problem Solving Group.

The eigenvalues are 0 (with multiplicity $m - n$) and 1 (with multiplicity n). We first note that both A and X have rank n: Since A is $m \times n$ and X is $n \times m$ and $m \geq n$, we have rank $A \leq n$ and rank $X \leq n$. But

$$n = \text{rank}(XA) \leq \min\{\text{rank } X, \text{rank}A)\}) \leq n.$$

Now, since X has rank n, there are $m-n$ vectors in a basis for the kernel (nullspace) of X. These vectors are also in the kernel of $A(XA)^{-1}X$, showing that 0 is an eigenvalue of $A(XA)^{-1}X$ with multiplicity at least $m - n$. Next, for any $\mathbf{v}$ in F^n (we are assuming these are matrices with entries in a field F),

$$A(XA)^{-1}X(A\mathbf{v}) = A(XA)^{-1}(XA)\mathbf{v} = A\mathbf{v},$$

so every vector in the range of A is an eigenvector with eigenvalue 1. But since $\text{rank}(A) = n$, there are n linearly independent such vectors, and hence 1 is an eigenvalue of $A(XA)^{-1}X$ of multiplicity n and 0 is an eigenvalue of multiplicity exactly $m - n$.

4303. *Proposed by Tung Hoang.*

Find the following limit

$$\lim_{n\to\infty} \left\{(6 + \sqrt{35})^n\right\},$$

where $\{x\} = x - [x]$ and $[x]$ is the greatest integer function.

We received 12 submissions, all correct. Almost all of these solutions made use of the Binomial Theorem and are very similar to one another. We present two solutions, both of which give stronger results and the second one is different from all the rest.

Solution 1, by Oliver Geupel.

We prove the more general result that for all $a, b \in \mathbb{N}$ with $(a - 1)^2 < b < a^2$, it is true that

$$\lim_{n\to\infty} \{(a + \sqrt{b})^n\} = 1.$$

The proposed problem is the special case when $a = 6$ and $b = 35$. By the Binomial Theorem, we have

$$(a + \sqrt{b})^n + (a - \sqrt{b})^n = \sum_{k=0}^{n} \binom{n}{k} \left(1 + (-1)^k\right) a^{n-k} b^{k/2}$$

which is an integer, say c_n. Since $0 < a - \sqrt{b} < 1$, we have

$$[(a+\sqrt{b})^n] = [c_n - (a-\sqrt{b})^n] = c_n - 1,$$

so

$$\begin{aligned}\{(a+\sqrt{b})^n\} &= (a+\sqrt{b})^n - [(a+\sqrt{b})^n] \\ &= c_n - (a-\sqrt{b})^n - (c_n - 1) \\ &= 1 - (a-\sqrt{b})^n\end{aligned}$$

from which $\lim\limits_{n\to\infty} \{(a+\sqrt{b})^n\} = 1$ follows.

Solution 2, by Missouri State University Problem Solving Group.

We prove in general that if $x \in \mathbb{R}, x > 1$ such that $x + \dfrac{1}{x} \in \mathbb{Z}$, then $\lim\limits_{n\to\infty} \{x^n\} = 1$. The proposed problem is the special case when $x = 6 + \sqrt{35}$.

We first prove that $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ for all integers $n \geq 0$. Since this is clear for $n = 0, 1$ it follows from the strong induction that for all $n \geq 2$,

$$x^n + \frac{1}{x^n} = \left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right) \in \mathbb{Z}.$$

Let $x^n + \dfrac{1}{x^n} = m \in \mathbb{Z}$. Since $x > 1$, it is obvious that $m - 1 < x^n < m$, so $[x^n] = m - 1$. Hence,

$$\{x^n\} = x^n - [x^n] = x^n - m + 1 = 1 - \frac{1}{x^n}$$

from which $\lim\limits_{n\to\infty} \{x^n\} = 1$ follows.

4304. *Proposed by Michel Bataille.*

Evaluate

$$\cot\frac{\pi}{7} + \cot\frac{2\pi}{7} + \cot\frac{4\pi}{7} + \cot^3\frac{\pi}{7} + \cot^3\frac{2\pi}{7} + \cot^3\frac{4\pi}{7}.$$

We received 12 submissions, of which 11 were correct and one was incomplete. We present the solution by Kee-Wai Lau.

We show that the sum of the problem, denoted by S, equals $\dfrac{32}{\sqrt{7}}$. Let $\theta = \dfrac{\pi}{7}$. Since

$$\cot k\theta + \cot^3 k\theta = \frac{\cos k\theta}{\sin^3 k\theta}$$

for $k = 1, 2, 4$, we can write $S = \dfrac{A}{B^3}$, where

$$A = \cos\theta \sin^3 2\theta \sin^3 4\theta + \cos 2\theta \sin^3 4\theta \sin^3 \theta + \cos 4\theta \sin^3 \theta \sin^3 2\theta,$$
$$B = \sin\theta \sin 2\theta \sin 4\theta.$$

Using the formulas

$$4\sin^3 x = 3\sin x - \sin 3x,$$
$$2\sin x \sin y = \cos(x - y) - \cos(x + y),$$
$$2\cos x \cos y = \cos(x - y) + \cos(x + y),$$

we readily obtain $\cos x \sin^3 y \sin^3 z$ in terms of cosines only. In this way we get

$$64\cos\theta \sin^3 2\theta \sin^3 4\theta = 8 + 12\cos\theta + 9\cos 2\theta + 5\cos 3\theta$$
$$64\cos 2\theta \sin^3 4\theta \sin^3 \theta = 8 + 5\cos\theta - 12\cos 2\theta - 9\cos 3\theta$$
$$64\cos 4\theta \sin^3 \theta \sin^3 2\theta = 8 - 9\cos\theta - 5\cos 2\theta + 12\cos 3\theta$$

It follows that $A = \dfrac{3 + C}{8}$, where $C = \cos\theta - \cos 2\theta + \cos 3\theta$. It is easy to check that

$$C^2 = \frac{3}{2} - \frac{5C}{2}.$$

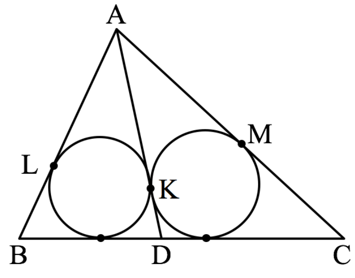
Since C is positive, $C = \dfrac{1}{2}$ and thus $A = \dfrac{7}{16}$. Hence to show that $S = \dfrac{32}{\sqrt{7}}$, it remains to show that $B = \dfrac{\sqrt{7}}{8}$. Changing product to sum, we obtain $4B = D$, where $D = \sin 3\theta + \sin 5\theta + \sin 8\theta$. It is again easy to check that

$$D^2 = \frac{3 + C}{2} = \frac{7}{4}.$$

Since $D > 0$, we have $D = \dfrac{\sqrt{7}}{2}$ and thus $B = \dfrac{\sqrt{7}}{8}$ indeed. This completes the solution.

4305. *Proposed by Moshe Stupel and Avi Sigler.*

Find a nice description of the point D on side BC of a given triangle ABC so that the incircles of the resulting triangles ABD and ADC are tangent to one another at a point of their common tangent line AD.



We received 11 submissions of which all but one were correct and complete. We present the solution by Joel Schlosberg, together with a construction by the problem proposers Moshe Stupel and Avi Sigler.

We use the fact that in a triangle XYZ, the distance from vertex X to the point of tangency on side XY of the circle inscribed in XYZ is $(XY + XZ - YZ)/2$.

Under the assumption that D is a point on BC such that the incircles of triangles $\triangle ABD$ and $\triangle ACD$ are tangent to each other, and the segment AD passes through their point of tangency, we have

$$KD = \frac{DA + DB - AB}{2} = \frac{DA + DC - AC}{2}.$$

It follows that $DB - AB = DC - AC$, and

$$DB = \frac{(DC - AC + AB) + (BC - DC)}{2} = \frac{AB + BC - AC}{2}.$$

That is, D is the point of tangency on BC of the incircle of $\triangle ABC$.

Conversely, assume that D is the point of tangency on BC of the incircle of $\triangle ABC$, and let K_1 and K_2 be the points of tangency on AD of the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. We have that

$$DK_1 = (DB + DA - AB)/2 \quad \text{and} \quad DK_2 = (DA + DC - AC)/2,$$

and their difference

$$\begin{aligned} DK_1 - DK_2 = \frac{DB - AB - DC + AC}{2} &= \frac{DB - AB + (BD - BC) + AC}{2} \\ &= DB - \frac{AB + BC - AC}{2} = 0. \end{aligned}$$

So, the points of tangency K_1 and K_2, are the same. Thus the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent to each other.

In conclusion, the point D can be characterized as the point of tangency on BC of the circle inscribed in $\triangle ABC$.

Editor's comment. Moshe Stupel and Avi Sigler presented a construction of the point D. Their construction is because D can also be viewed as the foot of the perpendicular from the center of the circle inscribed in $\triangle ABC$ to the side BC. Specifically, construct the bisectors of angles $\angle ABC$ and $\angle ACB$ and take their intersection to find the center of the circle inscribed in $\triangle ABC$. From the incircle center we drop a perpendicular to the side BC and obtain the point D.

Most solvers, starting from the assumption that the incircles of triangles $\triangle ABD$ and $\triangle ACD$ are tangent to each other, showed that the point D is the point of tangency on BC of the incircle of $\triangle ABC$. Two authors, Joel Schlosberg and C.R. Pranesachar, added that the reverse statement holds, namely, if D is the point of tangency on BC of the incircle of $\triangle ABC$ then the incircles of triangles $\triangle ABD$ and $\triangle ACD$ are tangent to each other.

4306. *Proposed by Marius Drăgan.*

Prove that

$$\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}\right]=\left[\sqrt{16n+20}\right]$$

for all $n \in \mathbb{N}$.

We received 9 solutions. We present the solution by Nghia Doan, slightly edited.

It is simple to verify the identity when $n = 0$, so assume that $n \geq 1$. By squaring both sides, one can easily check that $\sqrt{n}+\sqrt{n+3} \geq \sqrt{4n+5}$:

$$n+(n+3)+2\sqrt{n(n+3)} \geq 4n+5 \iff \sqrt{n(n+3)} \geq n+1 \iff n^2+3n \geq n^2+2n+1,$$

where the last inequality clearly holds for $n \geq 1$.

Moreover,

$$\sqrt{n+1}+\sqrt{n+2}-\sqrt{n}-\sqrt{n+3}=\frac{1}{\sqrt{n+1}+\sqrt{n}}-\frac{1}{\sqrt{n+3}+\sqrt{n+2}}>0,$$

whence $\sqrt{n+1}+\sqrt{n+2} > \sqrt{n}+\sqrt{n+3} \geq \sqrt{4n+5}$. Therefore

$$\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3} > 2\sqrt{4n+5}=\sqrt{16n+20}.$$

On the other hand, Jensen's inequality applied to the square root function (which is a concave function) gives us that

$$\begin{aligned}\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3} &\leq 4\sqrt{\frac{n+(n+1)+(n+2)+(n+3)}{4}}\\ &=\sqrt{16n+24}.\end{aligned}$$

Now suppose that there is an integer m such that

$$\sqrt{16n+20} < m \leq \sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3} \leq \sqrt{16n+24}.$$

Then, squaring, we get $16n+20 < m^2 \leq 16n+24$, which means m^2 must be one of $16n+21, \ldots, 16n+24$. However, the quadratic residues mod 16 are 0, 1, 4 and 9, which eliminates all the listed possibilities. So there is no integer m such that

$$\sqrt{16n+20} < m \leq \sqrt{16n+24},$$

which means that

$$[\sqrt{16n+20}] = [\sqrt{16n+24}].$$

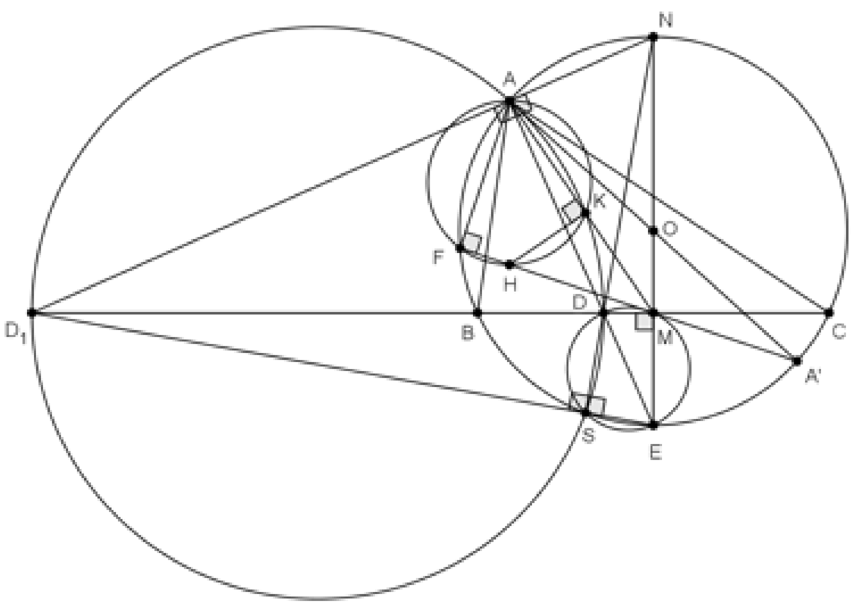
Therefore,

$$[\sqrt{n}++\sqrt{n+1}+\sqrt{n+2}+\sqrt{n+3}]=[\sqrt{16n+20}].$$

4307. *Proposed by Adnan Ibric and Salem Malikic.*

In a non-isosceles triangle ABC, let H and M denote the orthocenter and the midpoint of side BC, respectively. The internal angle bisector of $\angle BAC$ intersects BC and the circumcircle of triangle ABC at points D and E, $E \neq A$. If K is the foot of the perpendicular from H to AM and S is the intersection (other than E) of the circumscribed circles of triangles ABC and DEM, prove that quadrilateral $ASDK$ is cyclic.

We received five submissions, all of which were correct, and will feature the one from Stefan Lozanovski.



Let N be the midpoint of the arc BC that contains A. Then, because E is the midpoint of the arc BC opposite A, M lies on the diameter EN and $MD \perp EN$. Also, since

$$\angle DSE = 180^\circ - \angle DME = 90^\circ = \angle NSE,$$

point D must lie on the line SN and $ES \perp ND$. Thirdly, because $\angle EAN = 90^\circ$, we have $NA \perp DE$. Therefore the lines CD (which coincides with MD), ES, and NA concur at the orthocenter of triangle DEN, which we denote by D_1. Moreover, because of the right angles at S and at A, the quadrangle $SDAD_1$ is cyclic. It remains to show that K lies on this circle.

Since D and D_1 are the intersections of the line BC with the internal and external bisectors of $\angle BAC$, we know that D and D_1 are harmonic conjugates with respect to B and C; consequently, since M is the center of the circle with diameter BC, we get

$$MD \cdot MD_1 = MB^2.$$

Let A' be the antipode of A on the circumcircle (ABC). An old theorem of triangle geometry says that M is the midpoint of HA':

$$MH = MA'.$$

Let F be the second intersection of MH with (ABC). Because AA' is a diameter, $\angle AFA' = 90^\circ$; this with the given assumption that $\angle AKH = 90^\circ$ implies that $AFHK$ is cyclic and, therefore

$$MH \cdot MF = MK \cdot MA.$$

Note, finally, that the power of M with respect to (ABC) equals

$$MB \cdot MC = MF \cdot MA'.$$

In conclusion, we have

$$MD \cdot MD_1 = MB^2 = MB \cdot MC = MF \cdot MA' = MF \cdot MH = MK \cdot MA,$$

which implies that KAD_1D is cyclic. Since both S and K lie on the circle determined by A, D, D_1, we conclude that $ASDK$ is cyclic.

Comment by the proposers. The point K of our problem is known to have several nice properties. These were investigated in a recently published article, "A Special Point on the Median" by Anant Mudgal and Gunmay Handa [*Mathematical Reflections*, issue 2, 2017]. Among other things, the line BC is the perpendicular bisector of the segment KS, and AS is a symmedian of triangle ABC. While these results would quickly establish our result, we independently came upon our problem.

4308. *Proposed by Leonard Giugiuc and Sladjan Stankovik.*

Let a, b and c be positive real numbers. Prove that

$$27abc(a^2b + b^2c + c^2a) \leq (a + b + c)^2(ab + bc + ca)^2.$$

Eight correct solutions were received. One additional solution used Maple *and two others were wrong. We present the solution by AN-anduud Problem Solving Group.*

Let

$$A = a^2b + b^2c + c^2a, \quad B = ab^2 + bc^2 + ca^2, \quad C = 3abc.$$

The inequality can be written as

$$9CA \leq (A + B + C)^2.$$

Using the inequality

$$(u + v + w)^2 \geq 3(uv + vw + wu),$$

we find that

$$\begin{aligned} B &= \sqrt{(ca^2 + ab^2 + bc^2)(ab^2 + bc^2 + ca^2)} \\ &\geq \sqrt{3abc(a^2b + b^2c + c^2a)} = \sqrt{CA}, \end{aligned}$$

so that, by the arithmetic-geometric means inequality,

$$(A + B + C)^2 \geq (A + C + \sqrt{CA})^2 \geq (2\sqrt{CA} + \sqrt{CA})^2 = 9CA,$$

as desired.

4309. *Proposed by Daniel Sitaru.*

Let a, b and c be real numbers such that $a + b + c = 3$. Prove that

$$2(a^4 + b^4 + c^4) \geq ab(ab + 1) + bc(bc + 1) + ca(ca + 1).$$

Fifteen correct solutions were received. While it was not clear that everyone took on board the possibility that some of the variables could be negative, the inequalities they invoked turned out to be justifiable. We present three solutions.

Solution 1, by Šefket Arslanagić.

Using the inequality

$$3(x^2 + y^2 + z^2) \geq (x + y + z)^2,$$

we have that

$$\begin{aligned} a^4 + b^4 + c^4 &\geq \frac{1}{3}(a^2 + b^2 + c^2)(a^2 + b^2 + c^2) \\ &\geq \frac{1}{9}(a + b + c)^2(a^2 + b^2 + c^2) = a^2 + b^2 + c^2 \\ &\geq ab + bc + ca. \end{aligned}$$

Also $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$, so that the desired inequality holds.

Solution 2, by the AN-anduud Problem Solving Group.

Using the inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

we have that

$$a^4 + b^4 + c^4 \geq (ab)^2 + (bc)^2 + (ca)^2.$$

Since

$$x^4 - 4x + 3 = (x - 1)^2[(x + 1)^2 + 2] \geq 0,$$

we find that

$$\begin{aligned} a^4 + b^4 + c^4 &\geq 4(a + b + c) - 9 = 3 = \frac{1}{3}(a + b + c)^2 \\ &= \frac{1}{6}[(a - b)^2 + (b - c)^2 + (c - a)^2 + 6(ab + bc + ca)] \\ &\geq ab + bc + ca. \end{aligned}$$

Adding these two inequalities for $a^4+b^4+c^4$ yields the desired result. Equality holds iff $a=b=c=1$.

Solution 3, by Paolo Perfetti and Angel Plaza, independently.

Recall the Muirhead Inequalities for three variables. For $a,b,c>0$ and $p\geq q\geq r$, let

$$[p,q,r]=\frac{1}{6}(a^pb^qc^r+a^pb^rc^q+a^qb^pc^r+a^qb^rc^p+a^rb^pc^q+a^rb^qc^p).$$

Then,

$$p\geq u,\quad p+q\geq u+v\quad\text{and}\quad p+q+r=u+v+w$$

together imply that $[p,q,r]\geq[u,v,w]$.

Make the given inequality homogeneous by replacing each 1 by $\frac{1}{9}(a+b+c)^2$. Thus we have to prove that

$$\begin{aligned}18(a^4+b^4+c^4)\geq 11(a^2b^2+b^2c^2+a^2c^2)+(a^3b+ab^3+b^3c+bc^3+a^3c+ac^3)\\ +5(a^2bc+ab^2c+abc^2),\end{aligned}$$

or, equivalently,

$$9[4,0,0]\geq\frac{11}{2}[2,2,0]+[3,1,1]+\frac{5}{2}[2,1,1].$$

This is true since

$$\begin{aligned}[4,0,0]&\geq[2,2,0],\\ [4,0,0]&\geq[3,1,1],\\ [4,0,0]&\geq[2,1,1].\end{aligned}$$

4310. *Proposed by Steven Chow.*

Let $\triangle A_1B_1C_1$ be the incentral triangle with respect to $\triangle ABC$, i.e., A_1 is the point of intersection of $\overline{BC}$ and $\overleftrightarrow{AI}$ where I is the incentre of $\triangle ABC$, with B_1 and C_1 similarly defined. Let r be the inradius of $\triangle ABC$.

a) Prove that $AA_1\cdot BB_1\cdot CC_1\geq\frac{3\sqrt{3}}{2}(BC+CA+AB)\,r^2$.

b) $\star$ Prove or disprove that $B_1C_1\cdot C_1A_1\cdot A_1B_1\geq 3r^3\sqrt{3}$.

Remark. Curiously, this problem was discovered by the proposer when he misread problem 4203. See problem 4203 to appreciate the connection.

We received 6 submissions of which 4 were correct and complete. We present one solution for part a) and two solutions for part b).

Solution 1 to part a), by Michel Bataille.

Let

$$a=BC,\ b=CA,\ c=AB\text{ and }s=(a+b+c)/2.$$

Since AA_1, BB_1, and CC_1 are the angle bisectors of triangle $\triangle ABC$, we know that their lengths are

$$AA_1 = \frac{2\sqrt{bcs(s-a)}}{b+c}, \quad BB_1 = \frac{2\sqrt{cas(s-b)}}{c+a}, \quad CC_1 = \frac{2\sqrt{abs(s-c)}}{a+b}.$$

Therefore,

$$AA_1 \cdot BB_1 \cdot CC_1 = \frac{8s \cdot abc\sqrt{s(s-a)(s-b)(s-c)}}{(b+c)(c+a)(a+b)}.$$

Due to the link between the area of a triangle, the length of its three sides (Heron's formula) and the radius of its inscribed circle, r, we have

$$Area(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)} = rs,$$

and

$$AA_1 \cdot BB_1 \cdot CC_1 = \frac{8s^2 r \cdot abc}{(b+c)(c+a)(a+b)}.$$

Moreover, $abc = 4srR$, where R is the radius of circumscribed circle of $\triangle ABC$, leading to

$$AA_1 \cdot BB_1 \cdot CC_1 = \frac{32s^3r^2R}{(b+c)(c+a)(a+b)}.$$

From the concavity of the sine function on $[0, \pi]$, we have

$$2s = 2R(\sin A + \sin B + \sin C) \leq 2R \cdot 3 \cdot \sin\left(\frac{A+B+C}{3}\right) = 3R\sqrt{3} \qquad (1)$$

and, consequently

$$AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{64s^4r^2}{3\sqrt{3}(b+c)(c+a)(a+b)}.$$

Based on AM-GM inequality

$$(b+c)(c+a)(a+b) \leq \left(\frac{b+c+c+a+a+b}{3}\right)^3 = \frac{64s^3}{27} \qquad (2)$$

we obtain as required

$$AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{64s^4r^2}{3\sqrt{3}} \cdot \frac{27}{64s^3} = \frac{3\sqrt{3}}{2}(BC + CA + AB)r^2.$$

Solution 2 to part b), by Michel Bataille.

We prove the inequality. The angle bisector theorem can be used to calculate the following lengths:

$$BA_1 = ac/(b+c),\ CA_1 = ab/(b+c),\ AB_1 = bc/(a+c),$$
$$CB_1 = ab/(a+c),\ AC_1 = bc/(a+b),\ BC_1 = ac/(a+b).$$

Consequently,

$$\begin{aligned} B_1C_1^2 &= AC_1^2 + AB_1^2 - 2AC_1 \cdot AB_1 \cos A \\ &= \frac{b^2c^2}{(a+b)^2(a+c)^2}\left[(a+b)^2 + (a+c)^2 - 2(a+b)(a+c)\cos A\right] \\ &= \frac{b^2c^2}{(a+b)^2(a+c)^2}\left[(1-\cos A)(2a^2+2ac+2ab+2bc) + (b-c)^2\right] \\ &\geq \frac{4b^2c^2}{(a+b)(a+c)}\sin^2(A/2). \end{aligned}$$

Above we used $1 - \cos A = 2\sin^2(A/2)$. Similar inequalities are valid for C_1A_1 and A_1B_1, implying that

$$B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq \frac{8a^2b^2c^2 \sin(A/2)\sin(B/2)\sin(C/2)}{(a+b)(a+c)(b+c)}.$$

Since

$$r = 4R\sin(A/2)\sin(B/2)\sin(C/2),$$

$abc = 4srR$, and inequalities (1) and (2) at part a) we obtain, as claimed

$$B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq \frac{27}{64s^3} \cdot 16s^2r^2R^2 \cdot \frac{2r}{R} = 27r^3 \cdot \frac{R}{2s} \geq 27r^3 \cdot \frac{1}{3\sqrt{3}} = 3r^3\sqrt{3}.$$

Solution 3 to part b), by Leonard Giugiuc.

Notations used in part a) are maintained in what follows. We prove the inequality using two known results.

Lemma 1. If $\triangle XYZ$ is an arbitrary triangle then

$$(XY \cdot YZ \cdot ZX)^2 \geq \frac{64}{3\sqrt{3}}(Area(\triangle XYZ))^3.$$

Lemma 2. In $\triangle ABC$

$$Area(\triangle A_1B_1C_1) = \frac{2abc}{(a+b)(b+c)(c+a)} Area(\triangle ABC).$$

After combining the two lemmas we obtain

$$(B_1C_1 \cdot C_1A_1 \cdot A_1B_1)^2 \geq \frac{64}{3\sqrt{3}}\left(\frac{2abc}{(a+b)(b+c)(c+a)} Area(\triangle ABC)\right)^3.$$

Due to relations $abc = 4srR$ and $Area(\triangle ABC) = rs$ the last inequality becomes

$$(B_1C_1 \cdot C_1A_1 \cdot A_1B_1)^2 \geq \frac{64}{3\sqrt{3}}\left(\frac{8s^2r^2R}{(a+b)(b+c)(c+a)}\right)^3.$$

Moreover, using inequalities (1) and (2) mentioned in part a) we have, as required

$$(B_1C_1 \cdot C_1A_1 \cdot A_1B_1)^2 \geq \frac{64}{3\sqrt{3}}\left(\frac{16s^3r^2/(3\sqrt{3})}{64s^3/27}\right)^3 = 27r^6.$$

Editor's comment. Šefket Arslanagić pointed out that inequalities (1) and (2) were crucial to establishing the inequalities in parts a) and b). If part a) or part b) displays equality then (1) and (2) must become equalities as well. Then the triangle must be an equilateral triangle. In addition he mentioned that inequality (1) was established before and provided us the reference: Bottema O., Djordjević R.Ž., Janić R.R., Mitrinović D.S., and Vasić P.M. (1969) *Geometric inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands.